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ON THE GAUSSIAN MEASURE OF THE INTERSECTION

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The Gaussian correlation conjecture states that for any two symmetric, convex sets in n-dimensional space and for any centered, Gaussian measure on that space, the measure of the intersection is greater than or equal to the product of the measures. In this paper we obtain several results which substantiate this conjecture. For example, in the standard Gaussian case, we show there is a positive constant, c, such that the conjecture is true if the two sets are in the Euclidean ball of radius $c\sqrt{n}$. Further we show that if for every n the conjecture is true when the sets are in the Euclidean ball of radius $\sqrt{n}$, then it is true in general. Our most concrete result is that the conjecture is true if the two sets are (arbitrary) centered ellipsoids.

0. Introduction. The standard Gaussian measure on $\mathbb{R}^n$ is given by its density:

$$\mu_n(A) = \frac{1}{(2\pi)^{n/2}} \int_A \exp\left(\frac{-|x|^2}{2}\right) dx.$$

A general mean zero Gaussian measure on $\mathbb{R}^n$ is a linear image of the standard Gaussian measure.

Let $\mathscr{C}_n$ denote the collection of convex closed subsets of $\mathbb{R}^n$ which are symmetric about the origin.

CONJECTURE C. For any $n \geq 1$, if μ is a mean zero Gaussian measure on $\mathbb{R}^n$, then for all A, $B \in scon$,

$$\mu(A \cap B) \geq \mu(A)\mu(B).$$

Recall that a function f: $\mathbb{R}^n \to \mathbb{R}^+$ is called *quasiconcave* if for any $r \in \mathbb{R}$ the set $\{x \in \mathbb{R}^n \colon f(x) \geq r\}$ is convex. For such an f, let $A = \{(x, t) \colon f(x) \geq t\}$ and $A_t = \{x \colon f(x) \geq t\}$. Then, A_t is convex and symmetric if f is symmetric and further,

$$f(x) = \int_0^\infty I_{A_t}(x)\, dt.$$

By Fubini's theorem, Conjecture C has the following functional version.

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CONJECTURE C′. Let f, g be nonnegative, quasiconcave, and symmetric. Then

$$\mathbb{E}_{\mu_n}(f \cdot g) \geq \mathbb{E}_{\mu_n}(f) \cdot \mathbb{E}_{\mu_n}(g),$$

where $\mathbb{E}_{\mu_n}(f)$ denotes the expectation of f with respect to μ_n.

It is, of course, enough to show Conjecture C for symmetric and convex polytopes. Since convex, symmetric polytopes are images of the unit cube $[-1, 1]^m$ in some possibly higher dimensional space, $\mathbb{R}^m$, under a linear map an easy integral transformation shows that Conjecture C is equivalent to the following Conjecture C″, which is stated in a more probabilistic language.

CONJECTURE C″. If $\{X_i\}_{i=1}^n$ are jointly Gaussian, mean zero random variables, and $1 \leq k \leq n$, then

$$P\left(\max_{i \leq n} |X_i| \leq 1\right) \geq P\left(\max_{i \leq k} |X_i| \leq 1\right) P\left(\max_{k<i \leq n} |X_i| \leq 1\right).$$

According to Das Gupta, Eaton, Olkin, Perlman, Savage and Sobel [5], the history of this problem prior to 1970 starts with a paper of Dunnett and Sobel [7] in 1955 and after contributions by Dunn [6] in 1958, it culminated in papers of Khatri [10] and Šidák [14] (see also [15]), both in 1967, in which they independently obtained Conjecture C″ in the case $k = 1$.

THEOREM 1 ([10], [14]). *Let $\{X_i\}_{i=1}^n$ be jointly Gaussian mean zero random variables. Then*

$$P\left(\max_{i \leq n} |X_i| \leq 1\right) \geq P(|X_1| \leq 1) P\left(\max_{1<i \leq n} |X_i| \leq 1\right).$$

If a symmetric slab is defined to be a set of the form $\{x \in \mathbb{R}^n\colon |\langle x, u\rangle| \leq 1\}$ for some $u \in \mathbb{R}^n$, the theorem above is equivalent to the following theorem.

THEOREM 2. *If μ is a mean zero Gaussian measure on $\mathbb{R}^n$, $A \in \mathscr{C}_n$, and S is a symmetric slab, then*

$$\mu(A \cap S) \geq \mu(A)\mu(S).$$

As a corollary of the theorems above, Khatri and Šidák obtained a result which solved the problem studied by Dunnett and Sobel [7] and Dunn [6].

COROLLARY 1 ([10], [14]).

$$P\left(\max_{i \leq n} |X_i| \leq 1\right) \geq \prod_{i=1}^{n} P(|X_i| \leq 1).$$

Another important milestone for this problem was achieved by the work of Pitt in 1977, where the two-dimensional case was settled. For an extension of Pitt's result see [3].

THEOREM 3 ([12]). *For any* $A, B \in \mathscr{C}_2$ $\mu_2(A \cap B) \geq \mu_2(A)\mu_2(B)$.

In [5] and [8] measures other than Gaussian measures are considered. The problem can and has been attacked using measure theoretic, geometric and analytic techniques.

In this note we present several partial results using some of these techniques. In Proposition 1 (Section 1) we prove the conjecture for sets more general than sets having a common "orthogonal unconditional" basis. Our main result, Theorem 5, shows that the conjecture holds for arbitrary centered ellipsoids in $\mathbb{R}^n$.

In Section 2, we show that the conjecture is true for "small enough" sets. We also show, in Proposition 5, that the result holds "on the average." It follows from the remark after Proposition 3 that, if, in the statement of Conjecture C, one puts the factor $2^{n/2}$ on the left-hand side, then the resulting statement is true. By contrast, in Proposition 4, we prove that if one could replace the factor $2^{n/2}$ with $2^{o(n)}$, then the conjecture would follow.

We will need the following notations and concepts. In $\mathbb{R}^n$ the usual unit basis will be denoted by $e_1, e_2, \ldots, e_n$, $|\cdot|$ is the Euclidean norm and $\langle \cdot, \cdot \rangle$ the scalar product generated by $|\cdot|$. The expression $B_2^n = \{x \in \mathbb{R}^n\colon |x| \leq 1\}$ will be the Euclidean unit ball and $S^{n-1} = \{x \in \mathbb{R}^n\colon |x| \leq 1\}$ its sphere. The orthogonal group on $\mathbb{R}^n$, that is, the set of real unitary $n \times n$ matrices, will be denoted by $O(n)$. Lebesgue measure on $\mathbb{R}^n$ will be denoted by m_n.

1. Geometrical restrictions.

By induction on the dimension it is easy to see that the conjecture is true if the convex symmetric sets are 1-unconditional with respect to the same orthogonal basis $\{e_i\}_{i=1}^n$ (i.e., $(x_1, \ldots, x_n) \in A \Leftrightarrow (\pm x_1, \ldots, \pm x_n) \in A$). Here we relax somewhat the geometrical restrictions.

PROPOSITION 1. *Let* ν *be a product probability on* $\mathbb{R}^n$. *If* $A, B \in \mathscr{C}_n$ *satisfy:*

(i) $x \in A \cap B \Rightarrow x_i e_i \in A \cap B, \forall i \leq n$;

(ii) *for every pair of orthants,* Q *and* Q', $[\nu(A \cap Q) - \nu(A \cap Q')][\nu(B \cap Q) - \nu(B \cap Q')] \geq 0$ *(in particular, if* $\nu(B \cap Q)$ *are all equal);*

then $\nu(A \cap B) \geq \nu(A)\nu(B)$.

To prove this we need the following result. It can be found in [9] and is related to a result in [1].

THEOREM 4 ([9]). *Let* ν *be a product measure on* $\mathbb{R}^n$ *and let* $f_i, 1 \leq i \leq 4$, *be nonnegative functions on* $\mathbb{R}^n$ *satisfying*

$$f_1(x) f_2(y) \leq f_3(x \vee y) f_4(x \wedge y).$$

Then

$$\int f_1 \, d\nu \int f_2 \, d\nu \leq \int f_3 \, d\nu \int f_4 \, d\nu.$$

PROOF OF PROPOSITION 1. We shall first prove that the Karlin–Rinott theorem implies that, for each orthant Q,

$$\nu(A \cap Q)\nu(B \cap Q) \le \nu(Q)\nu(A \cap B \cap Q). \tag{1}$$

Let Q represent an orthant, say the first orthant, and let $f_1 = I_{A\cap Q}$, $f_2 = I_{B\cap Q}$, $f_3 = I_Q$ and $f_4 = I_{A\cap B\cap Q}$. To use the Karlin–Rinott theorem, we need to show that

$$x \in A \cap Q, \qquad y \in B \cap Q \Longrightarrow x \vee y \in Q \quad \text{and} \quad x \wedge y \in A \cap B.$$

Without loss of generality, we may assume that x and y are in the interiors of $A \cap Q$ and $B \cap Q$, respectively. We need to show that $x \wedge y \in A \cap B$. Assuming this were not true, we let w be the point in $A \cap B \cap Q$ which is the closest to $x \wedge y$. By the Pythagorean theorem, $w_i \le (x \wedge y)_i$ for every $1 \le i \le n$. By (i), the rectangular box $R = \{z \in Q; z_i \le w_i, \forall i\}$ is contained in $A \cap B$. Let U be an open set such that $x \in U \subseteq A \cap Q$ and similarly V an open set such that $y \in V \subseteq B \cap Q$. Then w is an interior point of the convex hull of U and R, which is a subset of A. Similarly, w is an interior point of the convex hull of V and R, which is a subset of B. Hence w is an interior point of $A \cap B \cap Q$. Therefore, if $x \wedge y$ is not already in $A \cap B \cap Q$, we reach a contradiction.

The Karlin–Rinott theorem now yields (1). Now apply (ii) in order to deduce that

$$2^{-n} \sum_{Q,\, Q'} \nu(A \cap Q)\nu(B \cap Q') \le \sum_{Q} \nu(A \cap Q)\nu(B \cap Q),$$

which implies together with (1) the claim. □

We now want to show the correlation conjecture for two ellipsoids (in arbitrary position).

THEOREM 5. *If A and B are centered ellipsoids in $\mathbb{R}^n$, then $\mu_n(A \cap B) \ge \mu_n(A)\mu_n(B)$.*

From Proposition 1 it follows that $\mu_n(E \cap F) \ge \mu_n(E)\mu_n(F)$ if E and F are ellipsoids with the same axis. Using the rotational invariance of μ_n we would be able to deduce Theorem 5 if we could show that for two ellipsoids E and F in the standard position, that is, $E = \{x \in \mathbb{R}^n\colon \sum_{i=1}^n (x_i^2/r_i^2) \le 1\}$ and $F = \{x \in \mathbb{R}^n\colon \sum_{i=1}^n (x_i^2/\rho_i^2) \le 1\}$, the minimum of $\mu_n(U(E) \cap F)$ over all $U \in O(n)$ is attained when U is some row permutation of the identity. Actually this is true for all rotational invariant measures on $\mathbb{R}^n$.

THEOREM 6. *Let ν be a rotation invariant measure on $\mathbb{R}^n$, and let*

$$E = \left\{x \in \mathbb{R}^n\colon \sum_{i=1}^n \frac{x_i^2}{r_i^2} \le 1\right\} \quad \text{and} \quad F = \left\{x \in \mathbb{R}^n\colon \sum_{i=1}^n \frac{x_i^2}{\rho_i^2} \le 1\right\}$$

be two ellipsoids in standard position. Then the value of $\min\{\nu(U(F) \cap E)\colon U \in O(n)\}$ *is achieved for some row permutation P of the identity; in particular this means that $P(F)$ and E are ellipsoids with the same axis.*

PROOF. Using a standard perturbation argument, we will make the following assumptions. Instead of considering the minimum of the mapping $O(n) \ni U \mapsto \int I_E(U(x)) I_F(x)\, d\nu(x)$ we let $f: [0,\infty) \to [0,\infty)$ be a continuously differentiable function with $f'(r) < 0$ whenever $r > 0$, define for $x \in \mathbb{R}^n$, $\tilde{F}(x) = f(|x|_F^2)$ where $|x|_F^2 = \sum_{i=1}^n x_i^2/\rho_i^2$ and we assume that $U_0 \in O(n)$ for which

$$\int I_E(U_0(x))\tilde{F}(x)\, d\nu(x) = \min_{U \in O(n)} \int I_E(U(x))\tilde{F}(x)\, d\nu(x).$$

We also assume that the radii $r_1, r_2, \dots, r_n$ of E and the radii $\rho_1, \rho_2, \dots, \rho_n$ of F are distinct. Finally, we will assume that ν has a strictly positive density $g(|x|)$ with respect to m_n.

In order to deduce the claim, we will show that the matrix

$$U_0^T \circ \begin{pmatrix} r_1^{-2} & & \\ & \ddots & \\ & & r_n^{-2} \end{pmatrix} \circ U_0$$

is diagonal. Since the values r_i^{-2} are distinct for $i = 1, 2 \dots n$, this would imply that U_0 must be a row permutation of some diagonal matrix J which has only the values 1 or -1 in its diagonal. Since $J(G) = G$ for any ellipsoid, we can assume that J is the identity.

We start with a variational argument. For $i < j$ in $\{1, 2, \dots, n\}$ and $\alpha \in \mathbb{R}$, let $V_{(i,j)}^{(\alpha)}$ be the matrix which acts on $\mathbb{R}^n$ in the following way. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we set

$$V_{(i,j)}^{(\alpha)}(x) := \big(x_1, \dots, x_{i-1}, x_i \cos\alpha - x_j \sin\alpha, x_{i+1}, \dots, x_{j-1}, x_i \sin\alpha + x_j \cos\alpha, x_{j+1}, \dots, x_n\big),$$

that is, $V_{(i,j)}^{(\alpha)}$ acts on the two-dimensional subspace of $\mathbb{R}^n$ spanned by e_i and e_j as a rotation by α, and on the orthogonal complement of that subspace, it is the identity.

Using the minimality of U_0 we deduce that

$$\begin{aligned} 0 &= \frac{\partial}{\partial\alpha}\left[\int I_E(U_0(x))\tilde{F}(V_{(i,j)}^{(\alpha)}(x)) g(|x|)\, dx\right]_{\alpha=0} \\ &= \int I_E(U_0(x)) f'(|x|_F^2) \\ &\quad \times \frac{\partial}{\partial\alpha}\left[\frac{(x_i\cos\alpha - x_j\sin\alpha)^2}{\rho_i^2} + \frac{(x_j\cos\alpha + x_i\sin\alpha)^2}{\rho_j^2}\right]_{\alpha=0} g(|x|)\, dx \\ &= 2(\rho_j^{-2} - \rho_i^{-2}) \int x_i x_j I_E(U_0(x)) f'(|x|_F^2) g(|x|)\, dx. \end{aligned}$$

We fix $i \le n$, and for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we let $x^{(i)} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \mathbb{R}^{n-1}$. Since the ρ_i's are distinct positive numbers, we deduce that for any linear map $L: \mathbb{R}^{n-1} \to \mathbb{R}$ we have

$$\int x_i L(x^{(i)}) I_E(U_0(x)) f'(|x|_E^2) g(|x|)\, dx = 0. \tag{2}$$

For $j \le n$ let u_j be the jth row of U_0 and $u_{(j,s)}$ the sth element of u_j. For $y \in \mathbb{R}^{n-1}$ we define

$$L(y) := \left(\sum_{j=1}^{n} \frac{u_{(j,i)}^2}{r_j^2} \right)^{-1} \sum_{j=1}^{n} \frac{u_{(j,i)}}{r_j^2} \langle u_j^{(i)}, y \rangle$$

and

$$Q(y) := \left(\sum_{j=1}^{n} \frac{u_{(j,i)}^2}{r_j^2} \right)^{-1} \left(\sum_{j=1}^{n} \frac{< u_j^{(i)}, y >^2}{r_j^2} - 1 \right).$$

For $x \in \mathbb{R}^n$ we observe that the following equivalences hold:

$$\begin{aligned}
&U_0(x) \in E \\
&\quad \Leftrightarrow \sum_{j=1}^{n} r_j^{-2} [u_{(j,i)} x_i + \langle u_j^{(i)}, x^{(i)} \rangle]^2 \le 1 \\
&\quad \Leftrightarrow x_i^2 \sum_{j=1}^{n} u_{(j,i)}^2 r_j^{-2} + 2x_i \sum_{j=1}^{n} u_{(j,i)} r_j^{-2} \langle u_j^{(i)}, x^{(i)} \rangle + \sum_{j=1}^{n} \langle u_j^{(i)}, x^{(i)} \rangle^2 r_j^{-2} \le 1 \\
&\quad \Leftrightarrow x_i^2 + 2x_i L(x^{(i)}) + Q(x^{(i)}) \le 0 \\
&\quad \Leftrightarrow L^2(x^{(i)}) \ge Q(x^{(i)}) \quad \text{and} \quad |x_i + L(x^{(i)})| \le \sqrt{L^2(x^{(i)}) - Q(x^{(i)})}.
\end{aligned}$$

We claim that $L \equiv 0$. Indeed, from the equivalences above and (2) we deduce that

$$\begin{aligned}
0 &= \int_{\{x:\ U_0(x) \in E\}} x_i L(x^{(i)}) f'(|x|_F^2) g(|x|)\, dx \\
&= \int_{L^2(x^{(i)}) \ge Q(x^{(i)})} L(x^{(i)}) \left[\int_{-L(x^{(i)}) - \sqrt{L^2(x^{(i)}) - Q(x^{(i)})}}^{-L(x^{(i)}) + \sqrt{L^2(x^{(i)}) - Q(x^{(i)})}} x_i f'(|x|_F^2) g(|x|)\, dx_i \right] dx^{(i)}.
\end{aligned}$$

Since for fixed $x^{(i)}$ the function $x_i \mapsto x_i f'(|x|_F^2) g(|x|)$ is odd and positive if and only if x_i is negative, we deduce that

$$\int_{-L(x^{(i)}) - \sqrt{L^2(x^{(i)}) - Q(x^{(i)})}}^{-L(x^{(i)}) + \sqrt{L^2(x^{(i)}) - Q(x^{(i)})}} x_i f'(|x|_F^2) g(|x|)\, dx_i$$

is positive (respectively, negative) if and only if $L(x^{(i)})$ is positive (respectively, negative). Thus we deduce that

$$L(x^{(i)}) \int_{-L(x^{(i)}) - \sqrt{L^2(x^{(i)}) - Q(x^{(i)})}}^{-L(x^{(i)}) + \sqrt{L^2(x^{(i)}) - Q(x^{(i)})}} x_i f'(|x|_F^2) g(|x|)\, dx_i$$

is positive if and only if $L(x^{(i)}) \ne 0$ and vanishes otherwise. Since $Q(0) < 0$, the inequality $L^2(x^{(i)}) \ge Q(x^{(i)})$ has solutions for a neighborhood of 0. This forces $L \equiv 0$. Going back to the definition of L we just showed that for $l \neq i$ the lth coordinate of

$$\sum_{j=1}^{n} \frac{u_{(j,i)}}{r_j^2} u_j$$

vanishes. But, on the other hand this coordinate is equal to the element in the ith row and lth column of the product

$$U_0^T \circ \begin{pmatrix} r_1^{-2} & & \\ & \ddots & \\ & & r_n^{-2} \end{pmatrix} \circ U_0.$$

Since $i \neq l$ are arbitrary elements of $\{1, \ldots, n\}$, this says that the above product is a diagonal matrix, which completes the proof of the theorem. □

While we do not know if Conjecture C′ holds for an arbitrary g and $f = I_E$, where E is an ellipsoid, we show below that it does hold for f being a Gaussian density, and g log-concave.

PROPOSITION 2. *If g is a nonnegative, symmetric, quasiconcave function on $\mathbb{R}^n$ and A is a non-negative definite matrix, then*

$$\mathbb{E}_\mu\big[\exp(-\tfrac{1}{2}\langle Ax, x\rangle)g(x)\big] \geq \mathbb{E}_\mu\big[\exp(-\tfrac{1}{2}\langle Ax, x\rangle)\big]\mathbb{E}_\mu\big[g(x)\big].$$

PROOF. It suffices to assume that $\mu = \mu_n$. Then,

$$\mathbb{E}_\mu\big[\exp(-\tfrac{1}{2}\langle Ax, x\rangle)g(x)\big] = (\det(I + A))^{-1/2}\mathbb{E}_\mu\big[g((I + A)^{-1/2}(x))\big].$$

We now diagonalize $(I + A)^{-1/2}$ with the unitary U, let $h = g \circ U$ and use the fact that μ is rotation invariant to allow us to write

$$\mathbb{E}_\mu\big[g((I + A)^{-1/2}(x))\big] = \mathbb{E}_\mu\big[g((UU^T(I + A)^{-1/2}UU^T(x))\big] = \mathbb{E}_\mu\big[h(D(x))\big].$$

So in order to show that

$$\mathbb{E}_\mu\big[\exp(-\tfrac{1}{2}\langle Ax, x\rangle)g(x)\big] \geq \mathbb{E}_\mu\big[\exp(-\tfrac{1}{2}\langle Ax, x\rangle)\big]\mathbb{E}_\mu\big[g(x)\big],$$

we need only show that

$$\mathbb{E}_\mu\big[h(D(x))\big] \geq E_\mu\big[h(x)\big].$$

Since $I - D$ is a nonnegative definite matrix, the result follows by a result of Anderson [2]. □

2. Restriction on size. We will make heavy use of the following concept from convex geometry. Recall that a nonnegative function $f\colon \mathbb{R}^n \to \mathbb{R}^+$ is called log-concave if for $x, y \in \mathbb{R}^n$ and $0 \le t \le 1$,

$$f(tx + (1-t)y) \ge f(x)^t f(y)^{1-t},$$

that is, $\log f$ is concave on its support.

Note that the indicator functions of convex sets are log-concave and that log-concave functions are quasiconcave. We also will need the following deep result of Prékopa and Leindler.

THEOREM 7 ([11] and [13]; see also [4]). *If f is log-concave on $\mathbb{R}^n$ and $1 \le k < n$, then the function $g\colon \mathbb{R}^k \to \mathbb{R}$, with*

$$g(x_1, \dots, x_k) = \int_{\mathbb{R}^{n-k}} f(x_1, \dots, x_k, z_1, \dots, z_{n-k})\, dz$$

is also log-concave.

Since $h \circ A$ is log-concave whenever h is log-concave and A is linear, and since the product of two log-concave functions is also log-concave, the corollary follows immediately.

COROLLARY 2. *If f and g are log-concave, so is $y \mapsto \int f(x+y)g(x)\, dx$.*

To get a glimpse of the mysterious power of the Prékopa–Leindler result, we will use it in order to give a very short proof of the result of Khatri and Šidák.

We first observe that the Conjectures C and thus C′ are trivially true in the case $n = 1$. Assume that $S = \{x \in \mathbb{R}^n\colon |x_1| \le s\}$ and that $A \in \mathscr{C}_n$. For $x_1 \in \mathbb{R}$, $f(x_1) := \int_{\mathbb{R}^{n-1}} I_A(x_1, y)\, d\mu_{n-1}(y)$. Since the density of μ_{n-1} and I_A are log-concave, we deduce from [11] and [13] that f is a log-concave function on $\mathbb{R}$ and thus

$$\mu(A \cap S) = \int_{\mathbb{R}} I_{[-s,s]}(x_1) f(x_1)\, d\mu_1(x_1) \ge \mu_1([-s,s])\, \mathbb{E}_{\mu_1}(f) = \mu(S)\mu(A),$$

where the inequality follows from the one-dimensional case.

Using the rotation on $\mathbb{R}^n \times \mathbb{R}^n$ given by $(x, y) \mapsto ((x+y)/\sqrt{2}, (x-y)/\sqrt{2})$ leads to the following observation.

PROPOSITION 3. *If A, $B \in \mathscr{C}_n$, we have*

$$\mu_n(A)\mu_n(B) \le \mu_n(\sqrt{2}(A \cap B))\mu_n\left(\frac{(A+B)}{\sqrt{2}}\right).$$

PROOF. Using the rotational invariance of the measure $\mu_n \otimes \mu_n$, we get

$$\begin{aligned}\mu_{2n}(A \times B) &= \int I_A(x) I_B(y)\, d\mu_n(x)\, d\mu_n(y)\\ &= \int I_A\left(\frac{u+v}{\sqrt{2}}\right) I_B\left(\frac{v-u}{\sqrt{2}}\right) \mu_n(du)\mu_n(dv)\\ &= \int \mu_n((\sqrt{2}A - u) \cap (\sqrt{2}B + u))\, \mu_n(du).\end{aligned}$$

Note that for $u \in \mathbb{R}^n$ it follows that $(\sqrt{2}A - u) \cap (\sqrt{2}B + u)$ is not empty if and only if there exists a $z \in \mathbb{R}^n$ for which $((z+u)/\sqrt{2}) \in A$ and $((z-u)/\sqrt{2}) \in B$. Since that can only happen if u lies in $(A - B)/\sqrt{2} = (A + B)/\sqrt{2}$, we deduce that the integrand can only be nonzero on $(A+B)/\sqrt{2}$. Furthermore, the mapping $u \mapsto \int \mu_n((\sqrt{2}A - u) \cap (\sqrt{2}B + u))\mu_n(du)$ is log-concave by the Prékopa–Leindler theorem. Since it is also symmetric, it is maximized at zero. Hence the integral is bounded by $\mu_n(\sqrt{2}(A \cap B))\mu_n((A+B)/\sqrt{2})$. □

REMARK. Note that for any measurable $K \subset \mathbb{R}^n$ and $c > 1$ it follows that

$$\begin{aligned}\mu_n(cK) &= (2\pi)^{-n/2} \int I_K(x/c) \exp(-|x|^2/2)\, dx\\ &= c^n (2\pi)^{-n/2} \int I_K(u) \exp(-c^2|u|^2/2)\, du \le c^n \mu_n(K).\end{aligned}$$

Thus Proposition 3 implies $\mu_n(A)\mu_n(B) \le 2^{n/2}\mu_n(A \cap B)$ if A, $B \in \mathscr{C}_n$.

Using $m_n(\cdot) \ge (2\pi)^{n/2}\mu_n(\cdot)$, we deduce the following corollaries.

COROLLARY 3. *For A, $B \in \mathscr{C}_n$ we have*

$$\mu_n(A \cap B) \ge \frac{(2\pi)^{n/2}}{m_n(A+B)} \mu_n(A)\mu_n(B).$$

COROLLARY 4. *Suppose ρ_n is chosen so that $m(2\rho_n B_2^n) = (2\pi)^{n/2}$. (Note that $\rho_n = (1/\sqrt{2})(\Gamma(1 + (n/2)))^{1/n} \sim \frac{1}{2}\sqrt{n/e}$.) Then, $\mu(A \cap B) \ge \mu(A)\mu(B)$, for all $A, B \in \mathscr{C}_n$ with $A, B \subset \rho_n B_2^n$.*

In Corollary 6 below we will show that, if we could replace the factor ρ_n by $\sqrt{n}$, then the conjecture would follow. We first make the following observation, which indicates that it would be enough to show Conjecture C approximately.

PROPOSITION 4. *Assume that there is a sequence of positive numbers (c_n) with $\lim_{n\to\infty} c_n^{1/n} = 1$, so that $\mu_n(A \cap B) \ge c_n \mu_n(A)\mu_n(B)$, for all $n \in \mathbb{N}$ and $A, B \in \mathscr{C}_n$. Then, for all $n \in \mathbb{N}$ and $A, B \in \mathscr{C}_n$,*

$$\mu_n(A \cap B) \ge \mu_n(A)\mu_n(B).$$

PROOF. For each N consider $A^N = A \times \cdots \times A$, and B^N. The assumption gives

$$\mu_n^N(A \cap B) = \mu_{Nn}(A^N \cap B^N) \geq c_{Nn}\, \mu_n(A)\mu_n(B).$$

Taking Nth roots, letting $N \to \infty$ and using the hypothesis, the result follows. □

We now show that the conjecture holds on the average. This is true for more general measures and more general sets.

PROPOSITION 5. *Let m be the Haar measure on the orthogonal group $O(n)$, and let ν be a rotational invariant probability on $\mathbb{R}^n$; assume that $A, B \subset \mathbb{R}^n$ are two star-shaped sets with 0 being a center, that is, for any $\theta \in \mathrm{S}^{n-1}$ the set $\{r \geq 0\colon r\theta \in A\}$ is an interval, which we will denote by A_θ.*

Then it follows that

$$\int_{O(n)} \nu(A \cap U(B))\, dm(U) \geq \nu(A)\nu(B).$$

PROOF. Since ν is rotational invariant, it is the image of some product probability $\nu_1 \otimes \sigma_n$ (ν_1 being a probability on $[0, \infty)$) under the map $\mathrm{S}^{n-1} \times [0, \infty) \ni (\theta, r) \mapsto \theta r$. We will also use the fact that for any θ_0 the measure σ_n is the image of m under the map $O(n) \ni U \mapsto U(\theta_0)$. Finally we observe that for two star-shaped sets A and B, with 0 being their center, and for any two θ, and θ', we deduce that $\nu_1(A_\theta \cap B_{\theta'}) = \min(\nu_1(A_\theta), \nu(B_{\theta'})) \geq \nu_1(A_\theta)\nu(B_{\theta'})$.

These observations allow us to make the following estimates:

$$\begin{aligned}\int_{O(n)} \nu(A \cap U(B))\, dm(U) &= \int_{\mathrm{S}^{n-1}} \int_{\mathrm{S}^{n-1}} \int_0^\infty I_{A_\theta}(r) I_{B_{\theta'}}(r)\, d\nu_1(r)\, d\sigma_n(\theta)\, d\sigma_n(\theta') \\ &= \int_{\mathrm{S}^{n-1}} \int_{\mathrm{S}^{n-1}} \nu_1(A_\theta \cap B_{\theta'})\, d\sigma_n(\theta)\, d\sigma_n(\theta') \\ &\geq \int_{\mathrm{S}^{n-1}} \int_{\mathrm{S}^{n-1}} \nu_1(A_\theta)\nu_1(B_{\theta'})\, d\sigma_n(\theta)\, d\sigma_n(\theta') \\ &= \int_{\mathrm{S}^{n-1}} \nu_1(A_\theta)\, d\sigma_n(\theta) \int_{\mathrm{S}^{n-1}} \nu_1(B_{\theta'})\, d\sigma_n(\theta') \\ &= \nu(A)\nu(B),\end{aligned}$$

which proves the claim. □

COROLLARY 5. *For any $r > 0$ and any $A \in \mathscr{C}_n$,*

$$\mu_n(A \cap rB_2^n) \geq \mu_n(A)\mu_n(rB_2^n).$$

Here is one example of how to use the above results.

COROLLARY 6. *If for all n, $\mu_n(A\cap B)\ge \mu_n(A)\mu_n(B)$ for all $A, B\in \mathscr{C}_n$ for which $A, B\subset \sqrt{n}B_2^n$, then the inequality holds for all n and $A, B\in\mathscr{C}_n$.*

PROOF. For $A, B\in\mathscr{C}_n$, we have

$$\begin{aligned}\mu_n(A\cap B) &\ge \mu_n(A\cap B\cap \sqrt{n}B_2^n)\ge \mu_n(A\cap\sqrt{n}B_2^n)\mu_n(B\cap\sqrt{n}B_2^n)\\ &\ge \mu_n(A)\mu_n(B)\mu_n^2(\sqrt{n}B_2^n),\end{aligned}$$

by Corollary 10. From the central limit theorem we deduce,

$$\mu_n(\sqrt{n}B_2^n)=\mu_n\left(\sum_{i=1}^n x_i^2\le n\right)=\mu_n\left(\frac{\sum_{i=1}^n(x_i^2-1)}{\sqrt{n}}\le 0\right)\to\frac{1}{2},$$

so the above proposition applies with $c_n=\mu_n(\sqrt{n}B_2^n)$. □

REMARK. In the above proof of Corollary 6, if $c<1$, one cannot substitute $c\sqrt{n}B_2^n$ for $\sqrt{n}B_2^n$.

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