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Research Article

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On the general position number of two classes of graphs

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Abstract: The general position problem is to find the cardinality of the largest vertex subset S such that no triple of vertices of S lies on a common geodesic. For a connected graph G, the cardinality of S is denoted by $\mathrm{gp}(G)$ and called the gp-number (or general position number) of G. In the paper, we obtain an upper bound and a lower bound regarding the gp-number in all cacti with k cycles and t pendant edges. Furthermore, the exact value of the gp-number on wheel graphs is determined.

Keywords: general position set, cactus, wheel

MSC 2020: 05C12, 05C69, 68Q25

1 Introduction

In the paper, all graphs are undirected, finite, and simple. Assume that $G = (V, E)$ is a connected graph with vertex set $V(G)$ and edge set $E(G)$. Let $u, v \in V(G)$. $d_G(u, v)$ denotes the number of edges on a shortest (u, v)-path in G. A (u, v)-path with length $d_G(u, v)$ is regarded as a v, r-geodesic. The interval $I_G(u, v)$ of G is the set of vertices w such that there exists a u, v-geodesic, which contains w. We refer the readers to [1] for undefined terminology and notations.

The classical Dudeney's no-three-in-line problem [2–4] is to determine as many as possible vertices that can be placed in the $m \times m$ grid such that no three vertices of them lie on a line. The famous problem has been further studied in several recent papers [5–8]. Later, Froese et al. [9] extended the problem in discrete geometry to general position subset selection problem, which is to obtain a maximal subset of vertices in general position. They also showed that the general position subset selection problem is NP-hard.

Motivated by the above two problems, the general position problem of graphs was introduced by Manuel and Klavžar [10] and Payne and Wood [11]. A subset R of $V(G)$ is a general position set in graph G if no three vertices of R lie on a common geodesic. A maximum general position set is named a gp-set of G. The *general position number* (the gp-number for short) of G is defined as the cardinality of a gp-set, denoted by $\mathrm{gp}(G)$. Hence, the aim of the general position problem is to determine bounds or exact value of the gp-number for some classes of graphs.

Anand et al. [12] deduced a formula for the general position number of the complement of an arbitrary bipartite graph. Klavžar and Yero [13] proved that $\mathrm{gp}(G) \geq \omega(G_{SR})$, where G_{SR} is the strong resolving graph of a connected graph G, and $\omega(G_{SR})$ is its clique number. In [14,15], the authors determined upper bounds of the gp-number on integer lattices and Cartesian grids and showed that the general position problem is

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NP-complete. For more properties of the general position problem, see [10,16–19]. Note that many researchers are highly concerned about the structural properties of the cactus and wheel graphs under graph parameters, such as [20–22]. Hence, it is interesting to explore the bounds of the gp-number for cactus and wheel graphs.

A connected graph G is called a cactus if its blocks consist of cycles and edges. A chain cactus is a cactus graph for which each block has at most two cut vertices and each cut vertex is shared by exactly two blocks. If a cycle of G has one cut vertex, we name it an *end-block*. A vertex of a cycle is said to be nontrivial if its degree is at least 3. Let v be a cut vertex of a cycle of G. If a component of $G - v$ contains an end-block C_0 (if v belongs to some cycle C_*, then the component does not contain vertices of C_*), then C_0 is defined as an end-block of v. Moreover, the path between two cycles is a *cyclic path*, i.e., (u, v)-path is a cyclic path, see Figure 1. Let $C_n^{t,k}$ be the class of all cacti of order n with k cycles and t pendant edges. Let C_n^k be the set of all cacti of order n with k cycles. The wheel graph W_n is the graph obtained from the cycle C_n of order n by adding a new vertex and connecting it to all vertices of C_n.

In the paper, we propose upper and lower bounds of the gp-number for cactus graphs, and the structure of the extremal graphs attaining the bounds is characterized. In addition, the exact value of the gp-number is also obtained for wheels.

2 The upper bound of cacti regarding gp-number

From [10], we know that $\mathrm{gp}(C_3) = 3$, $\mathrm{gp}(C_4) = 2$, and $\mathrm{gp}(C_n) = 3$ for $n \geq 5$.

Lemma 2.1. *Let G be a cactus with $k(\geq 2)$ cycles, and S be a* gp*-set of G. If C_0 is a cycle of G, then* $|V(C_0) \cap S| \leq 2$.

Proof. Let C_1 and C_2 be two cycles of G connected by a path P (maybe trivial). Assume that $V(C_1) \cap V(P) = u$ and $V(C_2) \cap V(P) = r$. Let u_1 and u_2, u_3 and u_4 be the adjacent vertices of u and r in C_1 and C_2, respectively. Note that $R = \{u_1, u_2, u_3, u_4\}$ is a general position set of G. Therefore, $\mathrm{gp}(G) \geq 4$.

Suppose $|V(C_0) \cap S| = 3$ and let $V(C_0) \cap S = \{x, y, z\}$. This means that $I(x, y) \cup I(y, z) \cup I(z, x) = V(C_0)$ and $I(x, y) \cap I(y, z) = y$, $I(y, z) \cap I(z, x) = z$, $I(x, y) \cap I(z, x) = x$. Since G is a cactus, the cycle C_0 possesses at least one nontrivial cut vertex, marked as v. Let H_0 be a nontrivial subgraph of G such that $V(H_0) \cap V(C_0) = v$. Without loss of generality, assume that $v \in I(x, y)$. For some $v_0 \in V(H_0)$, every path connecting the vertex v_0 and all vertices in C_0 goes through v. Therefore, x (or y) $\in I(v_0, z)$, and we deduce that $|S \cap V(H_0)| = 0$. Then, $\mathrm{gp}(G) = 3$, a contradiction. □

Manuel and Klavžar [10] determined the the gp-number of trees. We present it here.

Lemma 2.2. *If L is the set of leaves of a tree T, then* $\mathrm{gp}(T) = |L|$.

Suppose G is cactus and contains a cycle C_t and a cyclic path P_r. Set v a vertex of C_t (or P_r) with degree more than two and its two neighbors as v_1 and v_2. G_v denotes the subgraph of $G - vv_1 - vv_2$ containing v. T_k denotes the maximal subgraph induced by the vertices in acyclic subgraph of $G_v - v$ together with v. Clearly, T_k is a tree, we call it a *pendant tree* associated with v, and v is the *root* of T_k. A root vertex v is referred as a *nontrivial cut vertex* of G.

Lemma 2.3. *If $G \in C_n^{t,k}$ and $\mathcal{T}$ is the subgraph formed from pendant trees by removing their roots in G, then there exists a* gp*-set S such that* $|S \cap V(\mathcal{T})| = t$.

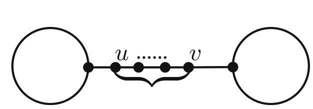

Figure 1: (u, v)-path is a acyclic path.

Proof. Let $G \in \mathcal{C}_n^{t,k}$ has w pendant trees labeled as $T_1, \dots, T_\ell, T_{\ell+1}, \dots, T_w$ with roots $v_1, \dots, v_\ell, v_{\ell+1}, \dots, v_w$, where the first ℓ roots (resp. the last $w-\ell$ roots) belong to cycles (resp. cyclic paths). Let $\mathcal{T}_1 = \bigcup_{i=1}^{\ell}\{T_i - v_i\}$, $\mathcal{T}_2 = \bigcup_{i=\ell+1}^{w}\{T_i - v_i\}$ and $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$. Let L_i $(1 \le i \le w)$ be the set of leaves of the pendant tree T_i and X_i $(1 \le i \le w)$ be the set of vertices with degree one in G. Obviously, $|L_i| = |X_i|$. Let S' be a gp-set of G.

Claim 1. S' does not contain any root v_i $(\ell+1 \le i \le w)$.

Assume that there is some $v_{i_0} \in S'$ for $\ell+1 \le i_0 \le w$. As shown in Figure 2, v_{i_0} is a nontrivial cut vertex of G, which, besides T_{i_0}, has p (≥ 2) components containing cycles labeled as $H_1, H_2, \dots, H_p$. For every triple including v_{i_0} of S' marked as $\{x, y, v_{i_0}\}$, we conclude that x and y simultaneously belong to some component (or the pendant tree) of v_{i_0}, assume that H_1. If not, suppose that $x \in V(H_j)$ and $y \in V(H_k)$ (or $V(T_{i_0} - v_{i_0})$), then any x, y-geodesic is via v_{i_0}, contradicted with S'. By the way, we obtain a new general position set $S = S' \cap V(H_1) \cup \{u_2, u_3, \dots, u_p, u_{p+1}\}$. Obviously, $|S| \ge |S'| + 2$, which is a contradiction.

We now consider the relationship between S' and the remaining roots of G.

If S' does not contain any root $v_i(1 \le i \le \ell)$, we take $S = S'$. Every vertex with degree 1 cannot lie on the geodesics of any other pair pendant vertices. Hence, $|S \cap V(\mathcal{T}_1)| = |\sum_{i=1}^{\ell} S \cap (V(T_i) - v_i)| = \sum_{i=1}^{\ell}|S \cap V(T_i)| = \sum_{i=1}^{\ell}|X_i| = \sum_{i=1}^{w}|L_i|$ by Lemma 2.2.

We hence suppose that S' includes some root v_{i_0}, which belongs to a cycle C_0. Evidently, $S' = (S' \cap V(G - T_{i_0})) \cup (S' \cap (V(T_{i_0})\backslash\{v_{i_0}\})) \cup \{v_{i_0}\}$. We claim that T_{i_0} is a path. Note that every path from the vertices in $V(G - T_{i_0})$ to the vertices of $V(T_{i_0})\backslash\{v_{i_0}\}$ is via v_{i_0}, which implies $S' \cap V(G - T_{i_0}) = \varnothing$ or $S' \cap (V(T_{i_0})\backslash\{v_{i_0}\}) = \varnothing$. First, suppose that $S' \cap (V(G - T_{i_0})) = \varnothing$. Mark the two adjacent vertices of v_{i_0} as x and y in C_0. Then, we obtain a new general position set $S'' = \{x, y\} \cup (S' \cap (V(T_{i_0})\backslash\{v_{i_0}\}))$. This means that $|S''| > |S'|$, which is a contradiction with the maximum of S'. That is to say, $S' \cap (V(T_{i_0})\backslash\{v_{i_0}\}) = \varnothing$. If $|X_{i_0}| \ge 2$, we have a new general position set $S''' = (S' \cap V(G - T_{i_0})) \cup X_{i_0}$. Then $|S'''| > |S'|$, contradicted with the maximum of S'. So $|X_{i_0}| = 1$, which implies that T_{i_0} is a path. Let $S = (S'\backslash\{v_{i_0}\}) \cup X_{i_0}$. Clearly, $|S| = |S'|$ and S is a gp-set of G.

Assume now that S' contains some of these roots. Then, repeating the aforementioned process, we obtain a new gp-set S of G such that all roots are not its elements.

By Lemma 2.2, we conclude that $|S \cap V(\mathcal{T})| = \sum_{i=1}^{w}|S \cap V(T_i)| = \sum_{i=1}^{w}|X_i| = \sum_{i=1}^{w}|L_i| = t$. □

Let C be the set of all cycles and $\mathcal{T}$ be the subgraph formed from pendant trees by removing their roots in G. From Lemmas 2.1 and 2.3, we deduce $\mathrm{gp}(G) = |S \cap V(C)| + |S \cap V(\mathcal{T})| \le 2k + t$. In order to demonstrate that the bound is best, we next exhibit the structure of cacti such that their gp-number equal $2k + t$.

For convenience, we introduce some notations. Let $G \in \mathcal{C}_n^{t,k}$ and $C_\ell = u_1u_2 \dots u_\ell u_1$ be a cycle of G. For two vertices u_i and u_j, clearly, C_ℓ consists of two (u_i, u_j)-paths. If all cut-vertices of C_ℓ is located on one (u_i, u_j)-path, then this path is referred to as a *(u_i, u_j)-cut-path* and use $d_c(u_i, u_j)$ to denote the number of edges in the (u_i, u_j)-cut-path. Let D_c be $\min_{u_i,u_j \in V(C_\ell)} d_c(u_i, u_j)$ of C_ℓ. We call cycle C_ℓ a *good cycle* if D_c is no more than either $\frac{\ell}{2} - 2$ for even ℓ or $\left\lfloor \frac{\ell}{2} \right\rfloor - 1$ for odd ℓ. Trivially, the cycle as an end-block of cactus is a good cycle. In addition, if D_c is at least either $\frac{\ell}{2}$ for even ℓ or $\left\lfloor \frac{\ell}{2} \right\rfloor + 1$ for odd ℓ, then the cycle is regarded as a *bad cycle*. Furthermore, C_ℓ is referred to a *normal cycle* if D_c equals to $\frac{\ell}{2} - 1$ for even ℓ or $\left\lfloor \frac{\ell}{2} \right\rfloor$ for odd ℓ.

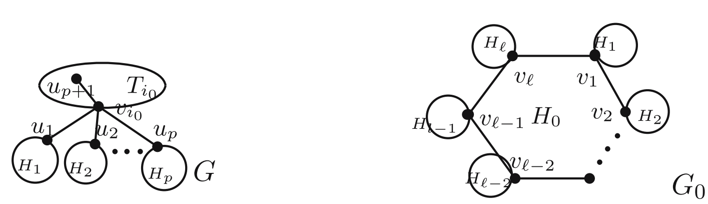

Figure 2: The two graphs G and G_0 used in Lemma 2.3 and Theorem 2.4.

Let $H_0 = v_1v_2 \dots v_\ell v_1$ be a cycle of G_0 (see Figure 1). Assume that $H_i (1 \le i \le \ell)$ is a connected subgraph with $V(H_i) \cap V(H_0) = \{v_i\}$ and $|V(H_i)| = n_i$ for $i \in \{0, 1, \dots, \ell\}$.

Theorem 2.4. *Let $G_0 \in C_n^{t,k}$. $\mathrm{gp}(G_0) = 2k + t$ if only all cycles of G_0 are good.*

Proof. Let $G_0 \in C_n^{t,k}$ with $\mathrm{gp}(G_0) = 2k + t$. Suppose S is a gp-set of G_0. Hence, $|S| = 2k + t$ and $|S \cap V(H_0)| = 2$ for every cycle H_0 of G_0 by Lemma 2.3. To verify the conclusion, it is sufficient to show $|S \cap V(H_0)| = 2$ if and only if H_0 is a good cycle.

We now assume that H_0 is a good cycle. Suppose $d_c(v_{i_0}, v_{j_0}) \le \frac{\ell}{2} - 2$ for even ℓ. We take the two vertices that are adjacent to v_{i_0} and v_{j_0} on another (v_{i_0}, v_{j_0})-path in H_0, and mark them as v'_{i_0} and v'_{j_0}, respectively. It is clear that the two vertices have degree two. For an arbitrary vertex u_0 in $G - V(H_0)$, v'_{i_0} is not on a (u_0, v'_{j_0})-shortest path and v'_{j_0} is also not on a (u_0, v'_{i_0})-shortest path. If $|S \cap V(H_0)| < 2$, then let S' be the set obtained from S by removing all the elements of its subset $S \cap V(H_0)$ and adding the two vertices v'_{i_0} and v'_{j_0}. We deduce that S' is a general position set with $|S'| \ge |S| + 1$, which contradicts the maximum of S. Hence, $|S \cap V(H_0)| = 2$.

Suppose $d_c(v_{i_0}, v_{j_0}) \le \left\lfloor \frac{\ell}{2} \right\rfloor - 1 = \frac{\ell - 1}{2} - 1$ for odd ℓ. Let v'_{i_0} and v'_{j_0} denote the two vertices of degree two, which are adjacent to v_{i_0} and v_{j_0} on another (v_{i_0}, v_{j_0})-path, respectively. Hence, for an arbitrary vertex w_0 in $G - V(H_0)$, v'_{i_0} is not on a (w_0, v'_{j_0})-shortest path and v'_{j_0} is also not on a (w_0, v'_{i_0})-shortest path. By a similar way as mentioned earlier, we obtain $|S \cap V(H_0)| = 2$.

Therefore, we obtain that $|S \cap V(H_0)| = 2$ if H_0 is a good cycle.

On the contrary, G includes at least a cycle H_0, which is not a good cycle. Note that end-blocks of G_0 are good. We thus assume that H_0 is not an end-block as shown in Figure 2. (It includes at least two cut vertices.) Hence, its D_c is no less than either $\frac{\ell}{2} - 1$ for even ℓ or $\left\lfloor \frac{\ell}{2} \right\rfloor$ for odd ℓ. Let v_{i_0} and v_{j_0} be two cut-vertices of H_0 for which $D_c = d_c(v_{i_0}, v_{j_0})$.

We first consider the special case that H_0 contains only two cut vertices. Then, $D_c \le \left\lfloor \frac{\ell}{2} \right\rfloor$, and we deduce that $|S \cap V(H_0)| \le 1$. Suppose now that there are at least three cut vertices in $V(H_0)$. If $d_c(v_{i_0}, v_{j_0}) \ge \left\lfloor \frac{\ell}{2} \right\rfloor + 1$, then there exists a cut vertex v_r which belongs to the (v_{i_0}, v_{j_0})-cut-path of the cycle H_0, such that one vertex is not on the geodesic of the other two vertices in the triple $\{v_{i_0}, v_{j_0}, v_r\}$. Let now $v_{k_0} (\ne v_{i_0}, v_{j_0}, v_r)$ be an arbitrary vertex of H_0. Assume that v_{k_0} is one vertex of the shortest (v_{i_0}, v_{j_0})-path. Then, it is easy to deduce that every shortest path from the vertices in $H_{v_{i_0}}$ to the vertices in $H_{v_{j_0}}$ goes through v_{k_0}. If $|S \cap V(H_0)| = 2$, then at least one of $|S \cap V(H_{v_{i_0}})|$, $|S \cap V(H_{v_{j_0}})|$ and $|S \cap V(H_{v_r})|$ is equal to zero, so $\mathrm{gp}(G_0) < 2k + t$ by Lemmas 2.1 and 2.3. Hence, $|S \cap V(H_0)| \ge 1$.

Assume that $d_c(v_{i_0}, v_{j_0}) = \left\lfloor \frac{\ell}{2} \right\rfloor$. If ℓ is even, then $\left\lfloor \frac{\ell}{2} \right\rfloor = \frac{\ell}{2}$. For a vertex u of $H_{v_{i_0}} - v_{i_0}$ and a vertex w of $H_{v_{j_0}} - v_{j_0}$, all vertices of H_0 lie on some (u, w)-geodesic. If $|S \cap V(H_0)| \ge 1$. Then, $|S \cap V(H_{v_{i_0}})| = 0$ or $|S \cap V(H_{v_{j_0}})| = 0$. According to Lemmas 2.1 and 2.3, $\mathrm{gp}(G_0) < 2k + t$. Hence, $|S \cap V(H_0)| = 0$ If ℓ is odd, then $\left\lfloor \frac{\ell}{2} \right\rfloor = \frac{\ell - 1}{2}$. Using a similar way for even ℓ, it is verified that $|S \cap V(H_0)| = 1$.

Suppose $d_c(v_{i_0}, v_{j_0}) = \left\lfloor \frac{\ell}{2} \right\rfloor - 1 = \frac{\ell}{2} - 1$ for even ℓ. We conclude that any vertex, which lies on a (v_{i_0}, v_{j_0})-cut-path is not an element of S. Let P_0 be another (v_{i_0}, v_{j_0})-path in H_0. Clearly, $|S \cap (V(P_0) \backslash \{v_{i_0}, v_{j_0}\})| = 1$.

Therefore, the proof is complete. □

By means of Theorem 2.4, we deduce that there are a lot of graphs belonging to $C_n^{t,k}$ for which the gp-number is equal to $2k + t$ and these graphs have different shapes.

Let A_1 be a cactus in which all cycles and all pendant trees share a common cut vertex v. Let R be the vertex set formed by all neighbors of v in cycles and the pendant vertices in A_1. In fact, R is a general position set, and $|R| = 2k + t$. Hence, $gp(A_1) \geq 2k + t$. Conversely, Lemmas 2.1 and 2.3 lead to $gp(A_1) \leq 2k + t$. Thus $gp(A_1) = 2k + t$.

In addition, let A_2 denote a chain cactus with the following two properties: (a) the length of every cycle in A_2 is at least 5 and its two cut vertices (if they exist) are adjacent, (b) there is one cut vertex of some cycle possessing pendant tree. Evidently, take two neighbors of the two cut vertices different with themselves for each cycle and all pendant vertices forming a general position set of A_2, denote it by R'. Then, $|R'| = 2k + t$ and $gp(A_2) \leq 2k + t$. So $gp(A_2) = 2k + t$ by Lemmas 2.1 and 2.3.

We now give two concrete examples. Let $B_1 \in C_n^{t,k}$ with $k = 3$, $t = 3$, $n = 17$, and $B_2 \in C_n^{t,k}$ with $k = 4$, $t = 2$, $n = 19$, see Figure 3. We have their general position sets S and S' that consists of all solid vertices, respectively. Evidently, $|S| = 9 = 2k + t$ and $|S'| = 10 = 2k + t$.

By means of Theorem 2.4, the main result holds.

Theorem 2.5. *If $G \in C_n^{t,k}$, then*

$$gp(G) \leq \max\{3, 2k + t\},$$

where the equality holds if and only if all cycles of G are good.

Proof.

Case 1. $k = 1$ and G with at most one pendant vertex. Note that G is a unicyclic graph, mark its unique cycle as C_0. Clearly, C_0 is a good cycle. We check that $gp(G) \leq 3$. In addition, $gp(G) = 3$ means that $G \cong C_n$ for $n \neq 4$ or G contains a pendant vertex.

Case 2. $k \geq 2$ or $k = 1$ and G with at least two pendant vertices. Observe that $3 \leq 2k + t$. Lemmas 2.1 and 2.3 result in $gp(G) \leq 2k + t$. Moreover, Theorem 2.4 implies that $gp(G) = 2k + t$ if and only if each cycle in G is good.

Together with the aforementioned two cases, the proof is complete. □

Corollary 2.6. *Let $G \in C_n^k$, we have*

$$gp(G) \leq \max\{3, n - 1\},$$

where equality holds if and only if $G \cong C_3$ or $G \cong B_0$.

Proof. Suppose $G \in C_n^k$. If G is isomorphic to C_3, then $gp(G) = 3$. We now assume that $G \ncong C_3$. Let t be the number of pendant edges in G. Clearly, $t \leq n - 2k - 1$. Since k is bounded, we deduce that $gp(G) \leq 2k + t \leq 2k + n - 2k - 1 = n - 1$.

$gp(G) = n - 1$ leads to that $t = n - 2k - 1$. That is to say, every cycle of G is a triangle, and all triangles share a common vertex. Thus, $G \cong B_0$, see Figure 3. Actually, it is easy to check that all vertices except for the cut vertex form a general position set of B_0. □

3 A lower bound for cacti

In this section, we will prove some lower bounds for cacti. Note that for a graph $G \in C_n^{t,k}$, its pendant vertices consist of a general position set. Hence, there exist an obvious lower bound, $gp(G) \geq t$. It is

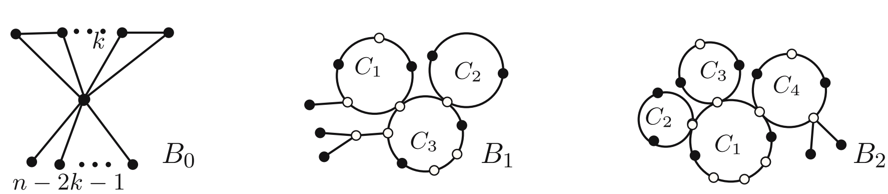

Figure 3: B_0 is used in Theorem 2.5, B_1 and B_2 attain the bound in Theorem 2.4.

interesting to determine all graphs attaining this bound. In addition, we also consider cactus graph with $t = 0$. In particular, if $G \in C_n^{0,1}$, then $G \cong C_n$, and hence, $\mathrm{gp}(G) = 3$ for $n = 3$ or $n \geq 5$ or $\mathrm{gp}(G) = 3$ for $n = 4$. Moreover, if $G \in C_n^{0,2}$, then $\mathrm{gp}(G) = 4$. We now consider $G \in C_n^{0,k}$ for $k \geq 3$.

Theorem 3.1. *Suppose k and k_1 are two integers with $k \geq 3$ and $k_1 \geq 0$. If $G \in C_n^{0,k}$ has k_1 odd cycles, then*

$$\mathrm{gp}(G) \geq \begin{cases} k_1 + 2 & k_1 \geq 3, \\ 4 & \textit{otherwise}. \end{cases}$$

Equality holds if and only if G is a chain cactus for which, except for two end-blocks, D_c of every even cycle equals to the half number of its vertices and D_c of every odd cycle equals to the floor of the half number of its vertices, or G is a cactus such that at least one cycle owns three cut vertices, which includes three properties: there are at least one odd cycle with three cut vertices, all even cycles have two cut vertices, and all end-blocks are odd.

Proof. Let k and k_1 are two integers and $k \geq 3$ and $k_1 \geq 0$. Suppose G is a cactus in $C_n^{0,k}$ having the minimal gp-number. Let S be a gp-set of G.

To exhibit clearly, we now introduce some notations. Let $C_{t_1}, C_{t_2}, \ldots, C_{t_k}$ be the k cycles of G. $ed(G)$ denotes the number of end-blocks in G. Let ℓ_i be the number of cut vertices on C_{t_i} for $i = 1, 2, \ldots, k$. Assume that $ed_2(G)$ denotes the number of the cycles with two cut vertices in G. Without loss of the generality, suppose $1 \leq \ell_1 \leq \ell_2 \leq \ldots \leq \ell_k$. In fact, $ed(G) = 2 + \sum_{i=ed(G)+1}^{k} \ell_i - 2$. Let v_{ed} be a cut vertex of $C_{t_{ed(G)}}$. In addition, set v'_{ed} the vertex on $C_{t_{ed(G)}}$ such that $d_c(v_{ed}, v'_{ed})$ is as big as possible and $d_c(v_{ed}, v'_{ed}) \leq \left\lfloor \frac{t_{ed}}{2} \right\rfloor$. To show the conclusion, we first show the claim.

Claim 1 G has the following properties.
(A) Each vertex in cyclic paths of G has degree two.
(B) For each vertex v on cycles of G, $G - v$ has at most two components.
(C) $\ell_r \leq 3$ for $1 \leq r \leq k$. In particular, $\ell_r \leq 2$ if C_r is even, $\ell_r \leq 3$ otherwise. If $\ell_r = 3$, then C_{t_r} is odd and end-blocks of G are odd.
(D) If G has an even end-block, then G is a chain cactus. □

Proof. (A) On the contrary, suppose that there exists a vertex u_0 in a cyclic path of G with $d_G(u_0) \geq 3$. Set its one neighbor u_1 such that u_1 isn't on the shortest (u_0, v_{ed})-path. Let G' be the new graph from G by deleting the edge u_0u_1 and joining $u_1v'_{ed}$. Set S' a gp-set of G'. Clearly, $|S| - |S'| = |V(C_{t_{ed}}) \cap S| - |V(C_{t_{ed}}) \cap S'| \geq 2 - 1 = 1$, is a contradiction.

Using the same way, we can verify (B); thus, the proof is omitted.

(C) On the contrary, suppose $\ell_r \geq 4$ with $r \geq ed(G) + 1$ as small as possible. Thus, there exists three cut vertices as v_r, v_{r_0}, and v_{r_1} of C_{t_r} such that v_r is a vertex in the (v_{r_0}, v_{r_1})-cut-path with length D_c. G' is the new graph from G by removing the subgraph from v_r to v'_{ed}. Let S' be a gp-set of G'. Let ℓ'_i be the number of cut vertices of C_{t_i} in G' for $i = 1, 2, \ldots, k$. Evidently, $\ell'_i = \ell_i$ for $i \neq r, ed(G)$, $\ell'_r = \ell_r - 1 \geq 3$, and $\ell'_{ed(G)} = \ell_{ed(G)} + 1 = 2$. Note that $|V(C_{t_r}) \cap S| = |V(C_{t_r}) \cap S'|$. Hence, $|S| - |S'| = |V(C_{t_{ed}}) \cap S| - |V(C_{t_{ed}}) \cap S'| \geq 2 - 1 = 1$, which contradicts with the choice of G.

Suppose there is an even cycle C_{t_r} with $\ell_r = 3$. Let u_0 is a cut vertex of C_{t_r}. G'' denotes the graph obtained from G by shifting the subgraph from u_0 to an end-block C_{t_h} ($h \leq ed(G)$) and adjusting the positions of cut vertices in two cycles for which C_{t_r} is bad and C_{t_h} is not good. Let S'' is a gp-set in G''. Observe that $|V(C_{t_r}) \cap S| = |V(C_{t_r}) \cap S''| = 0$. Thus, $|S| - |S'| = |V(C_{t_h}) \cap S| - |V(C_{t_h}) \cap S'| \geq 2 - 1 = 1$, which is a contradiction.

If $\ell_r = 3$, by the aforementioned discussion, we have C_{t_r} is odd. All end-blocks of G are odd. If not, using the same way,we also obtain a contradiction.

(D) Suppose G includes an even end-block. From (C), we deduce that $\ell_r \leq 2$. Together (A) with (B), it is deduced that G is a chain cactus. □

If $1 \le \ell_r \le 2$ for $r \le k$, then G is a chain cactus. Note that, except for two end-blocks, all others internal cycles contain two cut vertices. Furthermore, we discuss the contribution of internal cycles to S, as an bad even cycle C_0, $|S \cap V(C_0)| = 0$, and as a normal odd cycle C_0, $|S \cap V(C_0)| = 1$. Since G has the minimum with respect to the gp-number, odd cycles should be end-blocks as many as possible. Recall that the condition G owns k_1 odd cycles and $k - k_1$ even cycles, we hence arrive at $\mathrm{gp}(G) = k_1 - 2 + 4 = k_1 + 2$ for $k_1 \ge 3$ and $\mathrm{gp}(G) = 4$ for $k_1 = 0, 1, 2$. If $\ell_r = \ell_{r+1} = \cdots = \ell_k = 3$, then all end-blocks are odd and $ed(G) = 2 + \sum_{i=r}^{k} \ell_i - 2 = 2 + (k - r + 1)$. Therefore, $\mathrm{gp}(G) = 2ed(G) + k_1 - (k - r + 1) - ed(G) = k_1 + 2$.

Based on the aforementioned conclusion, we deduce the following two results.

Corollary 3.2. *If $G \in C_n^{0,k}$ has k odd cycles, then $\mathrm{gp}(G) \ge k + 2$ with equality if and only if G is either a chain cactus for which, except for two end-blocks, D_c of each cycle equals to the floor of the half number of its vertices, or a cactus such that each cycle owns at most three cut vertices, which includes three properties: there are odd cycles with three cut vertices, and all even cycles have two cut vertices and all end-blocks are odd.*

Corollary 3.3. *If $G \in C_n^{0,k}$ has k even cycles, then $\mathrm{gp}(G) \ge 4$ with equality if and only if G is a chain cactus for which, except for two end-blocks, D_c of each cycle equals to the half number of its vertices.*

Theorem 3.4. *If $G \in C_n^{t,k}$, then $\mathrm{gp}(G) \ge t$ with equality if and only if the cycles are bad.*

Proof. Suppose $G \in C_n^{t,k}$, S is the set of pendant vertices in G. By the property of the set of pendant vertices, $\mathrm{gp}(G) \ge t$. If equality holds, then S is the gp-set of G by Lemma 2.3. This means that the vertices of all cycles do not contribute to S. In other words, for any cycle C_a of G, $|V(C_a) \cap S| = 0$. Let h be the number of cut vertices of C_a. Clearly, $h \ge 2$. (On the contrary, C_a is an end-block. So $|V(C_a) \cap S| = 2$, a contradiction.)

Assume that $h = 2$, we deduce that $D_c \le \frac{a}{2}$ for an even a, $D_c \le \left\lfloor \frac{a}{2} \right\rfloor - 1$ otherwise. Observe that $|V(C_a) \cap S| = 0$ results in C_a is bad, which deduces that a is even and $D_c = \frac{a}{2}$. We next assume that $h \ge 3$. $|V(C_a) \cap S| = 0$ infers that C_a is bad from the definition of the bad cycle. By the choice of C_a, we finish the proof. □

4 Wheel graphs

In this section, we will deduce the gp-number of wheels. For convenience, we now introduce some notation. The vertex with degree n in W_n is called the center of W_n and denoted by w. Let $C_n = W_n - w$, and S denote a gp-set of W_n. Let $G_0 = C_n - S$ for short.

Theorem 4.1. *If $n \ge 3$, then* $\mathrm{gp}(W_n) = \begin{cases} 4, & \textit{if } n = 3, \\ 3, & \textit{if } n = 4, 5, \\ \left\lfloor \frac{2}{3}n \right\rfloor, & \textit{if } n \ge 6. \end{cases}$

Proof. Let $C_n = v_1v_2 \dots v_nv_1$. If $n = 3$, then $W_3 \cong K_4$. Clearly, $\{v_1, v_2, v_3, w\}$ forms a general position set of W_3, so $\mathrm{gp}(W_n) \ge 4$. Hence, $\mathrm{gp}(W_n) = 4$ by $|V(W_3)| = 4$. Suppose $n = 4, 5$. Evidently, $\{v_1, v_2, w\}$ is a general position set of W_n. In addition, $\forall S \subseteq V(W_n)$ with $|S| \ge 4$, $W_n[S]$ leads to an induced P_3. Thus, $\mathrm{gp}(W_n) \le 3$, which infers $\mathrm{gp}(W_n) = 3$. We next assume that $n \ge 6$. Let S be a gp-set of W_n. □

Claim 1. If $n \ge 6$, then the center $w \notin S$.

Proof of Claim 1. If $w \in S$, then $|S| = 3$. Since $n \ge 6$, there are two edges for which their distance is no less than 2 in C_n, and we denote their ends as x_1, x_2 and x_3, x_4, respectively. Then $S' = \{x_1, x_2, x_3, x_4\}$ is a general position set, which implies that $|S'| > |S|$. So $w \notin S$. □

Note that $C_n[S]$ contains no path P_3. Hence, it consists of some K_2 and K_1.

Claim 2. G_0 contains at most one K_2.

Proof of Claim 2. Suppose G_0 has at least two K_2. Let $e_1 = v_1v_2$ be an edge in G_0. It is not difficult to find another edge $e_2 = v_{x-1}v_x \in E(G_0)$ such that it is closest to e_1 in C_n. We now construct a new subset S' from S by the following process. If the vertex $v_i \in S(3 \le i \le x-2)$, then $v_{i-1} \in S'$, for others vertices of S, we copy them to S'. Since $v_{x-2} \in S$ we deduce $v_{x-2} \notin S'$, and add v_{x-1} to S'. In fact, every path in W_n possessing some triple of S' is not geodesic. Hence, S' is a general position set of W_n. Then, $|S'| = |S| + 1 > |S|$, which is a contradiction. □

Based on the aforementioned two claims, the gp-set S is a subset of $V(C_n)$ and $C_n[S]$ consists of some K_1 and K_2 for which every two consecutive components are separated by K_1 or K_2 (at most one). For convenience, the number of K_1 and K_2 in $C_n[S]$ are ℓ_1 and ℓ_2, respectively. Hence, $\text{gp}(W_n) = |S| = \ell_1 + 2\ell_2$. In addition, if G_0 contains one K_2, $|G_0| = \ell_1 + \ell_2 + 1$; otherwise, $|G_0| = \ell_1 + \ell_2$. In addition, $2\ell_1 + 3\ell_2 \le n$. We next take three cases to show the result.

Case 1. $n = 3k$.

From the fact $2\ell_1 + 3\ell_2 \le 3k$, we arrive at $\text{gp}(W_n) = |S| = \ell_1 + 2\ell_2 = \frac{2}{3}(2\ell_1 + 3\ell_2) - \frac{1}{3}\ell_1 \le \frac{2}{3}(3k) - \frac{1}{3}\ell_1 = 2k - \frac{1}{3}\ell_1$.

On the other hand, $R = \{v_2, v_3, v_5, v_6, v_8, v_9, \ldots, v_{3k-1}, v_{3k}\}$ is a general position set of W_n and $|R| = 2k$. Hence, $\text{gp}(W_n) \ge 2k$. That is to say, $2k \le 2k - \frac{1}{3}\ell_1$, which implies $\ell_1 = 0$. Consequently, $\ell_2 = k$ and $W_n[S]$ is made up of $2k$ copies of K_2.

Case 2. $n = 3k + 1$.

Using the fact $2\ell_1 + 3\ell_2 \le 3k + 1$, we deduce that $\text{gp}(W_n) = |S| = \ell_1 + 2\ell_2 = \frac{2}{3}(2\ell_1 + 3\ell_2) - \frac{1}{3}\ell_1 \le \frac{2}{3}(3k+1) - \frac{1}{3}\ell_1 = 2k + \frac{2}{3} - \frac{1}{3}\ell_1$.

Now let $R = \{v_2, v_3, v_5, v_6, v_8, v_9, \ldots, v_{3k-1}, v_{3k}\}$. As in Case 1, R is a general position set of W_n and $|R| = 2k$. So, $\text{gp}(W_n) \ge 2k$. Hence, $2k \le 2k + \frac{2}{3} - \frac{1}{3}\ell_1 < 2k + 1$. Namely, $\text{gp}(W_n) = 2k$. Moreover, R is a gp-set of W_n.

Case 3. $n = 3k + 2$.

From the fact $2\ell_1 + 3\ell_2 \le 3k + 2$, we obtain that $\text{gp}(W_n) = |S| = \ell_1 + 2\ell_2 = \frac{2}{3}(2\ell_1 + 3\ell_2) - \frac{1}{3}\ell_1 \le \frac{2}{3}(3k+2) - \frac{1}{3}\ell_1 = 2k + \frac{4}{3} - \frac{1}{3}\ell_1$.

Take $R = \{v_2, v_3, v_5, v_6, v_8, v_9, \ldots, v_{3k-1}, v_{3k}, v_{3k+2}\}$. It is not difficult to verify that R is a general position with $|R| = 2k + 1$. So $\text{gp}(W_n) \ge 2k + 1$. Hence, $2k + 1 \le 2k + \frac{4}{3} - \frac{1}{3}\ell_1 < 2k + 2$, which implies $\text{gp}(W_n) = 2k + 1$. Furthermore, $W_n[S]$ consists of one K_1 and $2kK_2$. Evidently, R is a gp-set of W_n.

Combining with the aforementioned three cases, the result holds.

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