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On the generalization of the Darboux theorem

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Abstract. The Darboux theorem asserts that every symplectic manifold (M^{2n}, ω) is locally symplectomorphic to (R^{2n}, ω_0), where ω_0 is the standard symplectic form on R^{2n}. This theorem was proved by Moser in 1965, the idea of proof, known as the Moser's trick, works in many situations. The Moser tricks is to construct an appropriate isotopy $\mathbb{F}_t$ generated by a time-dependent vector field X_t on M such that $\mathbb{F}_1^*\omega = \omega_0$. Nevertheless, it was showed by Marsden that Darboux theorem is not valid for weak symplectic Banach manifolds. However, in 1999 Bambusi showed that if we associate to each point of a Banach manifold a suitable Banach space (classifying space) via a given symplectic form then the Moser trick can be applied to obtain the theorem if the classifying space does not depend on the point of the manifold and a suitable smoothness condition holds.

If we want to try to generalize the Darboux theorem to more general context of Frechet manifolds we face an obstacle: in general vector fields do not have local flows. Recently, Fréchet geometry has been developed in terms of projective limit of Banach manifolds. In this framework under an appropriate Lipchitz condition local flows exist and with some restrictive conditions the Darboux theorem was proved by P. Mishra. In the present paper we consider the category of so-called bounded Fréchet manifolds and prove that in this category vector fields have local flows and following the idea of Bambusi we associate to each point of a manifold a Fréchet space independent of the choice of the point and with the assumption of bounded smoothness on vector fields we prove the Darboux theorem.

Анотація. Теорема Дарбу стверджує, що кожен симплектичний многовид (M^{2n}, ω) є локально симплектоморфним до (R^{2n}, ω_0), де ω_0 – стандартна симплектична форма на R^{2n}. Ця теорема була доведена Мозером у 1965 р. Ідея доведення, відома як трюк Мозера, працює у багатьох ситуаціях і полягає у побудові ізотопії $\mathbb{F}_t$ многовиду M, породженої залежним від часу векторним полем X_t на M, так, щоб $\mathbb{F}_1^*\omega = \omega_0$. Тим не

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менш, Марсден показав, що теорема Дарбу не вірна для слабких симплектичних многовидів Банаха. Однак у 1999 р. Бамбусі показав, що якщо з кожною точкою банахового многовиду пов'язати простір Банаха (класифікуючий простір) через задану симплектичну форму, то трюк Мозера може бути застосований для довдеення теореми Дарбу, за умови, що цей простір не залежить від точки і відповідної умови гладкості.

Однією з перешкод до узагальнення теореми Дарбу на випадок многовидів Фреше є те, що, взагалі кажучи, векторні поля не мають локальних потоків. Останнім часом геометрія Фреше була розроблена з точки зору проективних границь многовидів Банаха. У цьому контексті за відповідної умови Ліпшица векторні поля породжують локальні потоки, і за деяких додаткових сильних умов теорема Дарбу була отримана П. Мішра. В представленій роботі ми розглянемо категорію так званих обмежених многовидів Фреше і доводимо, що в цій категорії векторні поля мають локальні потоки. Слідуючи ідеї Бамбусі, ми пов'язуємо з кожною точкою многовиду Фреше простір, який не залежить від вибору точки і припускаючи обмежену гладкість на векторних полях доводимо теорему Дарбу.

1. Introduction

The Darboux theorem has been extended to weakly symplectic Banach manifolds by using Moser's method, see [1]. The essence of this method is to obtain an appropriate isotopy generated by a time dependent vector field that provides the chart transforming of symplectic forms to constant ones. In order to apply this method to a more general context of Fréchet manifolds we need to establish the existence of the flow of a vector filed which in general does not exist. One successful approach to the differential geometry in Fréchet context is to use projective limits of Banach manifolds, [2]. In this framework, a version of the Darboux theorem is proved in [6].

Another approach to Fréchet geometry is to apply a stronger notion of differentiability, [3]. This differentiability leads to a new category of generalized manifolds, the so called *bounded* (or MC^k) Fréchet manifolds. In this paper we prove that in that context the flow of a vector field exists (Theorem 2.4) and we will apply the Moser's method to obtain the Darboux theorem (Theorem 3.5).

The obtained theorem might be useful to study the topology of the space of Riemannian metrics $\mathcal{M}$ as it has the structure of a nuclear bounded Fréchet manifold. A theorem from [5, §48.9] asserts that if (M, σ) is a smooth weakly symplectic convenient manifold which admits smooth partitions of unity in $C^\infty_\sigma(M, \mathbb{R})$, and which admits 'Darboux charts', then the symplectic cohomology equals to the De Rham cohomology:

$$H^k_\sigma(M) = H^k_{DR}(M).$$

The manifold $\mathcal{M}$ admits smooth partition of unity in $C^{\infty}_{\sigma}(M, \mathbb{R})$ (this follows from [5, Theorem 16.10] and [5, Definition 16.1]) so it is interesting to ask if it has a Darboux chart. This, in turn, rises the question: how to construct weak symplectic forms on $\mathcal{M}$. It is known, [4], that expect Hilbert manifolds an infinite dimensional manifold may not admit a Lagrangian splitting so in general the Weinstein's construction, [12], is not applicable. Moreover, the Marsden's idea to construct a symplectic form on a manifold by using the canonical form on its cotangent bundle also is not applicable as there is no natural smooth vector bundle structure on the cotangent bundle [9, Remark I.3.9]. It is not clear yet how to construct symplectic forms on $\mathcal{M}$ but it seems that it might arise from a weak Riemannian metric and complex structure, however, that would require some assumptions and ingredients different from ones in Theorem 3.5.

2. Bounded differentiability

In this section we prove the existence of the local flow of a MC^k-vector field. We refer to [3] for more details on bounded Fréchet geometry.

Denote by (F, ρ) a Fréchet space whose topology is defined by a complete translational-invariant metric ρ. We consider only metrics with absolutely convex balls. Note that every Fréchet space admits such a metric, cf [3]. One reason to choose this particular metric is that a metric with this property can give us a collection of seminorms that defines the same topology. More precisely:

Theorem 2.1 ([7], Theorem 3.4)**.** *Assume that (F, ρ) is a Fréchet space and ρ is a metric with absolutely convex balls. Let*

$$B^{\rho}_{\frac{1}{i}}(0) \coloneqq \left\{ y \in F \mid \rho(y, 0) < \tfrac{1}{i} \right\},$$

and suppose U_i's, $i \in \mathbb{N}$, are convex subsets of $B^{\rho}_{\frac{1}{i}}(0)$. Define the Minkowski functionals

$$\|v\|^i \coloneqq \inf\left\{ \varepsilon > 0 \mid \varepsilon \in \mathbb{R},\ \tfrac{1}{\varepsilon} \cdot v \in U_i \right\}.$$

These Minkowski functionals are continuous seminorms on F. A collection $\{\|v\|^i\}_{i \in \mathbb{N}}$ of these seminorms generates the topology of F.

In the sequel we will assume that a Fréchet space F is graded with the collection of seminorms $\|v\|^n_F = \sum\limits_{k=1}^{n} \|v\|^k$ that defines its topology.

Let (E, g) be another Fréchet space and $\mathcal{L}_{g,\rho}(E, F)$ be the set of all linear maps $L : E \to F$ such that

$$\mathbf{Lip}(L)_{g,\rho} \coloneqq \sup_{x \in E \setminus \{0\}} \frac{\rho(L(x), 0)}{g(x, 0)} < \infty.$$

The transversal-invariant metric

$$D_{g,\rho} : \mathcal{L}_{g,\rho}(E,F) \times \mathcal{L}_{g,\rho}(E,F) \longrightarrow [0,\infty),$$
$$(L,H) \mapsto \mathbf{Lip}(L-H)_{g,\rho}, \tag{2.1}$$

on $\mathcal{L}_{\rho,g}(E,F)$ turns it into an Abelian topological group. Let U an open subset of E, and $P : U \to F$ a continuous map. If P is Keller-differentiable, $\mathrm{d}P(p) \in \mathcal{L}_{\rho,g}(E,F)$ for all $p \in U$, and the induced map

$$\mathrm{d}P(p) : U \to \mathcal{L}_{\rho,g}(E,F)$$

is continuous, then P is called bounded differentiable.

We say P is MC^0 and write $P^0 = P$ if it is continuous. We also say P is an MC^1 and write $P^{(1)} = P'$ if it is bounded differentiable. Recursively one can define maps of class MC^k for each $k > 1$, see [3]. If $\varphi(t)$ is a continuous path in a Fréchet space we denote its derivative by $\dfrac{d}{dt}\varphi(t)$.

Within this framework we define MC^k (bounded) Fréchet manifolds, MC^k-maps of manifolds and tangent bundles and their MC^k-vector fields. A MC^k-vector field X on a MC^k-Fréchet manifold M is a MC^k-section of the tangent bundle $\pi_{TM} : TM \to M$, i.e. a MC^k map $X : M \to TM$ with $\pi_{TM} \circ X = \mathrm{id}_M$. We write $\mathcal{V}(M)$ for the space of all vector fields on M. If $f \in MC^\infty(M,E)$ is a smooth function on M with values in a Fréchet space E and $X \in \mathcal{V}(M)$, then we obtain a smooth function on M via

$$X.f := \mathrm{d}f \circ X : M \to E.$$

For $X, Y \in \mathcal{V}(M)$, there exists a unique a vector field $[X,Y] \in \mathcal{V}(M)$ determined by the property that on each open subset $U \subset M$ we have

$$[X,Y].f = X.(Y.f) - Y.(X.f)$$

for all $f \in MC^\infty(U,\mathbb{R})$, see [8, Lemma II.3.1].

A vector field on an infinite dimensional Fréchet manifold may have no, one or multiple integral curves. However, a MC^k-vector field always has a unique integral curve.

Proposition 2.2. [3, Proposition 5.1] *Let $U \subseteq F$ be an open subset and $X : U \to F$ be a MC^k-vector field, $k \geqslant 1$. Then for each $p_0 \in U$ there exists an integral curve $\ell : I \to F$ at p_0. Furthermore, any two such curves coincide on the intersection of their domains.*

Corollary 2.3. [3, Corollary 5.1] *Let $U \subseteq F$ be an open subset and let $X : U \to F$ be a MC^k-vector field, $k \geqslant 1$. Let also $\mathbb{F}_t(p_0)$ be the solution of $\ell'(t) = X(\ell(t))$, $\ell(t_0) = p_0$. Then there is an open neighborhood U_0 of p_0 and a positive real number α such that for every $q \in U_0$ there exists a* *unique integral curve* $\ell(t) = \mathbb{F}_t(q)$ *satisfying* $\ell(0) = q$ *and* $\ell'(t) = X(\ell(t))$ *for all* $t \in (-\alpha, \alpha)$.

Theorem 2.4. *Let* X *be a* MC^k*-vector field on* $U \subset F$, $k \geqslant 1$. *Then one can find a real number* $\alpha > 0$ *such that for each* $x \in U$ *there exists a unique integral curve* $\ell_x(t)$ *satisfying* $\ell_x(0) = x$ *for all* $t \in I = (-\alpha, \alpha)$. *Furthermore, the mapping* $\mathbb{F} : I \times U \to F$ *given by* $\mathbb{F}_t(x) = \mathbb{F}(t, x) = \ell_x(t)$ *is of class* MC^k.

Proof. The first part of the proof follows from Corollary 2.3. We now prove the second part. Let $x, y \in U$ be arbitrary points. Define the functions $\varphi_n : I \to \mathbb{R}$, $n \in \mathbb{N}$, by

$$\varphi_n(t) = \|\mathbb{F}(t, x) - \mathbb{F}(t, y)\|_F^n.$$

Since X is MC^k, so it is globally Lipschitz. Let $\beta > 0$ be its Lipschitz constant. Then we have $\forall n \in \mathbb{N}$

$$\begin{aligned}\varphi_n(t) &= \left\| \int_0^t \big(X(\mathbb{F}(s, x)) - X(\mathbb{F}(s, y)\big)ds + x - y \right\|_F^n \leqslant \\ &\leqslant \|x - y\|_F^n + \beta \int_0^t \varphi(s)ds.\end{aligned}$$

Hence, by Gronwall's inequality, we obtain that

$$\|\mathbb{F}(t, x) - \mathbb{F}(t, y)\|_F^n \leqslant e^{\beta|t|}\|x - y\|_F^n, \quad \forall n \in \mathbb{N}. \tag{2.2}$$

Thereby $\mathbb{F}$ is Lipschitz continuous in the second variable and is jointly continuous.

Now, define $\mathbf{F}(t, x) \in \mathcal{L}_\rho(F)$ to be the solution of the equations

$$\frac{d\mathbf{F}(t, x)}{dt} = \mathrm{d}X(\mathbb{F}(t, x)) \circ \mathbf{F}(t, x), \qquad \mathbf{F}(0, x) = \mathrm{id},$$

where $\mathrm{d}X(\mathbb{F}(t, x)) : F \to F$ is derivative of X with respect to x at $\mathbb{F}(t, x)$. By Proposition 2.2 $\mathbf{F}(t, x)$ exits and is well defined. Since the vector field $\mathbf{F} \mapsto \mathrm{d}X(\mathbb{F}(t, x)) \circ \mathbf{F}$ on $\mathcal{L}_\rho(F)$ is Lipschitz in $\mathbf{F}$, uniformly in (t, x) in a neighborhood of every (t_0, x_0), by the above argument it follows that $\mathbf{F}(t, x)$ is continuous in (t, x). We will show that $\mathrm{d}\mathbb{F}(t, x) = \mathbf{F}(t, x)$.

Indeed, fix $t \in I$, for $h \in U$ define $\psi(s, h) = \mathbb{F}(s, x + h) - \mathbb{F}(s, x)$, then

$$\begin{aligned}\psi(t, h) - \mathbf{F}(t, x)(h) = &\int_0^t \big(X(\mathbb{F}(s, x + h)) - X(\mathbb{F}(s, x))\big)ds \\ &- \int_0^t \big[\mathrm{d}X(\mathbb{F}(s, x)) \circ \mathbf{F}(s, x)\big](h)ds\end{aligned}$$

$$= \int_0^t \mathrm{d}X(\mathbb{F}(s,x)(\big[(\psi(s,h) - \mathbf{F}(s,x)(h))\big])ds$$

$$+ \int_0^t \big(X(\mathbb{F}(s,x+h)) - X(\mathbb{F}(s,x))$$

$$- \mathrm{d}X(\mathbb{F}(s,x))(\mathbb{F}(s,x+h) - \mathbb{F}(s,x)\big)ds.$$

Since X is MC^k, for given $\varepsilon > 0$ there is a $\delta > 0$ such that $\|h\|_F^n < \delta$, $(n \in \mathbb{N})$, yields that the second term is less than

$$\int_0^t \varepsilon \|\mathbb{F}(s,x+h) - \mathbb{F}(s,x)\|_F^n, \quad \forall n \in \mathbb{N},$$

but by (2.2) this integral is less than $B\varepsilon \sup_{n\in\mathbb{N}} \|h\|_F^n$ for some positive constant B. Thus, by Gronwall's inequality we obtain

$$\|\psi(t,h) - \mathbb{F}(t,h)(h)\|_F^n \leqslant \varepsilon C \|h\|_F^n, \quad \forall n \in \mathbb{N},$$

where C is a positive constant. Whence $\mathrm{d}\mathbb{F}(t,x)(h) = \mathbf{F}(t,x)(h)$. Thus, both partial derivatives of $\mathbb{F}(t,x)$ exist and are continuous so $\mathbb{F}(t,x)$ is C^1. Moreover, $\mathbf{F}$ is globally Lipschitz and $x \mapsto \mathbf{F}(\cdot,x)$ is continuous therefore $\mathbb{F}(t,x)$ is MC^1. Using induction on k we obtain that $\mathbb{F}(t,x)$ is of class MC^k. By definition of $\mathbb{F}(t,x)$

$$\frac{d}{dt}\mathbb{F}(t,x) = X(\mathbb{F}(t,x)),$$

so

$$\frac{d}{dt}\frac{d}{dt}\mathbb{F}(t,x) = \mathrm{d}X(\mathbb{F}(t,x))\big(X(\mathbb{F}(t,x))\big)$$

and

$$\frac{d}{dt}\mathrm{d}\mathbb{F}(t,x) = \mathrm{d}X(\mathbb{F}(t,x))\big(\mathrm{d}\mathbb{F}(t,x)\big).$$

The right-hand sides are MC^{k-1}, so are the solutions by induction. Thus $\mathbb{F}(t,x)$ is MC^k. □

3. Darboux charts

In general for a Fréchet manifold differential forms cannot be defined as the sections of its cotangent bundle since we can not always define a manifold structure on the cotangent bundle, see [9, Remark I.3.9]. To define differential forms we follow the approach of Neeb [9].

Definition 3.1. Let M be a bounded Fréchet manifold. A p-form ω on M is a function ω which associates to each $x \in M$ a p-linear alternating map $\omega_x : T_x^p(M) \to \mathbb{R}$ such that in local coordinates the map

$$(x, v_1, \dots, v_p) \mapsto \omega_x(v_1, \dots, v_p)$$

is smooth. We write $\Omega^p(M,\mathbb{R})$ for the space of p-forms on M and identify $\Omega^0(M,\mathbb{R})$ with the space $C^\infty(M,\mathbb{R})$ of smooth functions.

The exterior differential $d_{dR}:\Omega^p(M,\mathbb{R})\to\Omega^{p+1}(M,\mathbb{R})$ is determined uniquely by the property that for each open subset $U\subset M$ we have for $X_0,\dots,X_p\in\mathcal{V}(U)$ in the space $C^\infty(U,\mathbb{R})$ the identity

$$(d_{dR}\omega)(X_0,\dots,X_p):=\sum_{i=0}^{p}(-1)^i X_i.\omega(X_0,\dots,\hat{X}_i,\dots,X_p)$$
$$+\sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_p).$$

Let $\omega\in\Omega^p(M,\mathbb{R})$, $Y\in\mathcal{V}(M)$ and $\mathbb{F}_t$ the local flow of Y. Define the usual *Lie derivative* by

$$\mathcal{L}_Y\omega=\frac{d}{dt}(\mathbb{F}_t^*\omega)\mid_{t=0},$$

which of course coincides by

$$(\mathcal{L}_Y\omega)(X_1,\dots,X_p)=Y.\omega(X_1,\dots,X_p)-\sum_{j=1}^{p}\omega(X_1,\dots,[Y,X_j],\dots,X_p)$$

for $X_i\in\mathcal{V}(U)$, $U\subset M$ open. For each $X\in\mathcal{V}(M)$ and $p\geqslant 1$ consider also the following linear map

$$i_X:\Omega^p(M,\mathbb{R})\to\Omega^{p-1}(M,\mathbb{R})\quad\text{with}\quad (i_X\omega)_x=i_{X(x)}\omega_x,$$

where $(i_v\omega_x)(v_1,\dots,v_{p-1}):=\omega_x(v,v_1,\dots,v_{p-1})$.

For $\omega\in\Omega^0(M,\mathbb{R})=C^\infty(M,\mathbb{R})$, we put $i_X\omega:=0$. Then for two vector fields $X,Y\in\mathcal{V}(M)$, we have on $\Omega(M,\mathbb{R})$ the Cartan formulas, [9, Proposition I.4.3]:

$$[\mathcal{L}_X,i_Y]=i_{[X,Y]},\qquad \mathcal{L}_X=d_{dR}\circ i_X+i_X\circ d_{dR},\qquad \mathcal{L}_X\circ d_{dR}=d_{dR}\circ\mathcal{L}_X.$$

Definition 3.2. Let M be a bounded Fréchet manifold. Say that M is *weakly symplectic* if there exists a closed smooth 2-form ω ($d_{dR}\omega=0$) being *weakly non-degenerate* in the sense that for all $x\in M$ and $v_x\in T_xM$

$$\omega_x(v_x,w_x)=0 \tag{3.1}$$

for all $w_x\in T_xM$ implies $v_x=0$.

The Darboux theorem is a local result so it suffices to consider the case when M is an open set U of the Fréchet model space F. For simplicity assume that $0\in U$. Let $x\in U$ be fixed and let F'_b be the strong dual of F. Define the map $\omega_x^\#:F\to F'_b$ by

$$\langle w,\omega_x^\#(v)\rangle=\omega_x(w,v),$$

where $\langle \cdot, \cdot \rangle$ is a duality pairing. Also, define $H_x := \{\omega_x(y, \cdot) \mid y \in F\}$. This is a subset of $F_b^{'}$ and its topology is induced from it.

Lemma 3.3. *Suppose $x \in U$ is fixed and the model space $F_x \simeq T_xM$ is nuclear. Then the map $\omega_x^{\#} : F_x \to H_x$ is an isomorphism.*

Proof. Condition (3.1) implies that $\omega_x^{\#}$ is injective and by the definition of H_x, it is surjective. The space F_x is nuclear so its strong dual is DFN-space and barreled. The dual space is Mackey (cf. [11, 5.3.4]) and H_x inherits its topology (see [11, 0.4.2, 0.4.3] so is ultrabornological and barrelled (cf.[11, 8.6.9]). Thus, by the open mapping theorem [10, Theorem 4.35] the inverse mapping is continuous, so $\omega_x^{\#}$ is isomorphism. □

We will need the following result.

Lemma 3.4. [8, (Poincaré Lemma) II.3.5] *Let E be locally convex, V a sequentially complete space and $U \subset E$ an open subset being star-shaped with respect to 0. Let also $\omega \in \Omega^{k+1}(U, V)$ be a V-valued closed $(k+1)$-form. Then ω is exact. Moreover, $\omega = d_{dR}\alpha$ for some $\alpha \in \Omega^k(U, V)$ with $\alpha(0) = 0$ given by*

$$\alpha(x)(v_1, \cdots v_k) = \int_0^1 t^k \omega(tk)(x, v_1, \cdots v_k)dt.$$

We assert the following theorem for some open neighborhood of $0 \in F$.

Theorem 3.5. *Suppsoe that the Fréchet model space F is nuclear. Assume also that*

(i) *there exits an open neighborhood $\mathcal{U}$ of zero such that all the spaces H_x are locally identical and $\omega_x^{t\#} : F \to H$ is an isomorphism for each $t \in [0, 1]$ and $x \in \mathcal{U}$.*
(ii) *for $x \in \mathcal{U}$ the map $(\omega_x^{t\#})^{-1} : H \to F$ is a field of isomorphism of class MC^∞.*

Then ω is locally isomorphic at zero to the constant form $\omega(0)$.

Proof. On $\mathcal{U}$ define $\omega^t = \omega_0 + t(\omega_0 - \omega)$ for $t \in [0, 1]$, where $\omega_0 = \omega(0)$. By Lemma 2.2 there exist a 1-form α being locally such that $d_{dR}\alpha = \omega_0 - \omega$ and $\alpha(0) = 0$. Consider a time-dependent vector field $X_t : \mathcal{U} \to F$ defined by

$$i_{X_t}\omega^t = -\alpha.$$

Thus, $\alpha = i_{X_1}\omega$, and so $\alpha \in H$. By Condition (i) for $x \in \mathcal{U}$ and all t, $\omega_x^{t\#}$ is an isomorphism hence

$$X_t := (\omega_x^{t\#})^{-1}\alpha$$

is well defined. By Condition (ii), X_t is MC^∞ so Theorem 2.4 implies that there exists a smooth isotopy $\mathbb{F}_t$ generated by X_t which for $t \in [0,1]$ satisfies

$$\mathbb{F}_t^* \omega^t = \omega_0. \tag{3.2}$$

To solve (3.2), we need to solve

$$\frac{d}{dt} \mathbb{F}_t^* \omega^t = 0. \tag{3.3}$$

We have by product rule of derivative and the Cartan formula that

$$\begin{aligned}\frac{d}{dt} \mathbb{F}_t^* \omega^t &= \mathbb{F}_t^* (\mathcal{L}_{X_t} \omega^t) + \mathbb{F}_t^* \frac{d}{dt} \omega^t \\ &= \mathbb{F}_t^* \Big(\frac{d}{dt} \omega^t - d_{dR}(i_{X_t} \omega^t) \Big) \\ &= \mathbb{F}_t^* (-d\alpha + \omega_0 - \omega) = 0.\end{aligned}$$

Thus, $\mathbb{F}_1^* \omega_1 = \mathbb{F}_0^* \omega_0$ and so $\mathbb{F}_1^* \omega = \omega_0$. □

Remark 3.6. In the projective limit approach despite the fact that many interesting results can be recovered for Fréchet manifolds there are some limitations. To construct geometric and topological objects we need to establish the existence of compatible projective limits of their corresponding Banach factors. This would not be easy in some cases and also we can not use some known results (e.g. the Poincare lemma for locally convex spaces). Therefore it is imposed the additional condition in [6, Theorem 4.2] for the existence of the required differential form as the Poincare lemma is not available in this setting. Also, we need a rather strong Lipschitz condition on mappings for the existence of local flows. In contrast, in metric approach we can apply known facts from the metric geometry and locally convex spaces that simplify proofs. There are some restrictions in this approach also; it is not easy to check MC^k-differentiability and the class of bounded maps can be very small. However as mentioned, manifolds of Riemannian metrics have the structure of nuclear bounded Fréchet manifolds and Theorem 3.5 can be used to study their cohomology, but it is not yet clear how to construct a symplectic structure that can be applied in this context.

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