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On the generalized lower bound conjecture for polytopes and spheres

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1. Introduction

The study of face numbers of polytopes is a classical problem. For a simplicial d-polytope P let $f_i(P)$ denote the number of its i-dimensional faces for $-1\leqslant i\leqslant d-1$ ($f_{-1}(P)=1$ for the empty set). The numbers $f_i(P)$ are conveniently described by the *h-numbers*, defined by $h_i(P)=\sum_{j=0}^{i}(-1)^{i-j}\binom{d-j}{i-j}f_{j-1}(P)$ for $0\leqslant i\leqslant d$. The Dehn–Sommerville relations assert that $h_i(P)=h_{d-i}(P)$ for all $0\leqslant i\leqslant\lfloor\frac{1}{2}d\rfloor$, generalizing the Euler–Poincaré formula.

In 1971, McMullen and Walkup [20] posed the following *generalized lower bound conjecture* (GLBC), generalizing the classical lower bound conjecture for simplicial polytopes (see [11, §10.2]).

Conjecture 1.1. (McMullen–Walkup) Let P be a simplicial d-polytope. Then

(a) $1=h_0(P)\leqslant h_1(P)\leqslant...\leqslant h_{\lfloor d/2\rfloor}(P)$;

(b) for an integer $1\leqslant r\leqslant\frac{1}{2}d$, the following are equivalent:

(i) $h_{r-1}(P)=h_r(P)$;

(ii) P is $(r-1)$-stacked, i.e. there is a triangulation K of P all of whose faces of dimension at most $d-r$ are faces of P.

The inequality $h_1(P)\leqslant h_2(P)$ was proved by Barnette [2], [3] in the early 1970s, and is called Barnette's *lower bound theorem.* Around 1980 the *g-theorem* was proved, giving a complete characterization of the face numbers of simplicial polytopes. It was conjectured by McMullen [18], sufficiency of the conditions was proved by Billera–Lee [4] and necessity by Stanley [28]. Stanley's result establishes part (a) of the GLBC.

As for part (b), the implication (ii) ⇒ (i) was shown in [20]. The implication (i) ⇒ (ii) is easy for $r=1$, and was proved for $r=2$ by Barnette [2] and Billera–Lee [4]. The main

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goal of this paper is to prove the remaining open part of the GLBC. In particular, it follows that $(r-1)$-stackedness of a simplicial d-polytope, where $r\leqslant\frac{1}{2}d$, only depends on its face numbers.

McMullen [19] proved that, to study Conjecture 1.1 (b), it is enough to consider combinatorial triangulations (see their definition below). Thus we write a statement in terms of (abstract) simplicial complexes. For a simplicial d-polytope P with boundary complex Δ (we regard Δ as an abstract simplicial complex), we say that a simplicial complex K is a (*combinatorial*) *triangulation* of P if its geometric realization is homeomorphic to a d-ball and its boundary is Δ. A triangulation K of P is *geometric* if in addition there is a geometric realization of K whose underlying space is P. For a simplicial complex Δ on the vertex set V and a positive integer i, let

$$\Delta(i) := \{F\subseteq V : \mathrm{skel}_i(2^F)\subseteq\Delta\},$$

where 2^F is the power set of F and $\mathrm{skel}_i(2^F)$ is the i-skeleton of 2^F, i.e. the collection of all subsets of F of size at most $i+1$.

THEOREM 1.2. *Let P be a simplicial d-polytope with h-vector $(h_0,h_1,...,h_d)$, Δ be its boundary complex and $1\leqslant r\leqslant\frac{1}{2}d$ be an integer. If $h_{r-1}=h_r$ then $\Delta(d-r)$ is the unique geometric triangulation of P all of whose faces of dimension at most $d-r$ are faces of P.*

Note that the uniqueness of such a triangulation was proved by McMullen [19]. Moreover, it was shown by Bagchi and Datta [1] that if Conjecture 1.1 (b) is true then the triangulation must be $\Delta(d-r)$.

Since the above theorem is described in terms of simplicial complexes, it would be natural to ask if a similar statement holds for triangulations of spheres, or more generally homology spheres. Indeed, we also prove an analogous result for homology spheres satisfying a certain algebraic property called the *weak Lefschetz property* (WLP, to be defined later).

THEOREM 1.3. *Let Δ be a homology $(d-1)$-sphere having the* WLP *over a field of characteristic* 0, *$(h_0,h_1,...,h_d)$ be the h-vector of Δ and $1\leqslant r\leqslant\frac{1}{2}d$ be an integer. If $h_{r-1}=h_r$ then $\Delta(d-r)$ is the unique homology d-ball with no interior faces of dimension at most $d-r$ and with boundary Δ.*

Note that an algebraic formulation of the g-conjecture (for homology spheres) asserts that any homology sphere has the WLP, see e.g. [29, Conjecture 4.22] for a stronger variation. If this conjecture holds, then Theorem 1.3 will extend to all homology spheres. Indeed, the case $r=2$ in Theorem 1.3 was proved by Kalai [13], without the WLP assumption, as part of his generalization of the lower bound theorem to homology manifolds and beyond. Further, note that for $r\leqslant\frac{1}{2}d$, if a homology $(d-1)$-sphere Δ satisfies that $\Delta(d-r)$ is a homology d-ball with boundary Δ, then Δ satisfies all the numerical conditions in the g-conjecture (including the non-linear Macaulay inequalities), as was shown by Stanley [27].

This paper is organized as follows. In §2 we give preliminaries on triangulations and prove the uniqueness claim in the above two theorems. In §3 we prove that $\Delta(d-r)$ satisfies a nice algebraic property called the Cohen–Macaulay property. In §4, by using this result together with a geometric and topological argument, we show that $\Delta(d-r)$ triangulates P in Theorem 1.2. In §5 we prove Theorem 1.3 based on the theory of canonical modules in commutative algebra. Lastly, in §6 we give some concluding remarks and open questions.

2. Triangulations

In this section, we provide some preliminaries and notation on triangulations, and prove the uniqueness statements in Theorems 1.2 and 1.3.

Let Δ be an (abstract) *simplicial complex* on the vertex set V, i.e. a collection of subsets of V such that, for any $F\in\Delta$ and $G\subset F$, one has $G\in\Delta$. An element $F\in\Delta$ is called a *face* of Δ, and a maximal face (under inclusion) is called a *facet* of Δ. A face $F\in\Delta$ is called an *i-face* if $\#F=i+1$, where $\#X$ denotes the cardinality of a finite set X. The *dimension* of Δ is $\dim\Delta=\max\{\#F-1:F\in\Delta\}$. For $0\leqslant k\leqslant\dim\Delta$, we write $\mathrm{skel}_k(\Delta)=\{F\in\Delta:\#F\leqslant k+1\}$ for the *k-skeleton* of Δ. Let $f_i=f_i(\Delta)$ be the number of i-faces of Δ. The *h-vector* $h(\Delta)=(h_0(\Delta),h_1(\Delta),...,h_d(\Delta))$ of Δ is the sequence of integers defined by

$$h_i(\Delta)=\sum_{j=0}^{i}(-1)^{i-j}\binom{d-j}{i-j}f_{j-1}$$

for $i=0,1,...,d$, where $d=\dim\Delta+1$ and where $f_{-1}=1$. If Δ is the boundary complex of a simplicial polytope P, we also call $h(\Delta)$ the h-vector of P.

Let Δ be a simplicial complex on the vertex set V. A subset $F\subset V$ is called a *missing face* of Δ if $F\notin\Delta$ and all proper subsets of F are faces of Δ. Note that the set of the missing faces of Δ determines Δ itself, since it determines all subsets of V which are not in Δ. It is not hard to see that, by definition, the simplicial complex $\Delta(i)$, defined in the introduction, is the simplicial complex whose missing faces are the missing faces F of Δ with $\#F\leqslant i+1$. In particular, for $j\leqslant i$, one has $\Delta(j)=\Delta(i)$ if and only if Δ has no missing k-faces for $j+1\leqslant k\leqslant i$.

The following relation between face numbers and missing faces will be used in the sequel. It was first proved by Kalai [15, Proposition 3.6] when $d>2r+1$, and was later generalized by Nagel [23, Corollary 4.8].

LEMMA 2.1. *Let Δ be the boundary complex of a simplicial d-polytope. If $h_{r-1}(\Delta)=h_r(\Delta)$ then $\Delta(r-1)=\Delta(d-r)$.*

Remark 2.2. Nagel [23] states this only for simplicial polytopes, but his proof works for homology spheres admitting the WLP, which we study in §5.

Next, we prove the uniqueness statements in Theorems 1.2 and 1.3. We start with some notation and definitions. Let $\mathbf{k}$ be a field. For a simplicial complex Δ, let $\widetilde{H}_i(\Delta;\mathbf{k})$ be the ith reduced homology group of Δ with coefficients in $\mathbf{k}$, and let

$$\mathrm{lk}_\Delta(F)=\{G\in\Delta : F\cup G\in\Delta \text{ and } F\cap G=\varnothing\}$$

be the *link* of F in Δ. A d-dimensional simplicial complex Δ is said to be a *homology d-sphere* (over $\mathbf{k}$) if the homology groups $\widetilde{H}_{d-\#F-i}(\mathrm{lk}_\Delta(F);\mathbf{k})$ are isomorphic to $\mathbf{k}$ for $i=0$ and vanish for all $i>0$, for all $F\in\Delta$ (including the empty face $\varnothing$). Also, a *homology d-ball* (over $\mathbf{k}$) is a d-dimensional simplicial complex Δ such that the homology groups $\widetilde{H}_{d-\#F-i}(\mathrm{lk}_\Delta(F);\mathbf{k})$ are either $\mathbf{k}$ or 0 for $i=0$ and vanish for $i>0$, for all $F\in\Delta$, and moreover, its boundary complex

$$\partial\Delta=\{F\in\Delta : \widetilde{H}_{d-\#F}(\mathrm{lk}_\Delta(F);\mathbf{k})=0\}$$

is a homology $(d-1)$-sphere. We say that a simplicial complex Δ is a *triangulation* of a topological space X if its geometric realization is homeomorphic to X. Note that a triangulation of a d-sphere (resp. d-ball) is a homology d-sphere (resp. d-ball) over any field. See e.g. [22, Lemma 63.2].

Let Δ be a homology d-ball. The faces in $\Delta-\partial\Delta$ are called the *interior faces* of Δ. If Δ has no interior k-faces for $k\leqslant d-r$ then Δ is said to be $(r-1)$-*stacked*. An $(r-1)$-*stacked* sphere (resp. homology sphere) is the boundary complex of an $(r-1)$-stacked triangulation of a ball (resp. homology ball).

Recall that a triangulation K of a simplicial d-polytope P with boundary complex Δ is a triangulation of a d-ball such that $\partial K=\Delta$. McMullen [19, Theorem 3.3] proved that, for $r\leqslant\frac{1}{2}(d+1)$, an $(r-1)$-stacked triangulation K of a simplicial d-polytope P is unique. Moreover, Bagchi and Datta [1, Corollary 3.6] proved that such a triangulation must be equal to $\Delta(d-r)$. We generalize these statements for homology spheres based on an idea of Dancis [6] who proved that a homology d-sphere is determined by its $\lceil\frac{1}{2}d\rceil$-skeleton (generalizing previous work of Perles who showed it for polytopes). In particular, our result answers [1, Question 6.4].

THEOREM 2.3. *Let* Δ *be a homology* $(d-1)$*-sphere and* $1\leqslant r\leqslant \frac{1}{2}(d+1)$ *be an integer.*
(i) *If* $\Delta(d-r)$ *is a homology* d*-ball with* $\partial\Delta(d-r)=\Delta$ *then it is* $(r-1)$*-stacked.*
(ii) *If* Δ' *is an* $(r-1)$*-stacked homology* d*-ball with* $\partial\Delta'=\Delta$, *then* $\Delta'=\Delta(d-r)$.

Proof. (i) is obvious, since $\Delta(d-r)$ and $\partial\Delta(d-r)=\Delta$ have the same $(d-r)$-skeleton. We prove (ii). Since Δ' is $(r-1)$-stacked, Δ' has the same $(d-r)$-skeleton as Δ, and therefore has the same $(d-r)$-skeleton as $\Delta(d-r)$ by definition. Thus, what we must prove is that Δ' has no missing faces of cardinality $>d-r+1$. Let F be a $(k+1)$-subset of $[n]$ with $k>d-r$ such that all its proper subsets are in Δ'. We claim that $F\in\Delta'$.

Consider the homology d-sphere $S=\Delta'\cup(\{v\}*\Delta)$, where v is a new vertex and where $\{v\}*\Delta=\Delta\cup\{\{v\}\cup G:G\in\Delta\}$ is the cone of Δ with vertex v. For a subset $W\subset V$, where V is the vertex set of S, let $S|_W=\{G\in S:G\subset W\}$ be the induced subcomplex of S on W. Since all proper subsets of F are in Δ' and Δ' is an induced subcomplex of S, to prove that $F\in\Delta'$ it is enough to show that $S|_F$ is not a $(k-1)$-sphere, equivalently that $\widetilde{H}_{k-1}(S|_F;\mathbf{k})=0$.

Since $S-S|_F$ is homotopy equivalent to $S|_{V\setminus F}$ (see e.g. [22, Lemma 70.1]), by Alexander duality (see e.g. [22, Theorem 71.1]) and the universal coefficient theorem with field coefficients, we have

$$\widetilde{H}_{k-1}(S|_F;\mathbf{k})\cong\widetilde{H}_{d-k}(S-S|_F;\mathbf{k})\cong\widetilde{H}_{d-k}(S|_{V\setminus F};\mathbf{k}),$$

so we need to show that $\widetilde{H}_{d-k}(S|_{V\setminus F};\mathbf{k})=0$. Since $d-k\leqslant r-1\leqslant d-r$, we have

$$\mathrm{skel}_{d-k}(S|_{V\setminus F})=\mathrm{skel}_{d-k}((\{v\}*\Delta)|_{V\setminus F})$$

and $S|_{V\setminus F}\supset(\{v\}*\Delta)|_{V\setminus F}$. Then, by the definition of the simplicial homology, we have

$$\dim_{\mathbf{k}}\widetilde{H}_{d-k}(S|_{V\setminus F};\mathbf{k})\leqslant\dim_{\mathbf{k}}\widetilde{H}_{d-k}((\{v\}*\Delta)|_{V\setminus F};\mathbf{k}).$$

Recall that $v\notin F$. The right-hand side of the above inequality is equal to zero since $(\{v\}*\Delta)|_{V\setminus F}=\{v\}*(\Delta|_{V\setminus(F\cup\{v\})})$ is a cone. Hence $\widetilde{H}_{d-k}(S|_{V\setminus F};\mathbf{k})=0$. $\square$

Unlike $(r-1)$-stacked polytopes with $r\leqslant\frac{1}{2}d$, $\frac{1}{2}(d-1)$-stacked simplicial d-polytopes cannot be characterized by their h-vectors, since $h_{(d-1)/2}=h_{(d+1)/2}$ holds for all simplicial d-polytopes when d is odd. On the other hand, Theorem 2.3 says that $\frac{1}{2}(d-1)$-stacked simplicial d-polytopes still have a nice combinatorial property. It would be of interest to have a nice combinatorial characterization of these polytopes.

3. Cohen–Macaulayness

In this section, we prove that the simplicial complexes $\Delta(d-r)=\Delta(r-1)$ in Theorems 1.2 and 1.3 (the equalities hold by Lemma 2.1 and Remark 2.2, respectively) satisfy a nice algebraic condition, called the Cohen–Macaulay property. We first introduce some basic tools in commutative algebra.

Stanley–Reisner rings

Let $S=\mathbf{k}[x_1, ..., x_n]$ be a polynomial ring over an infinite field $\mathbf{k}$. For a subset $F\subset[n]=\{1, ..., n\}$, we write $x_F=\prod_{k\in F} x_k$. For a simplicial complex Δ on $[n]$, the ring

$$\mathbf{k}[\Delta]=S/I_\Delta,$$

where $I_\Delta=(x_F : F\subset[n] \text{ and } F\notin\Delta)$, is called the *Stanley–Reisner ring* of Δ.

The simplicial complex $\Delta(i)$ has a simple expression in terms of Stanley–Reisner rings. For a homogeneous ideal $I\subset S$, let $I_{\leqslant k}$ be the ideal generated by all elements in I of degree $\leqslant k$. Since the missing faces of Δ correspond to the minimal generators of I_Δ and since $\Delta(i)$ is the simplicial complex whose missing faces are the missing faces F of Δ with $\#F\leqslant i+1$, one has

$$I_{\Delta(i)}=(I_\Delta)_{\leqslant i+1}.$$

Cohen–Macaulay property

Let $I\subset S$ be a homogeneous ideal and $R=S/I$. The *Krull dimension* $\dim R$ of R is the minimal number k such that there is a sequence of linear forms $\theta_1, ..., \theta_k\in S$ such that

$$\dim_{\mathbf{k}} S/(I+(\theta_1, ..., \theta_k))<\infty.$$

If $d=\dim R$, then a sequence $\Theta=(\theta_1, ..., \theta_d)$ of linear forms such that $\dim_{\mathbf{k}} S/(I+(\Theta))<\infty$ is called a *linear system of parameters* (l.s.o.p. for short) of R. A sequence of homogeneous polynomials $f_1, ..., f_r$ of positive degrees is called a *regular sequence* of R if f_i is a non-zero divisor of $S/(I+(f_1, ..., f_{i-1}))$ for all $i\in[r]$. We say that R is *Cohen–Macaulay* if every (equivalently, some) l.s.o.p. of R is a regular sequence of R.

A simplicial complex Δ is said to be *Cohen–Macaulay* (over $\mathbf{k}$) if $\mathbf{k}[\Delta]$ is a Cohen–Macaulay ring. The following topological criterion for the Cohen–Macaulay property was proved by Reisner [24].

LEMMA 3.1. (Reisner's criterion) *A simplicial complex Δ is Cohen–Macaulay* (*over* $\mathbf{k}$) *if and only if, for any face $F\in\Delta$, $\widetilde{H}_i(\mathrm{lk}_\Delta(F);\mathbf{k})=0$ for all $i<\dim\Delta-\#F$.*

The weak Lefschetz property

Let $I \subset S$ be a homogeneous ideal such that $R = S/I$ has Krull dimension 0. We write $R = \bigoplus_{i=0}^{s} R_i$, where R_i is the homogeneous component of R of degree i and where $R_s \neq 0$. We say that R has the *weak Lefschetz property* (WLP for short) if there is a linear form $w \in R_1$, called a *Lefschetz element* of R, such that the multiplication $\times w\colon R_k \to R_{k+1}$ is either injective or surjective for all k.

We say that a ring $R = S/I$ of Krull dimension $d > 0$, where I is a homogeneous ideal, has the WLP if it is Cohen–Macaulay and there is an l.s.o.p. Θ of R such that $S/(I + (\Theta))$ has the WLP. Also, a simplicial complex Δ is said to have the WLP (over $\mathbf{k}$) if $\mathbf{k}[\Delta]$ has the WLP. It is known that the boundary complex of a simplicial polytope has the WLP over $\mathbb{Q}$. See [8, §5.2].

For a homogeneous ideal $I \subset S$, the Hilbert series

$$H(S/I, t) = \sum_{i=0}^{\infty} (\dim_{\mathbf{k}} (S/I)_i) t^i$$

of the ring S/I can be written in the form

$$H(S/I, t) = \frac{h_0 + h_1 t + \ldots + h_s t^s}{(1-t)^d},$$

where $d = \dim S/I$ and $h_s \neq 0$. See [5, Corollary 4.1.8]. The vector $h(S/I) = (h_0, h_1, \ldots, h_s)$ is called the *h-vector* of S/I. If S/I has the WLP then its h-vector is *unimodal*, i.e. it satisfies $h_0 \leqslant \ldots \leqslant h_p \geqslant h_{p+1} \geqslant \ldots \geqslant h_s$ for some p. Indeed, let $R = S/(I + (\Theta))$, where Θ is an l.s.o.p. of S/I. Then we have $h_k = \dim_{\mathbf{k}} R_k$ for all k. Observe that the multiplication $\times w\colon R_k \to R_{k+1}$ is surjective if and only if $(S/(I + (\Theta, w)))_{k+1} = 0$. In particular, since S is generated by elements of degree 1, if the multiplication map is surjective for some $k = t$, then it is also surjective for all $k \geqslant t$. Thus, if R has the WLP then $h_p \geqslant h_{p+1}$ implies that $\times w\colon R_k \to R_{k+1}$ is surjective for all $k \geqslant p$, and we have $h_p \geqslant h_{p+1} \geqslant \ldots \geqslant h_s$.

Generic initial ideals

Here we briefly recall generic initial ideals. We do not give details on this subject. [10] and [12, §4] are good surveys on generic initial ideals.

Let $>_{\mathrm{rev}}$ be the degree reverse lexicographic order induced by $x_1 >_{\mathrm{rev}} \ldots >_{\mathrm{rev}} x_n$. For a homogeneous ideal $I \subset S$, let $\mathrm{in}_{>_{\mathrm{rev}}}(I)$ be the initial ideal of I with respect to $>_{\mathrm{rev}}$. Let $\mathrm{GL}_n(\mathbf{k})$ be the general linear group with coefficients in $\mathbf{k}$. Any $\varphi = (a_{ij}) \in \mathrm{GL}_n(\mathbf{k})$ induces an automorphism of S, again denoted by φ,

$$\varphi(f(x_1, \ldots, x_n)) = f\left(\sum_{k=1}^{n} a_{k1} x_k, \ldots, \sum_{k=1}^{n} a_{kn} x_k\right)$$

for any $f\in S$. It was proved by Galligo that $\mathrm{in}_{>_{\mathrm{rev}}}(\varphi(I))$ is constant for a generic choice of $\varphi\in \mathrm{GL}_n(\mathbf{k})$. See [10, Theorem 1.27]. This monomial ideal $\mathrm{in}_{>_{\mathrm{rev}}}(\varphi(I))$ is called the *generic initial ideal* of I with respect to $>_{\mathrm{rev}}$, and denoted by $\mathrm{gin}(I)$. We need the following well-known result on the WLP.

LEMMA 3.2. *Let $I\subset S$ be a homogeneous ideal and $d=\dim S/I$.*

(i) *S/I is Cohen–Macaulay if and only if $S/\mathrm{gin}(I)$ is Cohen–Macaulay.*

(ii) *S/I has the* WLP *if and only if $S/\mathrm{gin}(I)$ has the* WLP. *Moreover, if S/I has the* WLP, *then $x_n, ..., x_{n-d+1}$ is an l.s.o.p. of $S/\mathrm{gin}(I)$ and x_{n-d} is a Lefschetz element of $S/(\mathrm{gin}(I)+(x_n, ..., x_{n-d+1}))$.*

See [12, Corollary 4.3.18] for the first statement. The second statement follows from [12, Lemma 4.3.7] together with the facts that, for generic linear forms $\theta_1, ..., \theta_{d+1}\in S$, $\theta_1, ..., \theta_d$ is an l.s.o.p. of S/I and θ_{d+1} is a Lefschetz element of $S/(I+(\theta_1, ..., \theta_d))$, and that for a generic choice of $\varphi\in\mathrm{GL}_n(\mathbf{k})$ the linear forms $x_n, ..., x_{n-d}$ are generic for $S/\varphi(I)$.

The following result due to Green [10, Proposition 2.28] is crucial to proving the Cohen–Macaulay property of $\Delta(r-1)$.

LEMMA 3.3. (Crystallization principle) *Suppose that* $\mathrm{char}(\mathbf{k})=0$. *Let $I\subset S$ be a homogeneous ideal generated by elements of degree $\leqslant m$. If $\mathrm{gin}(I)$ has no minimal generators of degree $m+1$, then $\mathrm{gin}(I)$ is generated by elements of degree $\leqslant m$.*

THEOREM 3.4. *Suppose that* $\mathrm{char}(\mathbf{k})=0$. *Let $I\subset S$ be a homogeneous ideal such that S/I has the* WLP, *and let $h(S/I)=(h_0, h_1, ..., h_s)$. Suppose that $h_0\leqslant...\leqslant h_p$. If $h_{r-1}=h_r=h_{r+1}$ for some $1\leqslant r\leqslant p-1$, then $S/I_{\leqslant r}$ is Cohen–Macaulay of Krull dimension* $\dim S/I+1$.

Proof. Let $J=\mathrm{gin}(I)$ and $d=\dim S/I$. We first claim that $S/J_{\leqslant r}$ is Cohen–Macaulay. Observe that J is a monomial ideal. By Lemma 3.2, S/J is Cohen–Macaulay of Krull dimension d, and J has no minimal generators which are divisible by one of $x_n, ..., x_{n-d+1}$. Also, since $h_0\leqslant...\leqslant h_{r+1}$, the WLP shows that the multiplication

$$\times x_{n-d}\colon S/(J+(x_n, ..., x_{n-d+1}))_j \longrightarrow S/(J+(x_n, ..., x_{n-d+1}))_{j+1} \tag{1}$$

is injective for $j\leqslant r$, which implies that J has no minimal generators of degree $\leqslant r+1$ which are divisible by x_{n-d}. Indeed, if there is a minimal generator of the form ux_{n-d}, then u is in the kernel of the map (1). Thus $J_{\leqslant r}$ has no minimal generators which are divisible by one of $x_n, ..., x_{n-d}$. Hence $x_n, ..., x_{n-d}$ is a regular sequence of $S/J_{\leqslant r}$. In particular, we have $\dim S/J_{\leqslant r}\geqslant d+1$ since the length of a regular sequence is bounded by the Krull dimension ([5, Proposition 1.2.12]).

It is left to show that the quotient by this regular sequence is a finite-dimensional vector space over $\mathbf{k}$. Since the multiplication map (1) is surjective when $j=r-1$,

$$(S/J+(x_n,...,x_{n-d}))_r=0,$$

and J contains all monomials in $\mathbf{k}[x_1,...,x_{n-d-1}]$ of degree r. Thus

$$\dim_{\mathbf{k}} S/(J_{\leqslant r}+(x_n,...,x_{n-d}))<\infty,$$

and $S/J_{\leqslant r}$ is Cohen–Macaulay of Krull dimension $d+1$ with an l.s.o.p. $x_n,...,x_{n-d}$.

Next, we prove $\operatorname{gin}(I_{\leqslant r})=\operatorname{gin}(I)_{\leqslant r}$. By the crystallization principle, what we must prove is that $\operatorname{gin}(I_{\leqslant r})$ has no minimal generators of degree $r+1$. Since $I_{\leqslant r}\subset I$ and $(I_{\leqslant r})_r=I_r$, it is enough to prove that $\operatorname{gin}(I)$ has no minimal generators of degree $r+1$. Indeed, we already showed that $J=\operatorname{gin}(I)$ has no minimal generator of degree $r+1$ which is divisible by one of $x_n,...,x_{n-d+1},x_{n-d}$. We also showed that J contains all monomials in $\mathbf{k}[x_1,...,x_{n-d-1}]$ of degree r. These facts guarantee that $J=\operatorname{gin}(I)$ has no minimal generators of degree $r+1$, as desired.

We proved that $S/\operatorname{gin}(I_{\leqslant r})=S/\operatorname{gin}(I)_{\leqslant r}$ is Cohen–Macaulay of Krull dimension $d+1$. Then the desired statement follows from Lemma 3.2 (i). ☐

COROLLARY 3.5. *Suppose that* $\operatorname{char}(\mathbf{k})=0$. *Let* Δ *be a homology* $(d-1)$*-sphere having the* WLP *over* $\mathbf{k}$. *If* $h_{r-1}(\Delta)=h_r(\Delta)$ *for some* $r\leqslant\frac{1}{2}d$, *then* $\Delta(r-1)$ *is Cohen–Macaulay over* $\mathbf{k}$ *and has dimension* d.

Proof. Recall that the h-vector of Δ coincides with the h-vector of its Stanley–Reisner ring $\mathbf{k}[\Delta]$. Since the h-vector of Δ is symmetric, the WLP shows that

$$h_{r-1}(\Delta)=h_r(\Delta)=...=h_{d-r+1}(\Delta) \quad \text{and} \quad h_0(\Delta)\leqslant...\leqslant h_{r+1}(\Delta).$$

As $I_{\Delta(r-1)}=(I_\Delta)_{\leqslant r}$, Theorem 3.4 says that $\mathbf{k}[\Delta(r-1)]$ is Cohen–Macaulay of Krull dimension $d+1$. Thus $\Delta(r-1)$ is a Cohen–Macaulay simplicial complex of dimension d. ☐

Remark 3.6. The weaker assertion that $\dim\Delta(d-r)\leqslant d$ for $r\leqslant\frac{1}{2}d$ is true for any simplicial $(d-1)$-sphere Δ, and more generally for any simplicial complex Δ which embeds in the $(d-1)$-sphere.

This can be shown using van Kampen obstruction to embedability, see [16], [26], [30], and for cones over Flores complexes [7]. If we assume that $\dim\Delta(d-r)>d$ then, for d even, Δ contains $\operatorname{skel}_{d/2}(2^{[d+2]})$, and hence it contains the cone over Flores complex $L=\operatorname{skel}_{d/2-1}(2^{[d+1]})$. (Here $2^{[i]}$ is the power set of $[i]=\{1,...,i\}$.) By the non-vanishing on L of the van Kampen obstruction to embedability in the $(d-2)$-sphere, we conclude that the cone over L does not embed in the $(d-1)$-sphere, a contradiction. The argument for d odd is similar.

4. GLBC for polytopes

In this section we prove the existence part of Theorem 1.2.

THEOREM 4.1. *Let P be a simplicial d-polytope with h-vector $(h_0, h_1, ..., h_d)$, Δ be its boundary complex and $1 \leqslant r \leqslant \frac{1}{2}d$ be an integer. If $h_{r-1} = h_r$ then $\Delta(d-r)$ is a geometric triangulation of P.*

In the rest of this section, we fix a simplicial d-polytope P satisfying the assumption of Theorem 4.1, and prove the theorem for P.

We may assume that $P \subset \mathbb{R}^d$. Let $V = \{v_1, ..., v_n\} \subset \mathbb{R}^d$ be the vertex set of P and let Δ be the boundary complex of P. For a subset $T = \{v_{i_1}, ..., v_{i_k}\} \subset V$, we write $[T] = \operatorname{conv}(v_{i_1}, ..., v_{i_k})$ for the convex hull of the vertices in T. Let $\Delta' = \Delta(r-1)$. Recall that, under the assumptions of Theorem 4.1, $\Delta(d-r) = \Delta(r-1)$.

LEMMA 4.2. *The set $\{[F] : F \in \Delta'\}$ is a geometric realization of Δ', i.e.*
(i) *$[F_1] \cap [F_2] = [F_1 \cap F_2]$ for all $F_1, F_2 \in \Delta'$;*
(ii) *$\dim[F] = \#F - 1$ for all $F \in \Delta'$.*

Proof. The proof is similar to that of [1, Proposition 3.4].

(i) Assume by contradiction that $F_1, F_2 \in \Delta'$ form a counterexample to (i) with the size $\#F_1 + \#F_2$ minimal. Then, by Carathéodory's theorem, $[F_1]$ and $[F_2]$ are simplexes with $\dim[F_1] = \#F_1 - 1$ and $\dim[F_2] = \#F_2 - 1$. Also, the convex set $[F_1] \cap [F_2]$ is not contained in the boundary of P, as otherwise it would equal a single face $[F]$ with $F \in \Delta$ and thus $F_1 \cap F_2 = F$, which says that (i) holds for F_1 and F_2. In particular, we have $F_1 \notin \Delta$ and $F_2 \notin \Delta$. We prove the following properties for F_1 and F_2:
(a) any $p \in [F_1] \cap [F_2] \setminus [F_1 \cap F_2]$ is in the relative interior of both $[F_1]$ and $[F_2]$;
(b) $F_1 \cap F_2 = \varnothing$;
(c) $[F_1]$ and $[F_2]$ intersect in a single point.

We first prove (a). Suppose to the contrary that p is in the boundary of $[F_1]$. Then there is a $u \in F_1$ such that $p \in [F_1 \setminus \{u\}]$. Since $p \notin [F_1 \cap F_2]$, we have

$$p \in [F_1 \setminus \{u\}] \cap [F_2] \setminus [(F_1 \setminus \{u\}) \cap F_2],$$

contradicting the minimality of F_1 and F_2. Hence (a) holds.

Next we show (b). Let $p \in [F_1] \cap [F_2] \setminus [F_1 \cap F_2]$. By (a), there are convex combinations with positive coefficients $\sum_{v \in F_1} a_v v = p = \sum_{v \in F_2} b_v v$ with $\#F_1 \geqslant 2$ and $\#F_2 \geqslant 2$. If there is $u \in F_1 \cap F_2$, say with $a_u \leqslant b_u$, then by subtracting $a_u u$ from both sides and by normalizing them, we get a point q which is contained in $[F_1 \setminus \{u\}] \cap [F_2]$. Since q is in the relative interior of $[F_1 \setminus \{u\}]$ by the construction and since $F_1 \not\subset F_2$, we have $q \notin [(F_1 \setminus \{u\}) \cap F_2]$, contradicting the minimality. Hence (b) holds.

We finally prove (c). Suppose to the contrary that $[F_1]\cap[F_2]$ contains two different points p and q. Let ℓ be the line through them. Then the endpoints of the line segment $\ell\cap[F_1]\cap[F_2]$ must be on the boundary of either $[F_1]$ or $[F_2]$, contradicting (a) as $[F_1\cap F_2]$ is empty by (b). Hence (c) holds.

We now complete the proof of (i). By (a) and (c), the intersection of $[F_1]$ and $[F_2]$ equals the intersection of their affine hulls, as otherwise the neighborhood of p in $[F_1]\cap[F_2]$ is not a single point. This fact and (b) imply that $\#F_1+\#F_2\leqslant d+2$. However, since F_1 and F_2 are not in Δ and since $\Delta'=\Delta(d-r)$ and Δ have the same $(d-r)$-skeleton, we have $\#F_1\geqslant d-r+2$ and $\#F_2\geqslant d-r+2$, a contradiction. Hence we conclude that (i) holds.

(ii) Lemma 2.1 and Theorem 3.4 show that Δ' is d-dimensional and pure, i.e. all of its facets have cardinality $d+1$. Thus it is enough to show that if $F=\{v_{i_1},...,v_{i_{d+1}}\}$ is a facet of Δ' then $\dim[F]=d$. Suppose to the contrary that $\dim[F]<d$. Then $v_{i_1},...,v_{i_{d+1}}$ are in the same hyperplane in $\mathbb{R}^d$. Therefore, by Radon's theorem, there is a partition $F=F'\cup F''$ such that $[F']\cap[F'']\neq\varnothing$. This contradicts (i). $\square$

Let $[\Delta']=\bigcup_{F\in\Delta'}[F]$ be the underlying space of the geometric simplicial complex $\{[F]:F\in\Delta'\}$. To complete the proof of Theorem 4.1, it is left to show the following.

Lemma 4.3. $[\Delta']=P$.

Proof. Observe that $[\Delta']\subseteq P$. Assume by contradiction that there is $p\in P\setminus[\Delta']$. We assume that $[\Delta']$ and P are embedded in S^d via the natural homeomorphism $\mathbb{R}^d\cong S^d\setminus\{v\}\subset S^d$, where v is a point in S^d. Let $q\in\mathbb{R}^d\setminus P$. Since $[\Delta']$ contains the boundary of P, p and q are in different connected components in $S^d\setminus[\Delta']$. Thus $S^d\setminus[\Delta']$ is not connected. By Alexander duality, we have $\widetilde{H}_{d-1}([\Delta'];\mathbb{Q})\cong\widetilde{H}_0(S^d\setminus[\Delta'];\mathbb{Q})\neq 0$.

Recall that Δ has the WLP over $\mathbb{Q}$. Thus Δ' is Cohen–Macaulay over $\mathbb{Q}$ of dimension d by Corollary 3.5. By Lemma 4.2, $[\Delta']$ is the underlying space of a geometric realization of Δ'. By Reisner's criterion (Lemma 3.1), we have $\widetilde{H}_{d-1}([\Delta'];\mathbb{Q})=0$, a contradiction. $\square$

5. GLBC for Lefschetz spheres

In this section we prove the existence part in Theorem 1.3. The proof is algebraic and we assume familiarity with $\mathbb{Z}^n$-graded commutative algebra theory. See e.g. [21] for the basics of this theory.

First, we set some notation. Let $\mathbf{e}_i\in\mathbb{Z}^n$ be the ith unit vector of $\mathbb{Z}^n$. We consider the $\mathbb{Z}^n$-grading of $S=\mathbf{k}[x_1,...,x_n]$ defined by $\deg x_i=\mathbf{e}_i$. For a $\mathbb{Z}^n$-graded S-module M and for $\mathbf{a}=(a_1,...,a_n)\in\mathbb{Z}^n$, we denote by $M_{\mathbf{a}}$ the graded component of M of degree $\mathbf{a}\in\mathbb{Z}^n$.

Let $\mathfrak{m}=(x_1,...,x_n)$ be the graded maximal ideal of S. We regard $\mathbf{k}$ as a graded S-module by identification $\mathbf{k}=S/\mathfrak{m}$. We recall a few known properties on $\operatorname{Tor}_i^S(\mathbf{k},\cdot)$.

LEMMA 5.1. *Let C be a graded S-module. If $C_k=0$ for all $k\leqslant r$ then one has $\operatorname{Tor}_i(\mathbf{k},C)_{i+j}=0$ for all i and $j\leqslant r$.*

Proof. Let $\mathcal{K}_\bullet=\mathcal{K}_\bullet(x_1,...,x_n)$ be the Koszul complex of $x_1,...,x_n$ (see e.g. [5, §1.6]). Since $\mathcal{K}_\bullet$ is the minimal free resolution of $\mathbf{k}$,

$$\operatorname{Tor}_i(\mathbf{k},C)_{i+j}\cong H_i(\mathcal{K}_\bullet\otimes C)_{i+j}.$$

On the other hand, all the elements in $\mathcal{K}_i$ have degree $\geqslant i$ and all the elements in C have degree $\geqslant r+1$ by the assumption. These facts imply that $(\mathcal{K}_i\otimes C)_{i+j}=0$ for $j\leqslant r$. Hence $H_i(\mathcal{K}_\bullet\otimes C)_{i+j}=0$ for all $j\leqslant r$. ☐

The following fact on generic initial ideals is well known. See [10, Theorem 2.27].

LEMMA 5.2. (Bayer–Stillman) *Suppose that* $\operatorname{char}(\mathbf{k})=0$. *Let $I\subset S$ be a homogeneous ideal. If* $\operatorname{gin}(I)$ *is generated by monomials of degree $\leqslant m$ then $\operatorname{Tor}_i^S(\mathbf{k},S/I)_{i+j}=0$ for all $j\geqslant m$.*

We also recall some basic facts on canonical modules. For a subset $F\subset[n]$, let $\mathbf{e}_F=\sum_{i\in F}\mathbf{e}_i$. For a Cohen–Macaulay $\mathbb{Z}^n$-graded ring $R=S/I$ of Krull dimension d, the module $\omega_R=\operatorname{Ext}_S^{n-d}(R,S(-\mathbf{e}_{[n]}))$ is called the *canonical module* of R. An important property of a canonical module is that it is isomorphic to the Matlis dual of the local cohomology module $H_{\mathfrak{m}}^d(R)$ by the local duality (see [5, Theorem 3.6.19]). Now suppose that $R=\mathbf{k}[\Delta]$. Then the local duality and the Hochster's formula for local cohomology [5, Theorem 5.3.8] imply that, for any $F\in\Delta$, one has

$$\dim_{\mathbf{k}}(\omega_{\mathbf{k}[\Delta]})_{\mathbf{e}_F}=\dim_{\mathbf{k}}(H_{\mathfrak{m}}^d(\mathbf{k}[\Delta]))_{-\mathbf{e}_F}=\dim_{\mathbf{k}}\widetilde{H}_{d-1-\#F}(\operatorname{lk}_\Delta(F)). \tag{2}$$

Recall that, by Reisner's criterion, homology balls and spheres are Cohen–Macaulay.

The next result and Theorem 2.3 prove Theorem 1.3.

THEOREM 5.3. *Suppose that* $\operatorname{char}(\mathbf{k})=0$. *Let Δ be a homology $(d-1)$-sphere having the* WLP. *If $h_{r-1}(\Delta)=h_r(\Delta)$ for some $r\leqslant\frac{1}{2}d$, then $\Delta(r-1)$ is a homology d-ball whose boundary complex is Δ.*

Proof. Step 1. Let $\Delta'=\Delta(r-1)$ and $C=I_\Delta/I_{\Delta'}$. For a graded S-module M, let $\operatorname{ann}_S(M)=\{g\in S:gf=0 \text{ for all } f\in M\}$. We first show that C satisfies the following conditions:

(i) $\operatorname{ann}_S(C)=I_{\Delta'}$;
(ii) C is Cohen–Macaulay of Krull dimension $d+1$;
(iii) $\operatorname{Tor}_{n-d-1}^S(\mathbf{k},C)\cong\mathbf{k}(-\mathbf{e}_{[n]})$.

(i) The inclusion $\operatorname{ann}_S(C)\supset I_{\Delta'}$ is clear. It is enough to show that there is an element $f\in I_\Delta$ such that $gf\notin I_{\Delta'}$ for all $g\in S$ with $g\notin I_{\Delta'}$. Let $F_1, ..., F_s$ be the facets of Δ'. By Corollary 3.5, each F_i is of size $d+1$. We claim that the polynomial $f=\sum_{i=1}^s x_{F_i}\in I_\Delta$ satisfies the desired property.

To prove this, since C contains

$$\bigoplus_{i=1}^{s} x_{F_i}\cdot(S/(x_k : k\notin F_i))$$

as a submodule, it is enough to show that, for any $g\notin I_{\Delta'}$, $gx_{F_i}\neq 0$ in

$$x_{F_i}\cdot(S/(x_k : k\notin F_i))$$

for some i. Moreover, since $x_{F_i}\cdot(S/(x_k{:}k\notin F_i))$ is $\mathbb{Z}^n$-graded, we may assume that

$$g=x_{i_1}^{a_1}\, ...\, x_{i_t}^{a_t},$$

where $a_1, ..., a_t$ are non-zero. Since $x_{i_1}^{a_1}\, ...\, x_{i_t}^{a_t}\notin I_{\Delta'}$, we have $\{i_1, ..., i_t\}\in\Delta'$. Then, as Δ' is Cohen–Macaulay, Δ' is pure and there is a facet F_i which contains $\{i_1, ..., i_t\}$. Thus we have $x_{i_1}^{a_1}\, ...\, x_{i_t}^{a_t}x_{F_i}\neq 0$ in $x_{F_i}\cdot(S/(x_k{:}k\notin F_i))$ as desired.

(ii) Consider the short exact sequence

$$0\longrightarrow C\longrightarrow S/I_{\Delta'}\longrightarrow S/I_\Delta\longrightarrow 0. \tag{3}$$

Since $S/I_{\Delta'}$ is Cohen–Macaulay of Krull dimension $d+1$ and as S/I_Δ is Cohen–Macaulay of Krull dimension d, we conclude that C is Cohen–Macaulay of Krull dimension $d+1$ (e.g. use the depth lemma [5, Proposition 1.2.9]).

(iii) It remains to prove that

$$\operatorname{Tor}^S_{n-d-1}(\mathbf{k}, C)\cong\mathbf{k}(-\mathbf{e}_{[n]}).$$

Note that $\operatorname{Tor}^S_{n-d}(\mathbf{k}, S/I_{\Delta'})=0$ since $S/I_{\Delta'}$ is Cohen–Macaulay of Krull dimension $d+1$. Then the short exact sequence (3) induces the exact sequence

$$0\longrightarrow\operatorname{Tor}^S_{n-d}(\mathbf{k}, S/I_\Delta)_j\longrightarrow\operatorname{Tor}^S_{n-d-1}(\mathbf{k}, C)_j\longrightarrow\operatorname{Tor}^S_{n-d-1}(\mathbf{k}, S/I_{\Delta'})_j\longrightarrow ...$$

for all $j\geqslant 0$. As Δ is a homology $(d-1)$-sphere,

$$\operatorname{Tor}^S_{n-d}(\mathbf{k}, S/I_\Delta)\cong\mathbf{k}(-\mathbf{e}_{[n]})$$

by Hochster's formula for Betti numbers [5, Theorem 5.5.1]. On the other hand, since $\mathrm{gin}(I_{\Delta'})$ has no generators of degrees $\geqslant r+1$, as we showed in the proof of Theorem 3.4, we have $\mathrm{Tor}^S_{n-d-1}(\mathbf{k}, S/I_{\Delta'})_j=0$ for $j\geqslant n-d-1+r$ by Lemma 5.2. These facts and the exact sequence imply that

$$\bigoplus_{j\geqslant n-d-1+r} \mathrm{Tor}^S_{n-d-1}(\mathbf{k}, C)_j \cong \mathbf{k}(-\mathbf{e}_{[n]}).$$

On the other hand, since $I_{\Delta'}=(I_\Delta)_{\leqslant r}$, we have $C_k=0$ for $k\leqslant r$. This implies that

$$\mathrm{Tor}^S_{n-d-1}(\mathbf{k}, C)_j = 0$$

for $j<n-d-1+r$ by Lemma 5.1, and (iii) follows.

Step 2. We show that

$$C \cong \mathrm{Ext}^{n-d-1}_S(\mathbf{k}[\Delta'], S(-\mathbf{e}_{[n]})) = \omega_{\mathbf{k}[\Delta']}.$$

It is standard in commutative algebra that conditions (i)–(iii) imply this isomorphism, but we include its proof. Since C is Cohen–Macaulay of Krull dimension $d+1$, it follows from [5, Theorem 3.3.10] that

$$\mathrm{Ext}^{n-d-1}_S(\mathrm{Ext}^{n-d-1}_S(C, S(-\mathbf{e}_{[n]})), S(-\mathbf{e}_{[n]})) \cong C. \tag{4}$$

On the other hand, by the duality on resolutions of C and $\mathrm{Ext}^{n-d-1}_S(C, S(-\mathbf{e}_{[n]}))$, we have

$$\mathrm{Tor}^S_0(\mathbf{k}, \mathrm{Ext}^{n-d-1}_S(C, S(-\mathbf{e}_{[n]})))_{\mathbf{a}} \cong \mathrm{Tor}^S_{n-d-1}(\mathbf{k}, C)_{\mathbf{e}_{[n]}-\mathbf{a}} \quad \text{for all } \mathbf{a}\in\mathbb{Z}^n$$

(see [5, Corollary 3.3.9]). Then (iii) of Step 1 implies that $\mathrm{Ext}^{n-d-1}_S(C, S(-\mathbf{e}_{[n]}))$ has a single generator in degree 0, so $\mathrm{Ext}^{n-d-1}_S(C, S(-\mathbf{e}_{[n]}))\cong S/J$ for some ideal J.

We claim that $J=I_{\Delta'}$, or equivalently

$$\mathrm{ann}_S(\mathrm{Ext}^{n-d-1}_S(C, S(-\mathbf{e}_{[n]}))) = I_{\Delta'}.$$

Since $\mathrm{ann}_S(M)\subset\mathrm{ann}_S(\mathrm{Hom}_S(M,N))$ for all S-modules M and N, (4) says that

$$\mathrm{ann}_S(C) \subset \mathrm{ann}_S(\mathrm{Ext}^{n-d-1}_S(C, S(-\mathbf{e}_{[n]}))) \subset \mathrm{ann}_S(C),$$

which implies that $\mathrm{ann}_S(\mathrm{Ext}^{n-d-1}_S(C, S(-\mathbf{e}_{[n]})))=\mathrm{ann}_S(C)=I_{\Delta'}$ by (i) of Step 1. Now, the isomorphism

$$C \cong \mathrm{Ext}^{n-d-1}_S(\mathbf{k}[\Delta'], S(-\mathbf{e}_{[n]}))$$

follows from (4), as $\mathrm{Ext}_S^{n-d-1}(C, S(-\mathbf{e}_{[n]})) \cong S/I_{\Delta'} = \mathbf{k}[\Delta']$.

Step 3. We now prove the theorem. By Hochster's formula (2), for any $F \in \Delta'$ we have

$$\dim_{\mathbf{k}} \widetilde{H}_{d-\#F}(\mathrm{lk}_{\Delta'}(F)) = \dim_{\mathbf{k}}(\omega_{\mathbf{k}[\Delta']})_{\mathbf{e}_F} = \dim_{\mathbf{k}}(I_\Delta/I_{\Delta'})_{\mathbf{e}_F} = \begin{cases} 1, & \text{if } F \notin \Delta, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly the above equation together with Δ' being Cohen–Macaulay imply that Δ' is a homology ball whose boundary complex is equal to Δ. □

The proof given in this section is quite algebraic. It would be of interest to have a combinatorial or a topological proof of Theorem 5.3.

6. Concluding remarks

It is easy to see that (1-)stacked spheres are boundaries of stacked polytopes, and that their stacked triangulations are shellable. So it is natural to ask the following questions.

Question 6.1. Let Δ be an $(r-1)$-stacked d-ball with $r \leqslant \frac{1}{2}(d+1)$.
(i) Is it true that $\partial\Delta$ is polytopal?
(ii) Is it true that Δ is shellable?

The next examples show that the answers to the above questions are negative.

Example 6.2. Let B be Rudin's non-shellable triangulation of a 3-ball [25]. Its f-vector is $(1, 14, 66, 94, 41)$ and its h-vector is $(1, 10, 30, 0, 0)$. Let K be the join of B and a simplex σ of dimension $k \geqslant 2$. Then K is a $(k+4)$-ball. Also, the interior faces of K are exactly those containing both σ and an interior face of B. Then, since B contains no interior vertices, K is 2-stacked.

On the other hand, K is not shellable since B is not shellable. Indeed, a shelling order on K would induce a shelling order on B by deleting σ from all facets in the shelling order of K.

Also, ∂K is non-polytopal. Indeed, assume the contrary, then for a vertex v of σ, $\mathrm{lk}_{\partial K}(\sigma \setminus \{v\}) = B \cup (\{v\} * \partial B)$ is the boundary complex of a polytope. Thus, there is a Bruggesser–Mani *line shelling* of $\mathrm{lk}_{\partial K}(\sigma \setminus \{v\})$ which adds the facets with v last (see [31, §8.2] for details), so first it shells B, a contradiction.

Example 6.3. There exists a large number of shellable $(r-1)$-stacked d-balls with $r \leqslant \frac{1}{2}d$ whose boundary is non-polytopal. Indeed, fixing d, Goodman and Pollack [9] showed that the log of the number of combinatorial types of boundaries of simplicial d-polytopes on n vertices is at most $O(n \log(n))$. On the other hand, the log of the number of Kalai's squeezed $(d-1)$-spheres satisfying $h_{r-1}=h_r$, where $r\leqslant\frac{1}{2}d$, is at least $\Omega(n^{r-2})$ (see [14] for the details). Since Kalai's squeezed spheres satisfying $h_{r-1}=h_r$ are known to be the boundaries of $(r-1)$-stacked shellable balls (see [14] and [17] for details), they give a large number of $(r-1)$-stacked triangulations of a d-ball whose boundary is non-polytopal when $r\geqslant 4$.

Although the answers to Question 6.1 are negative in general, it would be of interest to study these problems for special cases. Below, we write a few open questions on stacked balls and spheres.

Conjecture 6.4. Let P be an $(r-1)$-stacked d-polytope with $r\leqslant\frac{1}{2}(d+1)$.
(i) (McMullen [19]) The $(r-1)$-stacked triangulation of P is *regular*.
(ii) (Bagchi–Datta [1]) The $(r-1)$-stacked triangulation of P is shellable.

Note that Conjecture 6.4 (i) implies Conjecture 6.4 (ii). McMullen's original conjecture considered the case $r\leqslant\frac{1}{2}d$, but we want to include the case $r=\frac{1}{2}(d+1)$, in view of Theorem 2.3. Also, it would be of interest to study the geometric meaning of the triangulation given in Theorem 1.2.

We see that there exists a non-shellable 2-stacked ball whose boundary is non-polytopal in Example 6.2, and that there even exists a shellable 3-stacked ball whose boundary is non-polytopal in Example 6.3. But the following question is open.

Question 6.5. Let Δ be a 2-stacked triangulation of a d-ball which is shellable. Is $\partial\Delta$ polytopal?

Finally, we raise the following question concerning Theorem 1.3.

Question 6.6. With the same notation as in Theorem 1.3, is it true that if Δ is a triangulation of a sphere then $\Delta(d-r)$ is a triangulation of a ball?

It seems to be plausible that if Δ is a PL-sphere then $\Delta(d-r)$ is a PL-ball. But we do not have an answer even for this case.

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