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**Abstract**

Let $A_k, k = 1, \ldots, m$ be $n \times n$ Hermitian matrices and let $f : \mathbb{C}^n \to \mathbb{R}^m$ have components $f^k(x) = x^H A_k x$, $k = 1, \ldots, m$. When $n \geq 3$ and $m = 3$, the set $W(A_1, \ldots, A_m) = \{f(x) : \|x\| = 1\}$ is convex. This property does not hold in general when $m > 3$. These particular cases of known results are proven here using a direct, geometric approach. A geometric characterization of the contact surfaces is obtained for any $n$ and $m$. Necessary conditions are given for $f(x)$ to be on boundary of $W(A_1, \ldots, A_m)$ or on certain subsets of this boundary. These results are of interest in the context of the computation of the structured singular value, a recently introduced tool for the analysis and synthesis of control systems.

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# 1 Introduction

Let $A_k, k = 1, \ldots, m$, be $n \times n$ Hermitian matrices and let $f : \mathbb{C}^n \to \mathbb{R}^m$ have components $f^k(x) = x^H A_k x$, $k = 1, \ldots, m$. The *generalized numerical range of matrices* $A_1, \ldots, A_m$ is the set $W(A_1, \ldots, A_m) = \{f(x) : \|x\| = 1\}$, a subset of $\mathbb{R}^m$ (e.g., [1–3]). It has been long known that, when $m = 2$, this set is always convex [1] and that, when $m = 3$, it still has a convex boundary [1,4]. Here a set is said to *have a convex boundary* if its intersection with each of its support hyperplane is convex [1,2,4]. More recently, it was shown [5–7], as a particular case of a more general result, that the generalized numerical range is still convex when $m = 3$ and $n > 2$, but that this property fails to hold in general if $m > 3$ or $n = 2$. In this note, a direct, geometric proof of convexity is given for the case $m = 3, n > 2$. For $m > 3$ or $n = 2$, a canonical family of examples is exhibited where $W(A_1, \ldots, A_m)$ is not convex. For any $m$ and $n$, a geometric characterization of the intersections of $W(A_1, \ldots, A_m)$ with its supporting hyperplanes is derived. Necessary conditions on $x$ are given for $f(x)$ to be (*i*) on the boundary of $W(A_1, \ldots, A_m)$, (*ii*) on the intersection of this boundary with the boundary of the cone $\hat{W}(A_1, \ldots, A_m)$ it generates and (*iii*) on a certain type of 'corner' of $W(A_1, \ldots, A_m)$. These results are of interest in the context of the computation of the structured singular value, a quantity recently introduced by Doyle [8] as a tool in control system analysis and synthesis (see [9]).

We will make repeated use of the concept of *3D-ellipsoid*, defined as follows.

**Definition 1.** We call *3D-ellipsoid* the image in $\mathbb{R}^m$ of the unit sphere in $\mathbb{R}^3$ under an affine map. A 3D-ellipsoid is *degenerate* if it is entirely contained in a two-dimensional affine set. □

With this definition, a 3D-ellipsoid is a compact set entirely contained in a subspace of $\mathbb{R}^m$ of dimension three (the range of the affine map). It can consist in either the boundary of a nondegenerate ellipsoid, a solid ellipse, a line segment, or a point.

In the sequel, $\partial B$ is the unit sphere in $\mathbb{C}^n$, $\Re$ and $\Im$ indicate the real and imaginary parts and, for any set $S$, co$S$ denotes its convex hull.

## 2 Main Results

The following two propositions hold for any $m$. The first one is a straightforward extension of a result in [8].

**Proposition 1.** If $n = 2$, $W(A_1, \ldots, A_m)$ is a 3D-ellipsoid. The $k$th coordinate of its center is trace$(A_k)/2$.

*Proof.* For $k = 1, \ldots, m$, let

$$A_k = \begin{bmatrix} a_k & b_k \\ \overline{b_k} & c_k \end{bmatrix},$$

where $a_k$, $c_k \in \mathbb{R}$, $b_k \in \mathbb{C}$, and $\overline{b_k}$ is the complex conjugate of $b_k$. The unit sphere in $\mathbb{C}^2$ can be expressed as

$$\left\{ e = \exp(i\phi) \begin{bmatrix} \cos\theta \\ \sin\theta \exp(i\psi) \end{bmatrix} : \theta, \phi, \psi \in \mathbb{R} \right\} \tag{1}$$

where $i$ is the square root of -1. For $e$ as in (1), elementary manipulations give

$$e^H A_k e = \frac{\text{trace}(A_k)}{2} + \begin{bmatrix} \frac{a_k - c_k}{2} & \Re b_k & -\Im b_k \end{bmatrix} \begin{bmatrix} \cos(2\theta) \\ \sin(2\theta)\cos\psi \\ \sin(2\theta)\sin\psi \end{bmatrix}.$$

Since $\left\{ \begin{bmatrix} \cos(2\theta) \\ \sin(2\theta)\cos\psi \\ \sin(2\theta)\sin\psi \end{bmatrix} : \theta, \psi \in \mathbb{R} \right\}$ is the unit sphere in $\mathbb{R}^3$, the claim is proven. $\square$

**Proposition 2.** If $n \geq 3$, $W(A_1, \ldots, A_m)$ is *not* a nondegenerate 3D-ellipsoid.

*Proof.* If $W(A_1, \ldots, A_m)$ is a singleton, the claim holds. Thus suppose it is not, i.e., suppose there exist $y, z' \in \partial B$ and $k_0 \in \{1, \ldots, m\}$ such that

$$y^H A_{k_0} y \neq z'^H A_{k_0} z'. \tag{2}$$

Since $n \geq 3$, there exists an $x \in \partial B$ such that

$$x^H y = x^H z' = 0$$

and, without loss of generality (in view of (2)),

$$x^H A_{k_0} x \neq y^H A_{k_0} y. \tag{3}$$

In view of (2), continuity implies that there exists a $z \in \partial B$ in the subspace of $\mathbb{C}^n$ generated by $y$ and $z'$ such that

$$z^H A_{k_0} z \neq x^H A_{k_0} x$$

and

$$z^H A_{k_0} z \neq y^H A_{k_0} y. \tag{4}$$

Now consider the sets

$$W_y = W([x\ y]^H A_1 [x\ y], \ldots, [x\ y]^H A_m [x\ y])$$

and

$$W_z = W([x\ z]^H A_1 [x\ z], \ldots, [x\ z]^H A_m [x\ z]).$$

Since both $y$ and $z$ are orthogonal to $x$, both $W_y$ and $W_z$ are subsets of $W(A_1, \ldots, A_m)$. By Proposition 1, both are 3D-ellipsoids and their centers have as $k_0$th coordinate respectively $(x^H A_{k_0} x + y^H A_{k_0} y)/2$ and $(x^H A_{k_0} x + z^H A_{k_0} z)/2$, so that, in view of (4), the two sets are distinct. Thus at least one of them is a proper subset of $W(A_1, \ldots, A_m)$. Since the known properties of $y$ and $z$ are identical, there is no loss of generality in assuming that this set is $W_y$. Also, clearly, $W_y$ passes through the two points in $\mathbb{R}^m$ whose $k$th coordinates are $x^H A_k x$ and $y^H A_k y$. Thus, in view of (3), $W_y$ is not a singleton. Since clearly a nondegenerate 3D-ellipsoid cannot have any 3D-ellipsoid but singletons as proper subsets, the proof is complete. $\square$

In proving the next proposition, we will make use of the following lemma, which extends a result in [8]. It holds for any $n$ and $m$.

**Lemma 1.** Given any $u, v_0, v_1 \in W(A_1, \ldots, A_m)$, there exists a point-to-set map $E_{uv_0v_1} : [0,1] \to 2^{\mathbb{R}^m}$, continuous in the Hausdorff topology, such that $u, v_0 \in E_{uv_0v_1}(0)$ and $u, v_1 \in E_{uv_0v_1}(1)$ and such that for all $t \in [0,1]$, $E_{uv_0v_1}(t)$ is a 3D-ellipsoid contained in $W(A_1, \ldots, A_m)$.

*Proof.* First, suppose that $v_0 \neq u \neq v_1$, and let $x, y_0, y_1 \in \partial B$ be unit vectors such that, for $k = 1, \ldots, m$, $u^k = x^H A_k x$, $v_0^k = y_0^H A_k y_0$, $v_1^k = y_1^H A_k y_1$. Clearly, $\{x, y_0\}$ and $\{x, y_1\}$ are both linearly independent over $\mathbb{C}$ and the vectors $y_0'$ and $y_1'$, given by

$$y_0' = \frac{1}{||y_0 - (x^H y_0)x||}(y_0 - (x^H y_0)x)$$

and

$$y_1' = \frac{1}{||y_1 - (x^H y_1)x||}(y_1 - (x^H y_1)x)$$

are both orthogonal to $x$ and have unit length. Let $y : [0,1] \to \partial B$ be any continuous map such that $y(0) = y_0'$ and $y(1) = y_1'$ and such that, for all

$t \in [0,1]$, $y(t)$ belongs to the subspace of $\mathbb{C}^n$ generated by $y'_0$ and $y'_1$. Next, for $k = 1, \ldots, m$, let $B_k : [0,1] \to \mathbb{C}^{2\times 2}$ be the continuous map defined by

$$B_k(t) = [\ x \quad y(t)\ ]^H A_k [\ x \quad y(t)\ ] \quad \forall t \in [0,1].$$

Proposition 1 implies that, for each $t \in [0,1]$, $W(B_1(t), \ldots, B_m(t))$ is a 3D-ellipsoid, say $E_{uv_0v_1}(t)$. It is easily checked that $E_{uv_0v_1}$ satisfies the required conditions. Finally, if $u = v_0$ (resp. $u = v_1$), pick $E_{uv_0v_1}$ to be the constant map whose value is any 3D-ellipsoid contained in $W(A_1, \ldots, A_m)$ and passing through $u$ and $v_1$ (resp. $u$ and $v_0$). $\square$

**Proposition 3.** If $n \geq 3$, $W(A_1, A_2, A_3)$ is convex.

*Proof.* Let $u, v \in W(A_1, A_2, A_3)$ and let $E \subset W(A_1, A_2, A_3)$ be a 3D-ellipsoid passing through $u$ and $v$ (see Lemma 1). We will show that the convex hull of $E$, denoted by $coE$, is contained in $W(A_1, A_2, A_3)$, thus proving convexity. If $E$ is degenerate, the result is clear. Thus assume $E$ is nondegenerate. In view of Proposition 2, $E$ must be a proper subset of $W(A_1, A_2, A_3)$. Thus let $\hat{w} \in W(A_1, A_2, A_3)$, $\hat{w} \notin E$, and let $w$ be any point in $coE$. We prove that $w \in W(A_1, A_2, A_3)$. If $w = \hat{w}$, the claims holds. Thus suppose that $w \neq \hat{w}$. Let $w_0$ and $w_1$ be the intersection points with $E$ of the straight line through $w$ and $\hat{w}$ and without loss of generality suppose that $w$ lies between $\hat{w}$ and $w_0$. Let $E_{\hat{w}w_0w_1} : [0,1] \to 2^{\mathbb{R}^m}$ be as specified by Lemma 1. Clearly $w \in \text{co}E_{\hat{w}w_0w_1}(0)$ and $w \notin \text{co}E_{\hat{w}w_0w_1}(1)$. Since $E_{\hat{w}w_0w_1}$ is a continuous map, there must exist a $t \in [0,1]$ such that $w \in E_{\hat{w}w_0w_1}(t)$. Thus $w \in W(A_1, A_2, A_3)$. $\square$

A canonical family of examples is easily constructed, showing that Proposition 3 cannot be extended to the case of more than three matrices. More precisely, for any $m \geq 4$, $n \geq 2$, one can find matrices $A_1, \ldots, A_m$ such that $W(A_1, \ldots, A_m)$ does not have a convex boundary (and thus is not convex). The construction is as follows. For $k = 1, \ldots, m-1$, let $B_k \in \mathbb{C}^{2\times 2}$ be Hermitian matrices such that $W(B_1, \ldots, B_{m-1})$ is a nondegenerate 3D-ellipsoid (see Proposition 1). Then, for $k = 1, \ldots, m-1$, let $A_k \in \mathbb{C}^{n\times n}$ be Hermitian matrices such that $A_k$ has $B_k$ as its top left corner and let $A_m = \mathrm{diag}(\sigma_1, \ldots, \sigma_m)$ with $\sigma_1 = \sigma_2 > \sigma_3 \geq \ldots \geq \sigma_m$. It is easily checked that the intersection of $W(A_1, \ldots, A_m)$ with its supporting hyperplane $\{u \in \mathbb{R}^m : u^m = \sigma_1\}$ is an $\mathbb{R}^m$-imbedding of $W(B_1, \ldots, B_{m-1})$, which is not convex.

Using the construction just described, the following proposition can be easily proved.

**Proposition 4.** The intersection of $W(A_1, \ldots, A_m)$ with any of its supporting hyperplanes is an $\mathbb{R}^m$-imbedding of the generalized numerical range of some matrices. $\square$

It is easy to see that, for any $m$ and $n$, points $f(x)$ on the intersection of $W(A_1, \ldots, A_m)$ with any supporting hyperplane are characterized by the fact that the corresponding $x$ is an eigenvector to the smallest eigenvalue of $\sum_{k=1}^m w^k A_k$ where the $w^k$'s are the components of a vector orthogonal to the hyperplane, pointing toward $W(A_1, \ldots, A_m)$. This fact is used by Doyle to construct the projection of the origin on $W(A_1, \ldots, A_m)$ when $W(A_1, \ldots, A_m)$ is convex ([8], see also [10]). Below, we derive properties of *any* point on the boundary of $W(A_1, \ldots, A_m)$ as well as properties of points on certain subsets of this boundary.

**Proposition 5.** If $x \in \partial B$ is such that $f(x)$ is on the boundary of $W(A_1, \ldots, A_m)$ then there exists a direction $w \in \mathbb{R}^m$ such that $x$ is an eigenvector of $\sum_{k=1}^m w^k A_k$. Moreover (*i*) if $\mathcal{H}$ is any supporting hyperplane to

$W(A_1, \ldots, A_m)$ at $f(x)$, then the direction orthogonal to $\mathcal{H}$ is a valid choice for $w$. (*ii*) if $f(x)$ is on the boundary of any cone containing $W(A_1, \ldots, A_m)$ (or, equivalently, of the cone generated by $W(A_1, \ldots, A_m)$), then $w$ can be chosen in such a way that

$$\sum_{k=1}^{m} w^k A_k x = 0.$$

(*iii*) if there exists no subset of $W(A_1, \ldots, A_m)$ containing $f(x)$ that is locally homeomorphic to $\mathbb{R}^{m-(q-1)}$, $1 \le q \le m$, around $f(x)$, then there is a $q$-dimensional subspace $\mathcal{S}$ of $\mathcal{V} = \{A \in \mathbb{C}^{n \times n} : A = \sum_{k=1}^{m} w^k A_k, w^i \in \mathbb{R}\}$ such that all matrices in $\mathcal{S}$ admit $x$ as an eigenvector.

*Proof.* Suppose that $x \in \partial B$ is such that $f(x)$ is on the boundary of $W(A_1, \ldots, A_m)$. Let

$$\partial B_x = \{z \in \partial B \mid x^H z = 0\}$$

and, for $k = 1, \ldots, m$, let $y_k$ be given by

$$y_k = A_k x - (x^H A_k x)x. \tag{5}$$

Clearly, for any $z \in \partial B_x$,

$$y_k^H z = x^H A_k z. \tag{6}$$

Next, for any $\theta \in \mathbb{R}$, $z \in \partial B_x$ define

$$\begin{aligned} f_x(\theta, z) &= f(\cos\theta x + \sin\theta z) \\ &= \cos^2\theta f(x) + \sin^2\theta f(z) + 2\cos\theta\sin\theta\,\Re \begin{bmatrix} x^H A_1 z \\ \vdots \\ x^H A_m z \end{bmatrix}. \end{aligned}$$

In view of (6), we can write

$$\frac{\partial f_x(0,z)}{\partial \theta} = 2\Re \begin{bmatrix} x^H A_1 z \\ \vdots \\ x^H A_m z \end{bmatrix} = 2M \begin{bmatrix} \Re z \\ -\Im z \end{bmatrix}$$

where

$$M = \begin{bmatrix} \Re y_1^T & \Im y_1^T \\ \vdots & \vdots \\ \Re y_m^T & \Im y_m^T \end{bmatrix}.$$

Let

$$F = \left\{ \frac{\partial f_x(0,z)}{\partial \theta} \mid z \in \partial B_x \right\}.$$

Since for all $k$, $y_k \in \partial B_x$, the ellipsoid $G$ given by

$$G = \left\{ 2M \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix} M^T b \mid \|M^T b\| = 1,\ b \in \mathbb{R}^m \right\}$$

is a subset of $F$. Clearly, since $f(x)$ is on the boundary of $W(A_1, \ldots, A_m)$, $F$ cannot contain any neighborhood of the origin, so that $G$ must be contained in an $m-1$ dimensional subspace of $\mathbb{R}^m$. This implies that $M$ is singular, i.e., $\sum_{k=1}^m w^k y_k = 0$ for some nonzero $w \in \mathbb{R}^m$. Thus it follows from (5) that $x$ is an eigenvector of $\sum_{k=1}^m w^k A_k$ as claimed. The corresponding eigenvalue is $x^H(\sum_{k=1}^m w^k A_k)x$. If $\mathcal{H}$ is any hyperplane supporting $W(A_1, \ldots, A_m)$ at $f(x)$, then $G$ must be contained in $\mathcal{H}$ and $(i)$ easily follows. Consider now, the cone $C$ generated by the ellipsoid $f(x) + G$ and suppose that $f(x)$ is on the boundary of a cone containing $W(A_1, \ldots, A_m)$. Clearly, since $G \subset F$, the ray $r = \{\alpha f(x) : \alpha > 0\}$ cannot be an interior ray of co$C$. Since $r$ passes through the center of every section of $C$, it implies that the interior of co$C$

is empty. Thus, $C$ must be entirely contained in a hyperplane $\mathcal{H}$ passing through the origin. Since $r$ belongs to $\mathcal{H}$, it follows that

$$\sum_{k=1}^{m} w^k f^k(x) = 0 ,$$

i.e.,

$$x^H (\sum_{k=1}^{m} w^k A_k) x = 0$$

for any $w$ normal to $\mathcal{H}$. Claim $(ii)$ follows. Finally, if no subset of $W(A_1, \ldots, A_m)$ containing $f(x)$ is homeomorphic to $\mathbb{R}^{m-(q-1)}$, $1 \leq q \leq m$, around $f(x)$, $G$ must be contained in subspace $\mathcal{T}$ of dimension $m - q$. The subspace $\mathcal{S} = \{A \in \mathcal{V} : A = \sum_{k=1}^{m} w^k A_k,\ w \perp \mathcal{T}\}$ satisfies claim $(iii)$. □

**Corollary.** If $W(A_1, A_2)$ is nonsmooth at a boundary point $f(x)$, then $x$ is an eigenvector of both $A_1$ and $A_2$. □

The well-known fact that such $x$ is an eigenvector of $A_1 + jA_2$ [3,11] is a direct consequence of this corollary.

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