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ON THE GENERALIZED ORDER-k FIBONACCI AND LUCAS NUMBERS

EMRAH KILIÇ and DURSUN TAŞCI

ABSTRACT. In this paper we consider the generalized order-k Fibonacci and Lucas numbers. We give the generalized Binet formula, combinatorial representation and some relations involving the generalized order-k Fibonacci and Lucas numbers.

1. Introduction. We consider the generalized *order-k* Fibonacci and Lucas numbers. In [**1**] Er defined k sequences of the generalized *order-k* Fibonacci numbers as shown:

$$g_n^i = \sum_{j=1}^{k} g_{n-j}^i, \quad \text{for} \quad n > 0 \quad \text{and} \quad 1 \le i \le k,$$

with boundary conditions for $1 - k \le n \le 0$,

$$g_n^i = \begin{cases} 1 & \text{if } i = 1 - n, \\ 0 & \text{otherwise,} \end{cases}$$

where g_n^i is the nth term of the ith sequence. For example, if $k = 2$, then $\{g_n^2\}$ is the usual Fibonacci sequence, $\{F_n\}$, and, if $k = 4$, then the fourth sequence of the generalized *order*-4 Fibonacci numbers is

$$1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, \dots .$$

In [**6**] the authors defined k sequences of the generalized *order-k* Lucas numbers as shown:

$$l_n^i = \sum_{j=1}^{k} l_{n-j}^i, \quad \text{for} \quad n > 0 \quad \text{and} \quad 1 \le i \le k,$$

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with boundary conditions for $1-k \le n \le 0$,

$$l_n^i = \begin{cases} -1 & \text{if } i = 1-n, \\ 2 & \text{if } i = 2-n, \\ 0 & \text{otherwise,} \end{cases}$$

where l_n^i is the nth term of the ith sequence. For example, if $k=2$, then $\{l_n^2\}$ is the usual Lucas sequence, $\{L_n\}$, and, if $k=4$, then the fourth sequence of the generalized *order*-4 Lucas numbers is

$$1, 3, 4, 8, 16, 31, 59, 114, 220, 424, 817, 1575, 30636, \dots .$$

Also, Er showed that

$$\begin{bmatrix} g_{n+1}^i \\ g_n^i \\ \vdots \\ g_{n-k+2}^i \end{bmatrix} = A \begin{bmatrix} g_n^i \\ g_{n-1}^i \\ \vdots \\ g_{n-k+1}^i \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix} \tag{1.1}$$

is a $k \times k$ companion matrix. Then he derived

$$G_{n+1} = AG_n,$$

where

$$G_n = \begin{bmatrix} g_n^1 & g_n^2 & \cdots & g_n^k \\ g_{n-1}^1 & g_{n-1}^2 & \cdots & g_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ g_{n-k+1}^1 & g_{n-k+1}^2 & \cdots & g_{n-k+1}^k \end{bmatrix}. \tag{1.2}$$

Moreover, Er showed that $G_1 = A$ and $G_n = A^n$. The matrix A is said to be the generalized *order-k* Fibonacci matrix. Furthermore, in [**3**], recently Karaduman proved that

$$\det G_n = \begin{cases} (-1)^n & \text{if } k \text{ is even,} \\ 1 & \text{if } k \text{ is odd.} \end{cases}$$

In [**5**], we showed

$$\begin{bmatrix} l_{n+1}^i \\ l_n^i \\ \vdots \\ l_{n-k+2}^i \end{bmatrix} = A \begin{bmatrix} l_n^i \\ l_{n-1}^i \\ \vdots \\ l_{n-k+1}^i \end{bmatrix}$$

and derived

$$H_{n+1} = AH_n$$

where

$$H_n = \begin{bmatrix} l_n^1 & l_n^2 & \dots & l_n^k \\ l_{n-1}^1 & l_{n-1}^2 & \dots & l_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ l_{n-k+1}^1 & l_{n-k+1}^2 & \dots & l_{n-k+1}^k \end{bmatrix}$$

also

$$H_1 = AK$$

where

$$K = \begin{bmatrix} -1 & 2 & 0 & 0 & \dots & 0 \\ 0 & -1 & 2 & 0 & \dots & 0 \\ 0 & 0 & -1 & 2 & \dots & 0 \\ 0 & 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 \end{bmatrix}.$$

Further, we proved

$$\det H_{n+1} = \begin{cases} -1 & \text{if } k \text{ is odd,} \\ (-1)^{n+1} & \text{if } k \text{ is even,} \end{cases}$$

and showed that

$$H_n = G_n K, \tag{1.3}$$

which is a well-known fact for $k = 2$, see [**6**].

2. Some relations involving the generalized *order-k* Fibonacci and Lucas numbers. In this section we present and extend some relationships between the generalized *order-k* Fibonacci and Lucas numbers by matrix methods. From [**1**], it is well known that, for all positive integers n, m and $1 \le i \le k$,

$$g_{m+n}^{i} = \sum_{j=1}^{k} g_n^j \, g_{m+1-j}^{i},$$

where g_n^i is the generalized order-k Fibonacci number.

We note that, for example, if $k = 2$, then $\{g_n^2\}$ is the usual Fibonacci sequence. For all n, $m \in Z^+$,

$$\begin{aligned} g_{m+n}^{2} &= \sum_{j=1}^{2} g_n^j \, g_{m+1-j}^{2} \\ &= g_n^1 \, g_m^2 + g_n^2 \, g_{m-1}^2 \end{aligned}$$

and, since $g_m^1 = g_{m+1}^2$ for $i = 1$, $k = 2$ and all $m \in Z^+$, we write

$$g_{m+n}^2 = g_{n+1}^2 \, g_m^2 + g_n^2 \, g_{m-1}^2.$$

Indeed, we generalize the following relation between Fibonacci numbers

$$F_{n+m} = F_{n+1}F_m + F_n F_{m-1},$$

see [**7**, p. 176].

Theorem 1. *Let l_n^i be the generalized* order-k *Lucas number. For all positive integers n, m and $1 \le i \le k$,*

$$l_{n+m}^{i} = \sum_{j=1}^{k} g_n^j \, l_{m+1-j}^{i}.$$

Proof. From [**6**], we know that $H_n = G_n K$, so we can write that

$$H_{n+m} = G_{n+m}K = A^{n+m}K = A^n A^m K = A^n H_m = G_n H_m$$

or

$$H_{n+m} = G_m \, H_n.$$

Since $H_{n+m} = G_n \, H_m$, $l^i_{n+m} = (H_{n+m})_{1,i}$,

$$\begin{aligned} l^i_{n+m} &= g^1_n \, l^i_m + g^2_n \, l^i_{m-1} + \cdots + g^k_n \, l^i_{m-k+1} \\ &= g^1_n \, l^i_m + (g^2_{n-1} + g^2_{n-2}) \, l^i_{m-1} + \cdots \\ &\quad + (g^k_{n-1} + g^k_{n-2} + \cdots + g^k_{n-k}) \, l^i_{m-k+1}. \end{aligned}$$

Thus, we obtain

$$l^i_{n+m} = \sum_{j=1}^{k} g^j_n \, l^i_{m+1-j},$$

so the proof is completed. □

For example, if $k = 2$, then g^2_n and l^2_n are the usual Fibonacci and Lucas number, respectively; then

$$\begin{aligned} l^2_{n+m} &= \sum_{j=1}^{2} g^j_n \, l^2_{m+1-j} \\ &= g^1_n \, l^2_m + g^2_n \, l^2_{m-1}, \end{aligned}$$

and, since $g^1_n = g^2_{n+1}$ for $i = 1$, $k = 2$, and all $n \in Z^+$, we write

$$l^2_{n+m} = g^2_{n+1} \, l^2_m + g^2_n \, l^2_{m-1}.$$

Indeed, we generalize the following relation involving Lucas and Fibonacci numbers,

$$\begin{aligned} L_{n+m} &= F_{n+1} \, L_m + F_n \, L_{m-1} \\ &= (F_n + F_{n-1}) \, L_m + F_n \, L_{m-1} \\ &= F_n(L_m + L_{m-1}) + F_{n-1} \, L_m \\ &= F_n \, L_{m+1} + F_{n-1} \, L_m, \end{aligned}$$

see [**7**, p. 176].

Note that $G_{n+m+p} = G_n G_{m+p}$, $g^i_{n+m} = (G_{n+m+p})_{p-1,i}$.

Then we have the following corollary.

Corollary 1. *Let* g_n^i *be the generalized* order-k *Fibonacci number. For all* $n, m, p \in \mathbf{Z}^+$ *and* $1 \le i \le k$,

$$g_{n+m+p}^i = \sum_{j=1}^{k} g_n^j \, g_{m+1-p-j}^i.$$

Also we note that $G_{n-p} G_{m+p} = G_{n+m}$, $g_{n+m}^i = (G_{n+m})_{1,i}$. Then we have the following corollary.

Corollary 2. *Let* g_n^i *be the generalized* order-k *Fibonacci number. Then, for* $n, m, p \in \mathbf{Z}^+$ *and* $1 \le i \le k$,

$$g_{m+n}^i = \sum_{j=1}^{k} g_{n-p}^j \, g_{m+p-j}^i.$$

In [**4**] Levesque gave a Binet formula for the Fibonacci sequence. In this paper, we derive a generalized Binet formula for the generalized *order*-k Fibonacci and Lucas sequence by using the determinant.

3. Generalized Binet formula. Let $f(\lambda)$ be the characteristic polynomial of the generalized order-k Fibonacci matrix A. Then $f(\lambda) = \lambda^k - \lambda^{k-1} - \cdots - \lambda - 1$, which is a well-known fact. Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be the eigenvalues of A. In [**5**], Miles also showed that $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct. Let V be a $k \times k$ Vandermonde matrix as follows:

$$V = \begin{bmatrix} \lambda_1^{k-1} & \lambda_2^{k-1} & \lambda_3^{k-1} & \dots & \lambda_k^{k-1} \\ \lambda_1^{k-2} & \lambda_2^{k-2} & \lambda_3^{k-2} & \dots & \lambda_k^{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1 & \lambda_2 & \lambda_3 & \dots & \lambda_k \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix}.$$

Let

$$d_k^i = \begin{bmatrix} \lambda_1^{n+k-i} \\ \lambda_2^{n+k-i} \\ \vdots \\ \lambda_k^{n+k-i} \end{bmatrix}$$

and $V_j^{(i)}$ be a $k \times k$ matrix obtained from V by replacing the jth column of V by d_k^i. Then we have the generalized Binet formula for the generalized order-k Fibonacci numbers with the following theorem.

Theorem 2. *Let g_n^i be the generalized* order-k *Fibonacci number, for $1 \le i \le k$. Then*

$$t_{ij} = \frac{\det(V_j^{(i)})}{\det(V)}$$

where $G_n = [t_{ij}]_{k\times k}$.

Proof. Since the eigenvalues of A are distinct, A is diagonalizable. It is easy to show that $AV = VD$, where $D = \mathrm{diag}\,(\lambda_1, \lambda_2, \dots, \lambda_k)$. Since V is invertible, $V^{-1}AV = D$. Hence, A is similar to D. So we have $A^nV = VD^n$. In [**1**], it is known that $G_n = A^n$. So we can write that $G_nV = VD^n$. Let $G_n = [t_{ij}]_{k\times k}$. Then we have the following linear system of equations:

$$\begin{aligned} t_{i1}\lambda_1^{k-1} + t_{i2}\lambda_1^{k-2} + \cdots + t_{ik} &= \lambda_1^{n+k-i} \\ t_{i1}\lambda_2^{k-1} + t_{i2}\lambda_2^{k-2} + \cdots + t_{ik} &= \lambda_2^{n+k-i} \\ &\vdots \\ t_{i1}\lambda_k^{k-1} + t_{i2}\lambda_k^{k-2} + \cdots + t_{ik} &= \lambda_k^{n+k-i}. \end{aligned}$$

And, for each $j = 1, 2, \dots, k$, we get

$$t_{ij} = \frac{\det(V_j^{(i)})}{\det(V)}. \tag{3.1}$$

So the proof is complete. □

Corollary 3. *Let g_n^k be the generalized* order-k *Fibonacci number, then*

$$g_n^k = t_{1k} = \frac{\det(V_k^{(1)})}{\det(V)}.$$

Proof. If we take $i = 1$ and $j = k$, then $t_{1,k} = g_n^k$. Also, employing Theorem 2, the proof is immediately seen. □

Now we are going to give a generalized Binet formula for the generalized *order-k* Lucas sequence. Firstly, we give a lemma for a relationship between the generalized *order-k* Fibonacci and Lucas numbers.

Lemma 1. *Let l_n^k and g_n^k be the generalized* order-k *Lucas and Fibonacci numbers, respectively. Then, for $k \geq 2$,*

$$l_n^k = g_n^k + 2g_{n-1}^k.$$

Proof. We will use the induction method to prove that $l_n^k = g_n^k + 2g_{n-1}^k$. From the definition of the generalized *order-k* Lucas and Fibonacci numbers, we know that, for all $k \in \mathbf{Z}^+$ with $k \geq 2$, $l_1^k = g_1^k = 1$ and $g_0^k = 0$. Then, it is true for $n = 1$, i.e.,

$$l_1^k = g_1^k + 2g_0^k = 1.$$

Suppose that the equation holds for n. So we have

$$l_n^k = g_n^k + 2g_{n-1}^k.$$

Now we show that the equation is true for $n+1$. From the definition of l_n^k, we have

$$l_{n+1}^k = l_n^k + l_{n-1}^k + \cdots + l_{n-k+1}^k. \tag{3.2}$$

Since $l_n^k = g_n^k + 2g_{n-1}^k$, we can write equation (3.2) as follows

$$\begin{aligned} l_{n+1}^k &= g_n^k + 2g_{n-1}^k + (g_{n-1}^k + 2g_{n-2}^k) + \cdots + (g_{n-k+1}^k + 2g_{n-k}^k) \\ &= (g_n^k + g_{n-1}^k + \cdots + g_{n-k+1}^k) + 2(g_{n-1}^k + g_{n-2}^k + \cdots + g_{n-k}^k) \\ &= g_{n+1}^k + 2g_n^k. \end{aligned}$$

So the equation holds for $n+1$. Thus the proof is complete. □

For example, if $k = 2$, $\{l_n^2\}$ is the usual Lucas sequence, $\{L_n\}$, and $\{g_n^2\}$ is the usual Fibonacci sequence, $\{F_n\}$, then it is a well-known fact that

$$L_n = F_n + 2F_{n-1}.$$

If $k = 4$ and $n = 9$, then

$$l_9^4 = g_9^4 + 2g_8^4 = 108 + 2.56 = 220.$$

Then we have the following theorem.

Theorem 3. *Let l_n^k be the generalized* order-k *Lucas number. Then, for $k \geq 2$,*

$$l_n^k = \frac{\det(V_k^{(1)}) + 2\det(V_k^{(2)})}{\det(V)}.$$

Proof. From Lemma 1, we know that $l_n^k = g_n^k + 2g_{n-1}^k$. Also, by Theorem 2, we have $G_n = [t_{ij}]_{k\times k}$ and $t_{ij} = \det(V_j^{(i)})/\det(V)$. If we take $i = 2$, $j = k$ in equation (3.1), and since $t_{2k} = g_{n-1}^k$, we write

$$\begin{aligned} l_n^k &= g_n^k + 2g_{n-1}^k \\ &= \frac{\det(V_k^{(1)}) + 2\det(V_k^{(2)})}{\det(V)}. \end{aligned}$$

So the proof is complete. □

For a further generalization of Theorem 3, we give a lemma which is immediately seen from equation (1.3).

Lemma 2. *Let l_n^i and g_n^i be the generalized order-k Lucas and Fibonacci numbers for $1 \leq i \leq k$. Then*

$$l_n^i = 2g_n^{i-1} - g_n^i$$

for $1 \leq i \leq k$.

Then we obtain the following theorem.

Theorem 4. *Let l_n^i be the generalized order-k Lucas number, for $1 \le i \le k$. Then*

$$l_n^i = \frac{2 \det V_{i-1}^{(1)} + \det V_i^{(1)}}{\det V}.$$

Proof. From $G_n = [t_{ij}] = [g_{n-i+1}^j]$ and Lemma 2, we obtain for $1 \le i \le k$,

$$l_n^i = 2t_{1,i-1} - t_{1,i},$$

and by using Theorem 2, the proof is immediately seen. □

4. Combinatorial representation of the generalized order-k Fibonacci and Lucas numbers. In this section, we consider a combinatorial representation of l_n^k and g_n^k for $n > 0$. Recall that A is the $k \times k$ $(0,1)$ matrix given by (1.1):

$$A = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

Also we know from [**1**] that G_n ia as in (1.2):

$$G_n = [t_{ij}] = \begin{bmatrix} g_n^1 & g_n^2 & \cdots & g_n^k \\ g_{n-1}^1 & g_{n-1}^2 & \cdots & g_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ g_{n-k+1}^1 & g_{n-k+1}^2 & \cdots & g_{n-k+1}^k \end{bmatrix}.$$

Lemma 3 [**4**].

$$t_{ij} = \sum_{(m_1,\dots,m_k)} \frac{m_j + m_{j+1} + \cdots + m_k}{m_1 + m_2 + \cdots + m_k} \times \binom{m_1 + m_2 + \cdots + m_k}{m_1, m_2, \dots, m_k}$$

where the summation is over nonnegative integers satisfying $m_1 + 2m_2 + \cdots + km_k = n - i + j$ and defined to be 1 if $n = i - j$.

Corollary 4. *Let* g_n^k *be the generalized* order-k *Fibonacci number. Then*

$$g_n^k = \sum_{(m_1,\dots,m_k)} \frac{m_k}{m_1+m_2+\cdots+m_k} \times \binom{m_1+m_2+\cdots+m_k}{m_1,m_2,\dots,m_k},$$

where the summation is over nonnegative integers satisfying $m_1+2m_2+\cdots+km_k = n-1+k$.

Proof. In Lemma 2, if we take $i=1$ and $j=k$, then the conclusion can be directly from (1.2). □

Corollary 5. *Let* g_n^k *be the generalized* order-k *Fibonacci number. Then*

$$g_{n-1}^k = \sum_{(d_1,\dots,d_k)} \frac{d_k}{d_1+d_2+\cdots+d_k} \times \binom{d_1+d_2+\cdots+d_k}{d_1,d_2,\dots,d_k}$$

where the summation is over nonnegative integers satisfying $d_1+2d_2+\cdots+kd_k = n-2+k$.

Proof. In Lemma 3, if we take $i=2$ and $j=k$, then the conclusion follows directly from (1.2). □

From Corollaries 4 and 5 and Lemma 1, we have the following corollary.

Corollary 6. *Let* l_n^k *be the generalized* order-k *Lucas number. Then*

$$l_n^k = \sum_{(m_1,\dots,m_k)} \frac{m_k}{m_1+m_2+\cdots+m_k} \times \binom{m_1+m_2+\cdots+m_k}{m_1,m_2,\dots,m_k} + 2\sum_{(d_1,\dots,d_k)} \frac{d_k}{d_1+d_2+\cdots+d_k} \times \binom{d_1+d_2+\cdots+d_k}{d_1,d_2,\dots,d_k}$$

where the summation is over nonnegative integers satisfying $m_1+2m_2+\cdots+km_k = n-1+k$ *and* $d_1+2d_2+\ldots+kd_k = n-2+k$.

Proof. From Lemma 1, we know that $l_n^k = g_n^k + 2g_{n-1}^k$ and if $i = 1$ and $j = k$ and, $i = 2$ and $j = k$, in Lemma 3, respectively, then the conclusion can be derived directly from (1.2). □

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Department of Mathematics, TOBB University of Economics and Technology, TR-06560 Södütözü, Ankara, Turkey
E-mail address: ekilic@etu.edu.tr

Gazi University, Department of Mathematics, 06500 Tekniokullar, Ankara, Turkey
E-mail address: dtasci@gazi.edu.tr