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ON THE GENERALIZED RIESZ B-DIFFERENCE SEQUENCE SPACES

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Abstract

In this paper, we define the new generalized Riesz B-difference sequence spaces $r_{\infty}^{q}(p,B), r_{c}^{q}(p,B)$, $r_{0}^{q}(p,B)$ and $r^{q}(p,B)$ which consist of the sequences whose $R^{q}B$-transforms are in the linear spaces $l_{\infty}(p), c(p)$, $c_0(p)$ and $l(p)$, respectively, introduced by I.J.Maddox[8],[9]. We give some topological properties and compute the $\alpha-,\beta-$ and $\gamma-$duals of these spaces. Also we determine the neccesary and sufficient conditions on the matrix transformations from these spaces into l_{∞} and c.

1 Introduction

By w, we denote the space of all real valued sequences. Any vector subspace of w is called as a sequence space. We write l_{∞}, c, c_0 for the sequence spaces of all bounded, convergent and null sequences, respectively. Also by, bs, cs, l_1 and l_p we denote the spaces of all bounded, convergent, absolutely and p-absolutely convergent series, respectively; where $1<p<\infty$.

A linear topological space X over the real field R is said to be a paranormed space if there is a subadditive function $g: X\to R$ such that $g(\theta)=0, g(x)=g(-x)$ and scalar multiplication is continuous, i.e.,$|\alpha_n-\alpha|\to 0$ and $g(x_n-x)\to 0$ imply $g(\alpha_n x_n-\alpha x)\to 0$ for all α's in R and all x's in X, where θ is the zero vector in the linear space X. Assume here and after that $p=(p_k)$ be a bounded sequence of strictly positive real numbers with $\sup p_k=H$ and $M=\max\{1,H\}$. Then the linear spaces $l_{\infty}(p), c(p), c_0(p)$ and $l(p)$ were defined by Maddox [8],[9] .

For simplicity notation, here and in what follows, the summation without limits runs from 0 to ∞. We assume throughout $(p_k)^{-1}+\left(p_k^{'}\right)^{-1}=1$ provided $1<$

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$\inf p_k \leq H < \infty$ and denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$, where $\mathbb{N} = \{0, 1, 2, ...\}$.

For the sequence spaces λ and μ, define the set $S(\lambda, \mu)$ by

$$S(\lambda, \mu) = \{z = (z_k) \in w : xz = (x_k z_k) \in \mu \text{ for all } x \in \lambda\} \tag{1.1}$$

With the notation (1.1), the $\alpha-, \beta-, \gamma-$duals of a sequence space λ, which are respectively denoted by λ^α, λ^β and λ^γ, are defined by

$$\lambda^\alpha = S(\lambda, l_1) \ , \ \lambda^\beta = S(\lambda, cs) \text{ and } \lambda^\gamma = S(\lambda, bs).$$

If a sequence space λ paranormed by h contains a sequence (b_n) with the property that for every $x \in \lambda$ there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} h\left(x - \sum_{k=0}^{n} \alpha_k b_k\right) = 0$$

then (b_n) is called a Schauder basis (or briefly basis) for λ. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as $x = \sum \alpha_k b_k$.

Let λ and μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ and we denote it by writing $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the $A-$transform of x, is in μ; where

$$(Ax)_n = \sum_k a_{nk} x_k \quad (n \in \mathbb{N}). \tag{1.2}$$

By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right side of (1.2) converges for each n$\in \mathbb{N}$ and every x$\in \lambda$ and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \mu$ for all x$\in \lambda$. A sequence x is said to be $A-$summable to α if Ax converges to α which is called as the $A-$limit of x.

The matrix domain λ_A of an infinite matrix A in sequence space λ is defined by

$$\lambda_A = \{x = (x_k) \in w : Ax \in \lambda\} \tag{1.3}$$

which is a sequence space. In the most cases, the new sequence space λ_A generated by the limitation matrix A from a sequence space λ is the expansion or the contraction of the original space λ.

Let (q_k) be a sequence of positive numbers and

$$Q_n = \sum_{k=0}^{n} q_k \ , \quad (n \in \mathbb{N}).$$

Then the matrix $R^q = (r_{nk}^q)$ of the Riesz mean is given by

$$r_{nk}^q = \begin{cases} \frac{q_k}{Q_n} & , \quad (0 \leq k \leq n) \\ 0 & , \quad (k > n) \end{cases}$$

The Riesz sequence space introduced in [1] is ;

$$r^q(p)=\left\{x=(x_k)\in w:\sum_k\left|\frac{1}{Q_k}\sum_{j=0}^{k}q_jx_j\right|^{p_k}<\infty\right\};\quad\text{with}\quad(0<p_k\leq H<\infty)$$

which is sequence space of the R^q-transform of x are in $l(p)$. Recently, Başarır and Öztürk [11] defined the Riesz difference sequence space $r^q(p,\triangle)$ which consist of the sequences whose $\triangle$-transforms are in the linear space $r^q(p)$, where $\triangle$ denotes the matrix $\triangle=(\triangle_{nk})$ defined by

$$\triangle_{nk}=\begin{cases}(-1)^{n-k} & , \quad (n-1\leq k\leq n),\\ 0 & , \quad (k<n-1)\text{ or }(k>n)\end{cases}.$$

Altay and Başar [3] introduced the generalized difference matrix $B=(b_{nk})$ by

$$b_{nk}=\begin{cases}r & , \quad (k=n)\\ s & , \quad (k=n-1)\\ 0 & , \quad (0\leq k<n-1)\text{ or }(k>n)\end{cases}$$

for all $k,n\in\mathbb{N}$, $r,s\in\mathbb{R}-\{0\}$. The matrix B can be reduced the difference matrix $\triangle$ in case $r=1,s=-1$. The results related to the matrix domain of the matrix B are more general and more comprehensive than the corresponding consequences of matrix domain of $\triangle$, and include them [11],[6].

Then main purpose of this paper is to introduce the Riesz $B-$difference sequence spaces $r^q_\infty(p,B)$, $r^q_c(p,B)$, $r^q_0(p,B)$ and $r^q(p,B)$ and to investigate some topological properties.

2 The Riesz B-Difference Sequence Spaces

Let define the sequence $y=\{y_k(q)\}$, which is used , as the $(R^qB)-$transform of a sequence $x=(x_k)$, i.e.,

$$y_k(q)=\frac{1}{Q_k}\left[\sum_{j=0}^{k-1}(q_j.r+q_{j+1}.s)\,x_j+q_k.r.x_k\right]\quad(k\in\mathbb{N}).\tag{2.1}$$

We define the Riesz $B-$difference sequence spaces $r^q_\infty(p,B)$, $r^q_c(p,B)$, $r^q_0(p,B)$ and $r^q(p,B)$ by

$$r^q_\infty(p,B)=\{x=(x_j)\in w:y_k(q)\in l_\infty(p)\},$$
$$r^q_c(p,B)=\{x=(x_j)\in w:y_k(q)\in c(p)\},$$
$$r^q_0(p,B)=\{x=(x_j)\in w:y_k(q)\in c_0(p)\}$$

and

$$r^q(p,B)=\{x=(x_j)\in w: y_k(q)\in l(p)\}.$$

Where the linear spaces $l_\infty(p), c(p), c_0(p)$ and $l(p)$ were defined as follows ;

$$l_\infty(p)=\left\{x=(x_k)\in w: \sup_{k\in\mathbb{N}}|x_k|^{p_k}<\infty\right\},$$

$$c(p)=\left\{x=(x_k)\in w: \lim_{k\to\infty}|x_k-l|^{p_k}=0 \text{ for some } l\in\mathbb{R}\right\},$$

$$c_0(p)=\left\{x=(x_k)\in w: \lim_{k\to\infty}|x_k|^{p_k}=0\right\}$$

which are the complete spaces paranormed by

$$g_1(x)=\sup_{k\in\mathbb{N}}|x_k|^{\frac{p_k}{M}}$$

and

$$l(p)=\left\{x=(x_k)\in w: \sum_k |x_k|^{p_k}<\infty\right\},$$

which is the complete spaces paranormed by

$$g_2(x)=\left(\sum_k |x_k|^{p_k}\right)^{\frac{1}{M}}.$$

If we take r=1 and s=-1 in the matrix B as in the Riesz $B-$difference sequence spaces $r_\infty^q(p,B)$, $r_c^q(p,B)$, $r_0^q(p,B)$ and $r^q(p,B)$ then these spaces reduce the sequence spaces $r_\infty^q(p,\Delta)$, $r_c^q(p,\Delta)$, $r_0^q(p,\Delta)$ and $r^q(p,\Delta)$.

If we take $p_k=p$ for all k then we denote $r_\infty^q(p,B)=r_\infty^q(B)$, $r_c^q(p,B)=r_c^q(B)$, $r_0^q(p,B)=r_0^q(B)$ and $r^q(p,B)=r^q(B)$.

We may begin with the following theorem .

Theorem 1. *(a) $r_0^q(p,B)$ is a complete linear metric space paranormed by g_B, defined by*

$$g_B(x)=\sup_{k\in\mathbb{N}}\left|\frac{1}{Q_k}\left[\sum_{j=0}^{k-1}(q_j.r+q_{j+1}.s)\,x_j+q_k.r.x_k\right]\right|^{\frac{p_k}{M}} \tag{2.2}$$

g is paranorm for the spaces $r_\infty^q(p,B)$ and $r_c^q(p,B)$ only in the trivial case with $\inf p_k>0$ when $r_\infty^q(p,B)=r_\infty^q(B)$ and $r_c^q(p,B)=r_c^q(B)$.

(b) $r^q(p,B)$ is a complete linear metric space paranormed by

$$g_B^*(x)=\left(\sum_k\left|\frac{1}{Q_k}\left[\sum_{j=0}^{k-1}(q_j.r+q_{j+1}.s)\,x_j+q_k.r.x_k\right]\right|^{p_k}\right)^{\frac{1}{M}} \tag{2.3}$$

with $0<p_k\le\sup p_k=H<\infty$ and $M=\max\{1,H\}$.

Proof. We only prove the theorem for the space $r_0^q(p,B)$. The proof of other spaces can be done similarly. The linearity of $r_0^q(p,B)$ with respect to the co-ordinatewise addition and scalar multiplication follows from the inequalities which are satisfied for $u,v\in r_0^q(p,B)$ [10].

$$\sup_{k\in\mathbb{N}}\left|\frac{1}{Q_k}\left[\sum_{j=0}^{k-1}(q_j.r+q_{j+1}.s)(u_j+v_j)+q_k.r.(u_k+v_k)\right]\right|^{\frac{p_k}{M}} \tag{2.4}$$

$$\leq \sup_{k\in\mathbb{N}}\left|\frac{1}{Q_k}\left[\sum_{j=0}^{k-1}(q_j.r+q_{j+1}.s)u_j+q_k.r.u_k\right]\right|^{\frac{p_k}{M}} \tag{2.1}$$

$$+\sup_{k\in\mathbb{N}}\left|\frac{1}{Q_k}\left[\sum_{j=0}^{k-1}(q_j.r+q_{j+1}.s)v_j+q_k.r.v_k\right]\right|^{\frac{p_k}{M}} \tag{2.2}$$

and for any $\alpha\in\mathbb{R}$ [8]

$$|\alpha_k|^{p_k}\leq\max\left\{1,|\alpha|^M\right\}. \tag{2.5}$$

It is clear that $g_B(\theta)=0$ and $g_B(-x)=g_B(x)$ for all $u\in r_0^q(p,B)$. Again the inequalities (2.4) and (2.5) yield the subadditivity of g_B and

$$g_B(\alpha u)\leq\max\{1,|\alpha|\}\,g_B(u). \tag{2.3}$$

Let $\{x^n\}$ be any sequence of the elements of the space $r_0^q(p,B)$ such that

$$g_B(x^n-x)\to 0 \tag{2.4}$$

and (λ_n) also be any sequence of scalars such that $\lambda_n\to\lambda$. Then, since the inequality

$$g_B(x^n)\leq g_B(x)+g_B(x^n-x) \tag{2.5}$$

holds by subadditivity of g_B, $\{g_B(x^n)\}$ is bounded, and thus we have

$$g_B(\lambda_n x^n-\lambda x)=\sup_{k\in\mathbb{N}}\left|\frac{1}{Q_k}\left[\sum_{j=0}^{k-1}(q_j.r+q_{j+1}.s)\left(\lambda_n x_j^n-\lambda x_j\right)+q_k.r\left(\lambda_n x_k^n-\lambda x_k\right)\right]\right|^{\frac{p_k}{M}} \tag{2.6}$$

$$=|\lambda_n-\lambda|^{\frac{1}{M}}\sup_{k\in\mathbb{N}}\left|\frac{1}{Q_k}\left[\sum_{j=0}^{k-1}(q_j.r+q_{j+1}.s)x_j^n+q_k.r.x_k^n\right]\right|^{\frac{p_k}{M}} \tag{2.7}$$

$$+|\lambda|^{\frac{1}{M}}\sup_{k\in\mathbb{N}}\left|\frac{1}{Q_k}\left[\sum_{j=0}^{k-1}(q_j.r+q_{j+1}.s)\left(x_j^n-x_j\right)+q_k.r\left(x_k^n-x_k\right)\right]\right|^{\frac{p_k}{M}} \tag{2.8}$$

$$\leq |\lambda_n - \lambda|^{\frac{1}{M}} g_B(x^n) + |\lambda|^{\frac{1}{M}} g_B(x^n - x) \tag{2.9}$$

which tends to zero as $n \to \infty$. Hence the continuity of the scalar multiplication has shown. Finally; it is clear to say that g_B is a paranorm on the space $r_0^q(p,B)$. Moreover; we will prove the completeness of the space $r_0^q(p,B)$. Let $\{x^i\}$ be a Cauchy sequence in the space $r_0^q(p,B)$, where $x^i = \left\{x_k^{(i)}\right\} = \{x_0^i, x_1^i, x_2^i, ...\} \in r_0^q(p,B)$. Then, for a given $\varepsilon > 0$ there exists a positive integer $n_0(\varepsilon)$ such that

$$g_B\left(x^i - x^j\right) < \varepsilon \tag{2.6}$$

for all $i, j \geq n_0(\varepsilon)$. If we use the definition of g_B we obtain for each fixed $k \in \mathbb{N}$ that

$$\left|\left(R^q B x^i\right)_k - \left(R^q B x^j\right)_k\right| \leq \sup_{k\in\mathbb{N}} \left|\left(R^q B x^i\right)_k - \left(R^q B x^j\right)_k\right|^{\frac{p_k}{M}} < \varepsilon \tag{2.7}$$

for $i, j \geq n_0(\varepsilon)$ which leads us to the fact that

$$\left\{\left(R^q B x^0\right)_k, \left(R^q B x^1\right)_k, \left(R^q B x^2\right)_k, ...\right\} \tag{2.10}$$

is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, so we write $\left(R^q B x^i\right)_k \to (R^q B x)_k$ as $i \to \infty$. Hence by using these infinitely many limits $(R^q B x)_0, (R^q B x)_1, (R^q B x)_2, ...$, we define the sequence $\{(R^q B x)_0, (R^q B x)_1, (R^q B x)_2, ...\}$. From (2.7) with $j \to \infty$ we have

$$\left|\left(R^q B x^i\right)_k - (R^q B x)_k\right| \leq \varepsilon \tag{2.8}$$

$i \geq n_0(\varepsilon)$ for every fixed $k \in \mathbb{N}$. Since $x^i = \left\{x_k^{(i)}\right\} \in r_0^q(p,B)$,

$$\left|\left(R^q B x^i\right)_k\right|^{\frac{p_k}{M}} < \varepsilon \tag{2.11}$$

for all $k \in \mathbb{N}$. Therefore, by (2.8) we obtain that

$$|(R^q B x)_k|^{\frac{p_k}{M}} \leq \left|(R^q B x)_k - \left(R^q B x^i\right)_k\right|^{\frac{p_k}{M}} + \left|\left(R^q B x^i\right)_k\right|^{\frac{p_k}{M}} < \varepsilon \tag{2.9}$$

for all $i \geq n_0(\varepsilon)$. This shows that the sequence $R^q B x$ belongs to the space $c_0(p)$. Since $\{x^i\}$ was an arbitrary Cauchy sequence, the space $r_0^q(p,B)$ is complete. □

If we take r=1, s=-1 in the theorem 1 then we have the following result.

Corollary 1. *(a)* $r_0^q(p,\Delta)$ *is a complete linear metric space paranormed by* g_Δ, *defined by* $g_\Delta(x) = \sup_{k\in\mathbb{N}} \left|\frac{1}{Q_k}\left[\sum_{j=0}^{k-1}(q_j - q_{j+1})x_j + q_k.x_k\right]\right|^{\frac{p_k}{M}}$.

g_Δ *is paranorm for the spaces* $r_\infty^q(p,\Delta)$ *and* $r_c^q(p,\Delta)$ *only in the trivial case with* $\inf p_k > 0$ *when* $r_\infty^q(p,\Delta) = r_\infty^q(\Delta)$ *and* $r_c^q(p,\Delta) = r_c^q(\Delta)$.

(b) [11] $r^q(p,\Delta)$ *is a complete linear metric space paranormed by*

$g_\Delta^\star(x) = \left(\sum_k \left|\frac{1}{Q_k}\left[\sum_{j=0}^{k-1}(q_j - q_{j+1}.)x_j + q_k.x_k\right]\right|^{p_k}\right)^{\frac{1}{M}}$ *with* $0 < p_k \leq \sup p_k = H < \infty$ *and* $M = \max\{1, H\}$.

Theorem 2. *Let* $rq_j + sq_{j+1} \neq 0$ *for all* j. *Then the Riesz* $B-$*difference sequence spaces* $r^q_\infty(p,B), r^q_c(p,B)$, $r^q_0(p,B)$ *and* $r^q(p,B)$ *are linearly isomorphic to the space* $l_\infty(p)$, $c(p)$, $c_0(p)$ *and* $l(p)$, *respectively; where* $0 < p_k \leq H < \infty$.

Proof. We establish this for the the space $r^q_\infty(p,B)$. For proof of the theorem, we should show the existence of a linear bijection between the space $r^q_\infty(p,B)$ and $l_\infty(p)$ for $0 < p_k \leq H < \infty$. With the notation of

$$y_k = \frac{1}{Q_k}\left[\sum_{j=0}^{k-1}(q_j.r + q_{j+1}.s)\,x_j + q_k.r.x_k\right]$$

define the transformation T from $r^q_\infty(p,B)$ to $l_\infty(p)$ by $x \mapsto y = Tx$. T is a linear transformation, morever; it is obviuos that $x = \theta$ whenever $Tx = \theta$ and hence T is injective.

Let $y = (y_k) \in l_\infty(p)$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{n=0}^{k-1}(-1)^{k-n}\left(\frac{s^{k-n-1}}{r^{k-n}q_{n+1}} + \frac{s^{k-n}}{r^{k-n+1}q_n}\right)Q_n y_n + \frac{Q_k y_k}{r.q_k} \qquad \text{for} \qquad k \in \mathbb{N}.$$

Then

$$\begin{aligned} g_B(x) &= \sup_{k\in\mathbb{N}}\left|\frac{1}{Q_k}\left[\sum_{j=0}^{k-1}(q_j.r + q_{j+1}.s)\,x_j + q_k.r.x_k\right]\right|^{\frac{p_k}{M}} \\ &= \sup_{k\in\mathbb{N}}\left|\sum_{j=0}^{k}\delta_{kj}y_j\right|^{\frac{p_k}{M}} = \sup_{k\in\mathbb{N}}|y_k|^{\frac{p_k}{M}} = g_1(y) < \infty \end{aligned}$$

where

$$\delta_{kj} = \begin{cases} 1 \ , & k = j \\ 0 \ , & k \neq j \end{cases}.$$

Thus, we have that $x \in r^q_\infty(p,B)$. Consequently; T is surjective and is paranorm preserving. Hence, T is linear bijection and this explains that the spaces $r^q_\infty(p,B)$ and $l_\infty(p)$ are linearly isomorphic, as was desired. □

Corollary 2. *Let* $q_j - q_{j+1} \neq 0$ *for all* j. *Then the* Δ*-Riesz sequence spaces* $r^q_\infty(p,\Delta), r^q_c(p,\Delta)$, $r^q_0(p,\Delta)$ *and* $r^q(p,\Delta)$ *are linearly isomorphic to the spaces* $l_\infty(p)$,$c(p)$, $c_0(p)$ *and* $l(p)$, *respectively; where* $0 < p_k \leq H < \infty$.

And now we shall quote some lemmas which are needed in proving our theorems.

Lemma 1. *[5]* $A \in (l_\infty(p) : l_1)$ *if and only if*

$$\sup_{K\in\mathcal{F}}\sum_n\left|\sum_{k\in K}a_{nk}K^{\frac{1}{p_k}}\right| < \infty \text{ for all integers } K > 1. \tag{2.10}$$

Lemma 2. *[7] Let* $p_k > 0$ *for every* $k \in \mathbb{N}$*. Then* $A \in (l_\infty(p) : l_\infty)$ *if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| K^{\frac{1}{p_k}} < \infty \text{ for all integers } K > 1. \tag{2.11}$$

Lemma 3. *[7] Let* $p_k > 0$ *for every* $k \in \mathbb{N}$*. Then* $A \in (l_\infty(p) : c)$ *if and only if*

$$\sum_k |a_{nk}| K^{\frac{1}{p_k}} \text{ convergence uniformly in } n \text{ for all integers } K > 1, \tag{2.12}$$

$$\lim_{n\to\infty} a_{nk} = \alpha_k \text{ for all } k \in \mathbb{N}. \tag{2.13}$$

Lemma 4. *[5] (i) Let* $1 < p_k \le H < \infty$ *for every* $k \in \mathbb{N}$*. Then* $A \in (l(p) : l_1)$ *if and only if there exists an integer* $K > 1$ *such that*

$$\sup_{K\in\mathcal{F}} \sum_k \left| \sum_{n\in K} a_{nk} K^{-1} \right|^{p_k'} < \infty.$$

(ii) Let $0 < p_k \le 1$ *for every* $k \in \mathbb{N}$*. Then* $A \in (l(p) : l_1)$ *if and only if*

$$\sup_{K\in\mathcal{F}} \sup_{k\in\mathbb{N}} \left| \sum_{n\in K} a_{nk} \right|^{p_k} < \infty.$$

Lemma 5. *[7] (i) Let* $1 < p_k \le H < \infty$ *for every* $k \in \mathbb{N}$*. Then* $A \in (l(p) : l_\infty)$ *if and only if there exists an integer* $K > 1$ *such that*

$$\sup_{n\in\mathbb{N}} \sum_k \left| a_{nk}^{-1} K^{-1} \right|^{p_k'} < \infty. \tag{2.14}$$

(ii) Let $0 < p_k \le 1$ *for every* $k \in \mathbb{N}$*. Then* $A \in (l(p) : l_\infty)$ *if and only if*

$$\sup_{n,k\in N} |a_{nk}|^{p_k} < \infty. \tag{2.15}$$

Lemma 6. *[7] Let* $0 < p_k \le H < \infty$ *for every* $k \in \mathbb{N}$*. Then* $A \in (l(p) : c)$ *if and only if* (2.6) *and* (2.7) *hold, and*

$$\lim_{n\to\infty} a_{nk} = \beta_k \quad \text{for} \quad k \in \mathbb{N} \tag{2.16}$$

also holds.

Theorem 3. *(a) Define the sets* $R_1(p)$, $R_2(p)$, $R_3(p)$, $R_4(p)$, $R_5(p)$ *and* $R_6(p)$ *as follows:*

$$R_1(p) = \bigcap_{K>1} \left\{ a = (a_k) \in w : \sup_{N\in\mathcal{F}} \sum_n \left| \sum_{k\in N} \left[\nabla(k,n) Q_k a_n + \frac{Q_n a_n}{r.q_n} \right] K^{\frac{1}{p_k}} \right| < \infty \right\},$$

$$R_2(p) = \bigcap_{K>1} \left\{ a = (a_k) \in w : \sum_k \left| \left(\frac{a_k}{r.q_k} + \nabla(k,n) \sum_{i=k+1}^{n} a_i \right) Q_k \right| K^{\frac{1}{p_k}} < \infty \right.$$
$$\left. \text{and } \left(\frac{a_k Q_k}{r.q_k} K^{\frac{1}{p_k}} \right) \in c_0 \right\},$$
$$R_3(p) = \bigcap_{K>1} \left\{ a = (a_k) \in w : \sum_k \left| \left(\frac{a_k}{r.q_k} + \nabla(k,n) \sum_{i=k+1}^{n} a_i \right) Q_k \right| K^{\frac{1}{p_k}} < \infty \right.$$
$$\left. \text{and } \left\{ \left(\frac{a_k}{r.q_k} + \nabla(k,n) \sum_{i=k+1}^{n} a_i \right) Q_n \right\} \in l_\infty \right\},$$
$$R_4(p) = \bigcup_{K>1} \left\{ a = (a_k) \in w : \sup_{N \in \mathcal{F}} \sum_n \left| \sum_{k \in N} \left[\nabla(k,n) Q_k a_n + \frac{Q_n a_n}{r.q_n} \right] K^{\frac{-1}{p_k}} \right| < \infty \right\},$$
$$R_5(p) = \left\{ a = (a_k) \in w : \sum_n \left| \sum_k \left[\nabla(k,n) Q_k a_n + \frac{Q_n a_n}{r.q_n} \right] \right| < \infty \right\}$$

and

$$R_6(p) = \bigcup_{K>1} \left\{ a = (a_k) \in w : \sum_k \left| \left(\frac{a_k}{r.q_k} + \nabla(k,n) \sum_{i=k+1}^{n} a_i \right) Q_k \right| K^{\frac{-1}{p_k}} < \infty \right\},$$

where

$$\nabla(k,n) = (-1)^{n-k} \left(\frac{s^{n-k-1}}{r^{n-k} q_{k+1}} + \frac{s^{n-k}}{r^{n-k+1} q_k} \right).$$

Then

$$\{r^q_\infty(p,B)\}^\alpha = R_1(p) \qquad \{r^q_\infty(p,B)\}^\beta = R_2(p) \qquad \{r^q_\infty(p,B)\}^\gamma = R_3(p),$$
$$\{r^q_c(p,B)\}^\alpha = R_4(p) \cap R_5(p) \qquad \{r^q_c(p,B)\}^\beta = R_6(p) \cap cs \qquad \{r^q_c(p,B)\}^\gamma = R_6(p) \cap bs,$$
$$\{r^q_0(p,B)\}^\alpha = R_4(p) \qquad \{r^q_0(p,B)\}^\beta = \{r^q_0(p,B)\}^\gamma = R_6(p).$$

(b) (i) Let $1 < p_k \le H < \infty$ for every $k \in \mathbb{N}$. Define the sets $R_7(p)$, $R_8(p)$ as follows:

$$R_7(p) = \bigcup_{K>1} \left\{ a = (a_k) \in w : \sup_{N \in \mathcal{F}} \sum_k \left| \sum_{n \in N} \left[\nabla(k,n) Q_k a_n + \frac{Q_n a_n}{r.q_n} \right] K^{-1} \right|^{p_k'} < \infty \right\}.$$
$$R_8(p) = \bigcup_{K>1} \left\{ a = (a_k) \in w : \sum_k \left| \left[\left(\frac{a_k}{r.q_k} + \nabla(k,n) \sum_{i=k+1}^{n} a_i \right) Q_k \right] K^{-1} \right|^{p_k'} < \infty \right\}.$$

Then; $[r^q(p,B)]^\alpha = R_7(p)$, $[r^q(p,B)]^\beta = R_8(p) \cap cs$, $[r^q(p,B)]^\gamma = R_8(p)$.

(ii) Let $0 < p_k \leq 1$ for every $k \in \mathbb{N}$. Define the sets $R_9(p)$, $R_{10}(p)$ by

$$R_9(p) = \left\{ a = (a_k) \in w : \sup_{N \in \mathcal{F}} \sup_{k \in \mathbb{N}} \left| \sum_{n \in N} \left[\nabla(k,n) Q_k a_n + \frac{Q_n a_n}{r.q_n} \right] K^{-1} \right|^{p_k} < \infty \right\}.$$

$$R_{10}(p) = \left\{ a = (a_k) \in w : \sup_{k \in N} \left| \left[\left(\frac{a_k}{r.q_k} + \nabla(k,n) \sum_{i=k+1}^{n} a_i \right) Q_k \right] \right|^{p_k} < \infty \right\}.$$

Then; $[r^q(p,B)]^\alpha = R_8(p)$, $[r^q(p,B)]^\beta = R_{10}(p) \cap cs$, $[r^q(p,B)]^\gamma = R_{10}(p)$.

Proof. We give the proof for the space $r^q_\infty(p,B)$. Let us take any $a = (a_n) \in w$. We easily derive with the notation

$$y_k = \frac{1}{Q_k} \left[\sum_{j=0}^{k-1} (q_j.r + q_{j+1}.s) x_j + q_k.r.x_k \right]$$

that

$$a_n x_n = \sum_{k=0}^{n-1} \nabla(k,n) a_n Q_k y_k + \frac{a_n Q_n y_n}{r.q_n} = \sum_{k=0}^{n} u_{nk} y_k = (Uy)_n ; \qquad (2.17)$$

$(n \in \mathbb{N})$, where $U = (u_{nk})$ is defined by

$$u_{nk} = \begin{cases} \nabla(k,n) a_n Q_k & , \quad (0 \leq k \leq n-1) \\ \frac{a_n Q_n}{r.q_n} & , \quad (k = n) \\ 0 & , \quad (k > n) \end{cases}$$

for all $k, n \in \mathbb{N}$. Thus we deduce from (2.17) that $ax = (a_n x_n) \in l_1$ whenever $x = (x_k) \in r^q_\infty(p,B)$ if and only if $Uy \in l_1$ whenever $y = (y_k) \in l_\infty(p)$. From $Lemma 1$, we obtain the desired result that

$$[r^q_\infty(p,B)]^\alpha = R_1(p).$$

Consider the equation

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n-1} \left(\frac{a_k}{r.q_k} + \nabla(k,n) \sum_{i=k+1}^{n} a_i \right) Q_k y_k + \frac{a_k Q_k y_k}{r.q_k} = (Vy)_n, (n \in \mathbb{N}); \qquad (2.18)$$

where $V = (v_{nk})$ defined by

$$v_{nk} = \begin{cases} \left(\frac{a_k}{r.q_k} + \nabla(k,n) \sum_{i=k+1}^{n} a_i \right) Q_k & , \quad (0 \leq k \leq n-1) \\ \frac{a_k Q_k}{r.q_k} & , \quad (k = n) \\ 0 & , \quad (k > n) \end{cases}$$

for all $k, n \in \mathbb{N}$. Thus we deduce by with (2.18) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in r^q_\infty (p, B)$ if and only if $Vy \in c$ whenever $y = (y_k) \in l_\infty (p)$. Therefore we derive from *Lemma*3 that

$$\sum_k \left| \left(\frac{a_k}{r.q_k} + \nabla (k,n) \sum_{i=k+1}^{n} a_i \right) Q_k \right| K^{\frac{1}{p_k}} < \infty$$

and

$$\lim_{k\to\infty} \frac{a_k Q_k}{r.q_k} K^{\frac{1}{p_k}} = 0$$

which shows that $[r^q_\infty (p, B)]^\beta = R_2 (p)$.

As this, we deduce by (2.18) that $ax = (a_k x_k) \in bs$ whenever $x = (x_k) \in r^q_\infty (p, B)$ if and only if $Vy \in l_\infty$ whenever $y = (y_k) \in l_\infty (p)$. Therefore we obtain by *Lemma*2 that $[r^q_\infty (p, B)]^\gamma = R_3 (p)$ and this completes proof. □

Corollary 3. *Define the sets* $T_1 (p)$, $T_2 (p)$, $T_3 (p)$, $T_4 (p)$, $T_5 (p)$ *and* $T_6 (p)$ *as follows:*

$$T_1 (p) = \bigcap_{K>1} \left\{ a = (a_k) \in w : \sup_{N\in\mathcal{F}} \sum_n \left| \sum_{k\in N} \left[\Lambda (k,n) Q_k a_n + \frac{Q_n a_n}{q_n} \right] K^{\frac{1}{p_k}} \right| < \infty \right\},$$

$$T_2 (p) = \bigcap_{K>1} \left\{ a = (a_k) \in w : \sum_k \left| \left(\frac{a_k}{q_k} + \Lambda (k,n) \sum_{i=k+1}^{n} a_i \right) Q_k \right| K^{\frac{1}{p_k}} < \infty \right.$$
$$\left. \textit{and } \left(\frac{a_k Q_k}{q_k} K^{\frac{1}{p_k}} \right) \in c_0 \right\},$$

$$T_3 (p) = \bigcap_{K>1} \left\{ a = (a_k) \in w : \sum_k \left| \left(\frac{a_k}{q_k} + \Lambda (k,n) \sum_{i=k+1}^{n} a_i \right) Q_k \right| K^{\frac{1}{p_k}} < \infty \right.$$
$$\left. \textit{and } \left\{ \left(\frac{a_k}{q_k} + \Lambda (k,n) \sum_{i=k+1}^{n} a_i \right) Q_n \right\} \in l_\infty \right\},$$

$$T_4 (p) = \bigcup_{K>1} \left\{ a = (a_k) \in w : \sup_{N\in\mathcal{F}} \sum_n \left| \sum_{k\in N} \left[\Lambda (k,n) Q_k a_n + \frac{Q_n a_n}{q_n} \right] K^{\frac{-1}{p_k}} \right| < \infty \right\},$$

$$T_5 (p) = \left\{ a = (a_k) \in w : \sum_n \left| \sum_k \left[\Lambda (k,n) Q_k a_n + \frac{Q_n a_n}{r.q_n} \right] \right| < \infty \right\}$$

and

$$T_6 (p) = \bigcup_{K>1} \left\{ a = (a_k) \in w : \sum_k \left| \left(\frac{a_k}{q_k} + \Lambda (k,n) \sum_{i=k+1}^{n} a_i \right) Q_k \right| K^{\frac{-1}{p_k}} < \infty \right\},$$

where

$$\Lambda(k,n) = (-1)^{n-k}\left(\frac{(-1)^{n-k-1}}{q_{k+1}} + \frac{(-1)^{n-k}}{q_k}\right).$$

Then

$$\{r_\infty^q(p,\Delta)\}^\alpha = T_1(p) \qquad \{r_\infty^q(p,\Delta)\}^\beta = T_2(p) \qquad \{r_\infty^q(p,\Delta)\}^\gamma = T_3(p),$$

$$\{r_c^q(p,\Delta)\}^\alpha = T_4(p)\ \cap T_5(p) \qquad \{r_c^q(p,\Delta)\}^\beta = T_6(p)\cap cs \qquad \{r_c^q(p,\Delta)\}^\gamma = T_6(p)\cap bs,$$

$$\{r_0^q(p,\Delta)\}^\alpha = T_4(p) \qquad \{r_0^q(p,\Delta)\}^\beta = \{r_0^q(p,\Delta)\}^\gamma = T_6(p).$$

3 The Basis for the Spaces $r_0^q(p,B)$ and $r_c^q(p,B)$

In the present section, we give two sequences of the points of the spaces $r_0^q(p,B)$ and $r_c^q(p,B)$ which form the basis for those spaces.

Theorem 4. *Let $\mu_k(t) = (R^qBx)_k$ for all $k\in\mathbb{N}$ and $0 < p_k \le H < \infty$. Define the sequence $b^{(k)}(q) = \left\{b_n^{(k)}(q)\right\}_{n\in\mathbb{N}}$ of the elements of the space $r_0^q(p,B)$ for every fixed $k\in\mathbb{N}$ by*

$$b_n^{(k)}(q) = \begin{cases} \nabla(k,n)\,Q_k & , \quad (0\le n\le k-1) \\ \frac{Q_k}{r.q_k} & , \quad (k=n) \\ 0 & , \quad (n > k-1) \end{cases} \tag{3.1}$$

where

$$\nabla(k,n) = (-1)^{n-k}\left(\frac{s^{n-k-1}}{r^{n-k}q_{k+1}} + \frac{s^{n-k}}{r^{n-k+1}q_k}\right).$$

Then,
(a) The sequence $\{b^{(k)}(q)\}_{k\in\mathbb{N}}$ is a basis for the space $r_0^q(p,B)$ and any $x\in r_0^q(p,B)$ has a unique representation of the form

$$x = \sum_k \mu_k(q)\, b^{(k)}(q). \tag{3.2}$$

(b) The set $\left\{(R^qB)^{-1}e, b^{(k)}(q)\right\}$ is a basis for the space $r_c^q(p,B)$ and any $x\in r_c^q(p,B)$ has a unique representation of the form

$$x = le + \sum_k |\mu_k(q) - l|\, b^{(k)}(q); \tag{3.3}$$

where

$$l = \lim_{k\to\infty}(R^qBx)_k. \tag{3.4}$$

Proof. It is clear that $\left\{b^{(k)}(q)\right\} \subset r_0^q(p,B)$, since

$$R^q B b^{(k)}(q) = e^{(k)} \in c_0(p)\,, \quad (\text{for} \quad k \in \mathbb{N}) \tag{3.5}$$

for $0 < p_k \leq H < \infty$; where $e^{(k)}$ is the sequence whose only non-zero term is a 1 in k^{th} place for each $k \in \mathbb{N}$.

Let $x \in r_0^q(p,B)$ be given. For every non-negative integer m, we put

$$x^{[m]} = \sum_{k=0}^{m} \mu_k(q)\, b^{(k)}(q)\,. \tag{3.6}$$

Then, we obtain by applying $R^q B$ to (3.6) with (3.5) that

$$R^q B x^{[m]} = \sum_{k=0}^{m} \mu_k(q)\, R^q B b^{(k)}(q) = \sum_{k=0}^{m} \left(R^q B\right)_k e^{(k)}$$

and

$$\left(R^q B\left(x - x^{[m]}\right)\right)_i = \begin{cases} 0 & , \quad (0 \leq i \leq m) \\ \left(R^q B x\right)_i & , \quad (i > m) \end{cases} ;(i,m \in \mathbb{N})\,.$$

Given $\varepsilon > 0$, then there exists an integer m_0 such that

$$\sup_{i \geq m} \left|\left(R^q B x\right)_i\right|^{\frac{p_k}{M}} < \frac{\varepsilon}{2}$$

for all $m \geq m_0$. Hence,

$$g_B\left(x - x^{[m]}\right) = \sup_{i \geq m} \left|\left(R^q B x\right)_i\right|^{\frac{p_k}{M}} \leq \sup_{i \geq m_0} \left|\left(R^q B x\right)_i\right|^{\frac{p_k}{M}} < \frac{\varepsilon}{2} < \varepsilon$$

for all $m \geq m_0$ which proves that $x \in r_0^q(p,B)$ is represented as in (3.2).

To show the uniqueness of this representation, we suppose that

$$x = \sum_k \lambda_k(q)\, b^{(k)}(q)\,.$$

Since the linear transformation T, from $r_0^q(p,B)$ to $c_0(p)$ used in *Theorem* 2, is continuous we have

$$\left(R^qBx\right)_n=\sum_k\lambda_k(q)\left\{R^qBb^{(k)}(q)\right\}_n=\sum_k\lambda_k(q)\,e_n^{(k)}=\lambda_n(q)\,;\;n\in\mathbb{N}$$

which contradicts the fact that $\left(R^qBx\right)_n=\mu_k(q)$ for all $n\in\mathbb{N}$. Hence, the representation (3.2) of $x\in r_0^q(p,B)$ is unique. Thus the proof of the part (a) of Theorem is completed.

(b) Since $\left\{b^{(k)}(q)\right\}\subset r_0^q(p,B)$ and $e\in c$, the inclusion $\left\{e,b^{(k)}(q)\right\}\subset r_c^q(p,B)$ trivially holds. Let us take $x\in r_c^q(p,B)$. Then, there uniquely exists an l satisfying (3.4). We thus have the fact that $u\in r_0^q(p,B)$ whenever we set $u=x-le$. Therefore, we deduce by part (a) of the present theorem that the representation of x given by (3.3) is unique and this step concludes the proof of the part (b) of Theorem. □

Now we characterize the matrix mappings from the spaces $r_\infty^q(p,B)\,,r_c^q(p,B)$, $r_0^q(p,B)$ and $r^q(p,B)$ to the spaces l_∞ and c. The following theorems can be proved by used standart methods and we omit the detail.

Theorem 5. (i) *$A\in\left(r_\infty^q(p,B):l_\infty\right)$ if and only if*

$$\lim_{k\to\infty}\frac{a_{nk}}{q_k}Q_kM^{\frac{1}{p_k}}=0,(\forall n,M\in\mathbb{N}) \tag{3.7}$$

and

$$\sup_{n\in\mathbb{N}}\sum_k\left|\frac{a_{nk}}{r.q_k}+\nabla(k,n)\sum_{i=k+1}^{n}a_{ni}\right|Q_kM^{\frac{1}{p_k}}<\infty,(\forall M\in\mathbb{N}) \tag{3.8}$$

hold.

(ii) $A\in\left(r_c^q(p,B):l_\infty\right)$ if and only if (3.7),

$$\sup_{n\in\mathbb{N}}\sum_k\left|\left(\frac{a_{nk}}{r.q_k}+\nabla(k,n)\sum_{i=k+1}^{n}a_{ni}\right)Q_k\right|M^{\frac{1}{p_k}}=0,(\exists M\in\mathbb{N}) \tag{3.9}$$

and

$$\sup_{n\in\mathbb{N}}\sum_k\left|\left(\frac{a_{nk}}{r.q_k}+\nabla(k,n)\sum_{i=k+1}^{n}a_{ni}\right)Q_k\right|<\infty \tag{3.10}$$

hold.

(iii) $A\in\left(r_0^q(p,B):l_\infty\right)$ if and only if (3.7) and (3.9) hold.

(iv) Let $1 < p_k \leq H < \infty$ *for every* $k \in \mathbb{N}$. *Then* $A \in (r^q(p,B) : l_\infty)$ *if and only if there exists an integer* $K > 1$ *such that*

$$R(K) = \sup_{n\in\mathbb{N}} \sum_k \left| \left[\left(\frac{a_k}{r.q_k} + \nabla(k,n) \sum_{i=k+1}^{n} a_{ni} \right) Q_k \right] K^{-1} \right|^{p_k'} < \infty \tag{3.11}$$

and

$$\{a_{nk}\}_{k\in\mathbb{N}} \in cs$$

for each $n \in \mathbb{N}$.

(v) Let $0 < p_k \leq 1$ *for every* $k \in \mathbb{N}$. *Then* $A \in (r^q(p,B) : l_\infty)$ *if and only if*

$$\sup_{n,k\in\mathbb{N}} \left| \left[\left(\frac{a_k}{r.q_k} + \nabla(k,n) \sum_{i=k+1}^{n} a_{ni} \right) Q_k \right] \right|^{p_k} < \infty \qquad 3.12$$

and

$$\{a_{nk}\}_{k\in\mathbb{N}} \in cs$$

for each $n \in \mathbb{N}$.

Theorem 6. *(i)* $A \in (r^q_\infty(p,B) : c)$ *if and only if* (3.7),

$$\sup_{n\in\mathbb{N}} \sum_k \left| \left(\frac{a_{nk}}{r.q_k} + \nabla(k,n) \sum_{i=k+1}^{n} a_{ni} \right) Q_k \right| M^{\frac{1}{p_k}} < \infty, (\forall M \in \mathbb{N}) \tag{3.13}$$

and

$$\exists (\alpha_k) \subset \mathbb{R} \text{ such that } \lim_{n\to\infty} \left[\sum_k \left| \left(\frac{a_{nk}}{r.q_k} + \nabla(k,n) \sum_{i=k+1}^{n} a_{ni} \right) Q_k - \alpha_k \right| M^{\frac{1}{p_k}} \right] = 0, \tag{3.14}$$

$(\forall M \in \mathbb{N})$ *hold.*

(ii) $A \in (r^q_c(p,B) : c)$ *if and only if* (3.7), (3.9),

$$\exists \alpha \in \mathbb{R} \text{ such that } \lim_{n\to\infty} \left| \left(\frac{a_{nk}}{r.q_k} + \nabla(k,n) \sum_{i=k+1}^{n} a_{ni} \right) Q_k - \alpha \right| = 0, \tag{3.15}$$

$$\exists (\alpha_k) \subset \mathbb{R} \text{ such that } \lim_{n\to\infty} \left| \left(\frac{a_{nk}}{r.q_k} + \nabla(k,n) \sum_{i=k+1}^{n} a_{ni} \right) Q_k - \alpha_k \right| = 0, (\forall k \in \mathbb{N}) \tag{3.16}$$

and

$$\exists\left(\alpha_k\right)\subset\mathbb{R}\quad such\ that\ \sup_{n\in\mathbb{N}} L\sum_k\left|\left(\frac{a_{nk}}{r.q_k}+\nabla\left(k,n\right)\sum_{i=k+1}^{n}a_{ni}\right)Q_k-\alpha_k\right|M^{\frac{-1}{p_k}}<\infty, \tag{3.17}$$

$(\forall L,\exists M\in\mathbb{N})$ *hold.*

(iii) $A\in\left(r_0^q\left(p,B\right):c\right)$ *if and only if* $(3.7)\,,(3.9)\,,(3.16)$ *and* $(3.17)\,.$

Corollary 4. *(i)* $A\in\left(r_\infty^q\left(p,\Delta\right):l_\infty\right)$ *if and only if*

$$\lim_{k\to\infty}\frac{a_{nk}}{q_k}Q_kM^{\frac{1}{p_k}}=0,(\forall n,M\in\mathbb{N}) \tag{3.18}$$

and

$$\sup_{n\in\mathbb{N}}\sum_k\left|\frac{a_{nk}}{q_k}+\Lambda\left(k,n\right)\sum_{i=k+1}^{n}a_{ni}\right|Q_kM^{\frac{1}{p_k}}<\infty,(\forall M\in\mathbb{N}) \tag{3.19}$$

hold.

(ii) $A\in\left(r_c^q\left(p,\Delta\right):l_\infty\right)$ *if and only if* $(3.18)\,,$

$$\sup_{n\in\mathbb{N}}\sum_k\left|\left(\frac{a_{nk}}{q_k}+\Lambda\left(k,n\right)\sum_{i=k+1}^{n}a_{ni}\right)Q_k\right|M^{\frac{1}{p_k}}=0,(\exists M\in\mathbb{N}) \tag{3.20}$$

and

$$\sup_{n\in\mathbb{N}}\sum_k\left|\left(\frac{a_{nk}}{q_k}+\Lambda\left(k,n\right)\sum_{i=k+1}^{n}a_{ni}\right)Q_k\right|<\infty \tag{3.21}$$

hold.

(iii) $A\in\left(r_0^q\left(p,\Delta\right):l_\infty\right)$ *if and only if* (3.18) *and* (3.20) *hold.*

Corollary 5. *(i)* $A\in\left(r_\infty^q\left(p,\Delta\right):c\right)$ *if and only if* (3.18),

$$\sup_{n\in\mathbb{N}}\sum_k\left|\left(\frac{a_{nk}}{q_k}+\Lambda\left(k,n\right)\sum_{i=k+1}^{n}a_{ni}\right)Q_k\right|M^{\frac{1}{p_k}}<\infty,(\forall M\in\mathbb{N}) \tag{3.22}$$

and

$$\exists\left(\alpha_k\right)\subset\mathbb{R}\ such\ that\ \lim_{n\to\infty}\left[\sum_k\left|\left(\frac{a_{nk}}{q_k}+\Lambda\left(k,n\right)\sum_{i=k+1}^{n}a_{ni}\right)Q_k-\alpha_k\right|M^{\frac{1}{p_k}}\right]=0, \tag{3.23}$$

$(\forall M \in \mathbb{N})$ *hold.*

(ii) $A \in (r_c^q(p,\Delta) : c)$ *if and only if* $(3.18), (3.20)$,

$$\exists \alpha \in \mathbb{R} \text{ such that } \lim_{n\to\infty} \left| \left(\frac{a_{nk}}{q_k} + \Lambda(k,n) \sum_{i=k+1}^{n} a_{ni} \right) Q_k - \alpha \right| = 0, \tag{3.24}$$

$$\exists (\alpha_k) \subset \mathbb{R} \text{ such that } \lim_{n\to\infty} \left| \left(\frac{a_{nk}}{q_k} + \Lambda(k,n) \sum_{i=k+1}^{n} a_{ni} \right) Q_k - \alpha_k \right| = 0, (\forall k \in \mathbb{N}) \tag{3.25}$$

and

$$\exists (\alpha_k) \subset \mathbb{R} \text{ such that } \sup_{n\in\mathbb{N}} L \sum_k \left| \left(\frac{a_{nk}}{q_k} + \Lambda(k,n) \sum_{i=k+1}^{n} a_{ni} \right) Q_k - \alpha_k \right| M^{\frac{-1}{p_k}} < \infty, \tag{3.26}$$

$(\forall L, \exists M \in \mathbb{N})$ *hold.*

(iii) $A \in (r_0^q(p,\Delta) : c)$ *if and only if* $(3.18), (3.20), (3.25)$ *and* (3.26).

References

[1] B. Altay, F. Başar, *On The Paranormed Riesz Sequence Spaces Of Non-Absolute Type*, Southeast Asian Bull. Math. 26 (2002), 5, 701–715.

[2] B. Altay, F. Başar, *Some Euler Sequence Spaces of Non-Absolute Type*, Ukranian Math. J. 57 (2005), 1-17.

[3] B. Altay, F. Başar, *On The Space of Sequences of p-Bounded Variation and Related Matrix Mappings*, Ukranian Math. J. 55 (2003), 1, 136-147.

[4] B. Altay, H. Polat, *On Some Euler Difference Sequence Spaces*, Southeast Asian Bull. Math. 30 (2006), 209-220.

[5] K.G.,Grosse-Erdmann, *Matrix Transformations Between The Sequnce Space Of Maddox*, J. Math. Anal. Appl.180 1993, 223-238.

[6] H.Kızmaz, *On Certain Sequence Space*, Canad. Math. Bull. 24(2) 1981,169-176.

[7] C.G.Lascarides, I.J.Maddox, *Matrix Transformations Between Some Classes Of Sequences*, Proc.Comb. Phil. Soc. 68 1970, 99-104.

[8] I.J.Maddox,*Paranormed Sequence Spaces Generated By Infinite Matrices*, Proc.Comb. Phil. Soc. 64 1968, 335-340.

[9] I.J.Maddox, *Spaces Of Strongly Summable Sequences*,Quart.J.Math. Oxford, 18(2) 1967,345-355.

[10] I.J.Maddox, *Element Of Functional Analysis* , The University Press, Cambridge, 1970.

[11] M.Başarır,M.Öztürk, On The Riesz Difference Sequence Space, Rendiconti del Circolo Matematico di Palermo, 2008, **57**, 377-389.

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