Commun. math. Phys. 13, 154—162 (1969)

On the Generalized Saletan Contractions

MARIAN KUPCZYŃSKI

Institute of Theoretical Physics, Warsaw University

Received February 18, 1969

Abstract. The generalized Saletan contractions leaving some subalgebra unchanged are investigated. By use of a simple basis the necessary and sufficient conditions for these contractions and the brackets of the contracted algebra expressed only by the structure constants of the algebra to be contracted are obtained. Many basis independent results are derived. Some of them give as a special case Saletan results. Similar contractions but for $\varepsilon \to \infty$ are proposed and investigated.

1. Introduction

The continuous process deforming one Lie group into another nonisomorphic Lie group is believed to have many physical applications. One postulates [1, 2] that some physical processes: like switching off interactions or passing to the limit with some physical quantities allowing to disregard some physical effects (for example velocity of light $\nearrow \infty$) should be reflected in the group theoretical description of the physical systems by some continuous deforming processes called contractions.

From many possible mathematical formulations of group-contraction [1–7] we choose the following one:

Let us consider N-dimensional Lie algebra $R^{(0)}$ on the vector space S. The corresponding Lie brackets will be denoted by $[,]^{(0)}$. We say that $R^{(0)}$ can be contracted by $A(\varepsilon)$ if there exists an analytic curve in a vector space $L(S, S)$ of all linear mappings of S:

$$E^1 \ni \varepsilon \to A(\varepsilon) \in L(S, S)$$

such that for $\varepsilon = \varepsilon_1 - A(\varepsilon_1) = I$ and for all $\varepsilon \neq \varepsilon_0 - A(\varepsilon)$ are one to one mappings of S such that for all "$\boldsymbol{a}$" $\in S$ and "$\boldsymbol{b}$" $\in S$ there exists a limit, as $\varepsilon \to \varepsilon_0$, of series $[\boldsymbol{a}, \boldsymbol{b}]^{(0)}(\varepsilon)$ where

$$[\boldsymbol{a}, \boldsymbol{b}]^{(0)}(\varepsilon) = A(\varepsilon)^{-1}[A(\varepsilon)\,\boldsymbol{a}, A(\varepsilon)\,\boldsymbol{b}]^{(0)}, \tag{1}$$

$$[\boldsymbol{a}, \boldsymbol{b}]^{(1)} = \lim_{\varepsilon \to \varepsilon_0} [\boldsymbol{a}, \boldsymbol{b}]^{(0)}(\varepsilon)\,. \tag{2}$$

The limit $[\boldsymbol{a}, \boldsymbol{b}]^{(1)}$ defines new Lie algebra $R^{(1)}$ on S called a contraction of $R^{(0)}$. Usually one takes $\varepsilon_0 = 0$ or $\varepsilon_0 = \infty$. If one chooses a basis in S one obtains a representation of $A(\varepsilon)$ by a set of $(N \times N)$-matrices and the series (1) has a corresponding one for the structure constants of $R^{(0)}(\varepsilon)$:

$$C^{\gamma}_{(0)\,\alpha,\beta}(\varepsilon) = A^{-1}(\varepsilon)^{\gamma}_{\gamma'} C^{\gamma'}_{(0)\alpha',\beta'} A(\varepsilon)^{\beta'}_{\beta} A(\varepsilon)^{\alpha'}_{\alpha} .^{1} \tag{3}$$

The necessary and sufficient conditions for this contraction [4] one obtains demanding the existence of the limit of the series (3)

$$C^{\gamma}_{(1)\,\alpha,\beta} = \lim_{\varepsilon\to\varepsilon_0} C^{\gamma}_{(0)\,\alpha,\beta}(\varepsilon) . \tag{4}$$

The problems of deformations of Lie algebras are investigated in their full generality by means of modern differential geometry but the general investigations give the general results, which [5–7] are difficult sometimes to apply to the practical problems, so many simple realizations of the contraction have been proposed. Though they have been presented in different ways, all of them can be investigated by the use of Eq. (4).

Now we adopt the following notation: $\oplus$ is used for the direct sum of vector spaces and for the direct sum of linear operators or matrices for example: an operator $A = A_1 \oplus A_2$ acts in a vector space $S = S_1 \oplus S_2$ in a way $AS = A_1 S_1 \oplus A_2 S_2$; $\ominus$ is used in the following meaning $(S \ominus S_1) \oplus S_1 = S$; $\widehat{\boldsymbol{a},\boldsymbol{b}} \in S$ means for all "$\boldsymbol{a}$" and "$\boldsymbol{b}$"-elements of S.

We obtain the Inonü-Wigner contraction [3] if we choose

$$\varepsilon_0 = 0, \qquad A(\varepsilon) = (I + \varepsilon w) \oplus \varepsilon I \tag{5}$$

where "w" is a nonsingular matrix.

The Saletan contraction [4] is obtained by the choice:

$$\varepsilon_0 = 0, \qquad A(\varepsilon) = u + \varepsilon I, \qquad u = u_R \oplus u_N \tag{6}$$

where u_R is a nonsingular matrix and u_N is a nilpotent one of a rank q.

The generalized Saletan contractions one obtains [6] by taking

$$A(\varepsilon) = \varepsilon^{p} A_s(\varepsilon) \tag{7}$$

where $A_s(\varepsilon)$ is defined by (6) and p is a natural number.

The p-contractions [1, 8] are obtained by taking

$$\varepsilon_0 = 0, \qquad A(\varepsilon)^{i}_{j} = \varepsilon^{m_i} \delta^{i}_{j}, \qquad 1 \leqq i, j \leqq N \tag{8}$$

where for m_i can be chosen real numbers[2].

We are interested in the generalized Saletan contractions (7) which will be denoted (S, p) and $(S, 0)$ if $p = 0$.

For (S, p) Eq. (1) and (2) were investigated for $A(\varepsilon)$ in the general form (6) and (7), Saletan obtained the necessary and sufficient conditions for $(S, 0)$ in the form of the general equations:

$$\widehat{\boldsymbol{a},\boldsymbol{b}} \in S \quad u^2[\boldsymbol{a},\boldsymbol{b}]^{(0)}_{N} - u[u\boldsymbol{a},\boldsymbol{b}]^{(0)}_{N} - u[\boldsymbol{a},u\boldsymbol{b}]^{(0)}_{N} + [u\boldsymbol{a},u\boldsymbol{b}]^{(0)}_{N} = 0 \tag{9}$$

[1] We use Einstein summation convention $\sum_i a_i b_i = a^i b_i$.

[2] We use in this paper δ^{i}_{j} and δ_{ij} for Kronecker δ' s.

and the new bracket in the form:

$$\widehat{\boldsymbol{a},\boldsymbol{b}} \in S \quad [\boldsymbol{a},\boldsymbol{b}]^{(1)} = u^{-1}[u\boldsymbol{a}, u\boldsymbol{b}]^{(0)}_R - u[\boldsymbol{a},\boldsymbol{b}]^{(0)}_N + [u\boldsymbol{a},\boldsymbol{b}]^{(0)}_N + [\boldsymbol{a}, u\boldsymbol{b}]^{(0)}_N, \tag{10}$$

where $[\boldsymbol{a},\boldsymbol{b}]^{(0)}_N$ and $[\boldsymbol{a},\boldsymbol{b}]^{(0)}_R$ are projections of $[\boldsymbol{a},\boldsymbol{b}]^{(0)}$ into subspaces S_N and S_R, $S_N \oplus S_R = S$. For (S, p) Monique Levy-Nahas obtained more complicated results: new bracket is given by

$$\widehat{\boldsymbol{a},\boldsymbol{b}} \in S \quad [\boldsymbol{a},\boldsymbol{b}]^{(1)}_p = (-u)^{p-1}([u\boldsymbol{a}, u\boldsymbol{b}]^{(0)} - u[\boldsymbol{a},\boldsymbol{b}]^{(1)}) \tag{11}$$

and the necessary and sufficient condition by

$$u[\boldsymbol{a},\boldsymbol{b}]^{(1)}_p = 0 \tag{12}$$

where $p \geqq 1$ and $[\boldsymbol{a},\boldsymbol{b}]^{(1)}$ is defined by (10). For $(S, 0)$ Saletan found some simple corollaries expressible in the group theoretical language. For (S, p) the same proofs seem to be very difficult.

2. The Construction of the Basis

We are going to investigate (S, p) in a special basis which enables us to find the necessary and sufficient conditions for (S, p) involving only the structure constants of $R^{(0)}$. To simplify calculations we put $u_R = I$ in (6), for doing it we have two arguments: u_R being nonsingular transforms a subalgebra generated by S_R in $R^{(0)}$ into an isomorphic subalgebra in $R^{(1)}$ so by our choice we don't loose much of generality. Besides for physical reasons we want to have a model fo contraction leaving some subalgebra unchanged.

Now we construct our basis in S, from matrix theory it is known that every nilpotent matrix u_N of rank q can be written in a very simple form in a basis consisting of all independent Jordan chains in S_N, where Jordan chain generated by $\boldsymbol{a} \in S_N$ is a set of linearly independent vectors: $\boldsymbol{a}, u_N\boldsymbol{a}, u_N^2\boldsymbol{a}, \ldots u_N^{n_a-1}\boldsymbol{a}\,(n_a \leqq q, u_N^{n_a}\boldsymbol{a} = 0)$. In this basis $S_N = \bigoplus_a S_{n_a}$, $\sum_a n_a = q_1$ [9], where S_{n_a} is a vector space spanned by Jordan chain generated by "$\boldsymbol{a}$". The elements of our new basis will be denoted by I_{lk}. They are defined as follows:

$$u_N^k I_{ls} = I_{ls+k}, \quad I_{l\,l+1} = 0\,,$$
$$1 \leqq s \leqq l, \quad l = l_1, \quad l_2 \ldots l_p, \quad \sum_{i=1}^{p} l_i = q_1\,. \tag{13}$$ [3]

To have all our basis in S we add $N - q_1$ linearly independent vectors in S_R which we signify I_s, $s = 1, \ldots, N - q_1$. Now we let I_{lk} and I_s to

[3] Here l_i is a dimension of vector space spanned by Jordan chain generated by vector $I_{l_i 1}$.

satisfy the following commutation relations in $R^{(0)}$:

$$[I_s, I_t]^{(0)} = C^n_{(0)s,t} I_n + C^{wi}_{(0)s,t} I_{wi},$$
$$[I_s, I_{lk}]^{(0)} = C^n_{(0)s,lk} I_n + C^{wi}_{(0)s,lk} I_{wi}, \tag{14}$$
$$[I_{rs}, I_{lk}]^{(0)} = C^n_{(0)rs,lk} I_n + C^{wi}_{(0)rs,lk} I_{wi}.$$

3. The Main Results

The contracting series of the matrices $A(\varepsilon) = \varepsilon^p((I \oplus u_N) + \varepsilon I)$ acts on our basis in the following way: $A(\varepsilon) I_{lk} = \varepsilon^p I_{lk+1}$ for $k \leqq l-1$, $A(\varepsilon) I_{ll} = \varepsilon^{p+1} I_{ll}$, $A(\varepsilon) I_s = \varepsilon^p I_s$ so $A(\varepsilon)^\alpha_\beta$ matrix, acting on the vectors in S in our basis, has the form:

$$A(\varepsilon)^{s'}_s = \delta^{s'}_s \varepsilon^p, \qquad A(\varepsilon)^{w'i'}_{wi} = \delta^{w'}_w (\delta^{i'}_{i+1} + \varepsilon \delta^{i'}_i) \varepsilon^p, \tag{15}$$
$$A^{-1}(\varepsilon)^s_{s'} = \delta^s_{s'} \varepsilon^{-p}, \qquad A^{-1}(\varepsilon)^{wi}_{w'i'} = \delta^w_{w'} (-1)^{i-i'} \varepsilon^{-(i-i'+1+p)} \text{ where } i \geqq i'.$$

Using (15) from (3) we obtain

$$C^n_{(0)s,t}(\varepsilon) = \varepsilon^p C^n_{(0)s,t}, \; C^{wi}_{(0)s,t}(\varepsilon) = C^{wi'}_{(0)s,t}(\varepsilon) \frac{(-1)^{i-i'}}{\varepsilon^{i-i'+1-p}}, \text{[4]}$$
$$C^n_{(0)s,lk}(\varepsilon) = (C^n_{(0)s,lk+1} + \varepsilon C^n_{(0)s,lk}) \varepsilon^p,$$
$$C^{wi}_{(0)s,lk}(\varepsilon) = (C^{wi'}_{(0)s,lk+1} + \varepsilon C^{wi'}_{(0)s,lk}) \frac{(-1)^{i-i'}}{\varepsilon^{i-i'+1-p}}, \tag{16}$$
$$C^n_{(0)rs,lk}(\varepsilon) = (C^n_{(0)rs+1,lk+1} + \varepsilon C^n_{(0)rs+1,lk} + \varepsilon C_{(0)rs,lk+1} + \varepsilon^2 C^n_{(0)rs,lk}) \varepsilon^p,$$
$$C^{wi}_{(0)rs,lk}(\varepsilon) = (C^{wi'}_{(0)rs+1,lk+1} + \varepsilon C^{wi'}_{(0)rs+1,lk} + \varepsilon C^{wi'}_{(0)rs,lk+1} + \varepsilon^2 C^{wi'}_{(0)rs,lk}) \frac{(-1)^{i-i'}}{\varepsilon^{i-i'+1-p}}.$$

From the Eq. (16), demanding all coefficients standing at ε^{-k} for $k = 1, 2, \ldots$ to vanish, we obtain:

Theorem 1. *The algebra $R^{(0)}$ can be contracted by (S, p) if and only if there exists a basis in $R^{(0)}$ such that the structure constants of $R^{(0)}$ in this basis fulfil the following equations:*

$$C^{wi}_{(0)s,lk+1} - C^{wi-1}_{(0)s,lk} = 0, \tag{17}$$
$$C^{wi}_{(0)s,t} = 0, \tag{18}$$
$$C^{wi}_{(0)rs+1,lk+1} - C^{wi-1}_{(0)rs+1,lk} - C^{wi-1}_{(0)rs,lk+1} + C^{wi-2}_{(0)rs,lk} = 0 \tag{19}$$

for $1 \leqq i \leqq w - p = N_1$, $1 \leqq s, t \leqq N - q_1$, $r1 \leqq rs \leqq rr$, $l1 \leqq lk \leqq ll$, we assume that $C^\gamma_{(0)\beta,ll+1} = C^\gamma_{(0)\beta,l0} = C^{ww+1}_{(0)\beta,\alpha} = C^{w0}_{(0)\beta,\alpha} = 0$ where α, β, γ are arbitrary indices labelling elements of the basis.

Now assuming that (17–19) are fulfilled we read from (16) the structure constants of $R^{(1)}$.

[4] Here and in the following formulae we have summation over i' from 1 to i.

Theorem 2. *There exists a basis in the contracted algebra $R^{(1)}$ that the structure constants of $R^{(1)}$ in this basis are given by the following equalities:*

$$
\begin{aligned}
&C^{n}_{(1)s,t} = \delta_{p0} C^{n}_{(0)s,t}, \quad C^{wi}_{(1)s,t} = \delta_{iw}\delta_{p1}(-1)^{p-1} C^{wi+1-p}_{(0)s,t},\\
&C^{n}_{(1)s,lk} = \delta_{p0} C^{n}_{(0)s,lk+1}, \; C^{wi}_{(1)s,lk} = \delta_{p0} C^{wi}_{(0)s,lk} \quad \text{for} \quad i \leqq w-1,\\
&C^{ww}_{(1)s,lk} = (-1)^{p-1}[(1-\delta_{p0})\, C^{ww+1-p}_{(0)s,lk+1} - C^{ww-p}_{(0)s,lk}],\\
&C^{n}_{(1)rs,lk} = \delta_{p0} C^{n}_{(0)rs+1,lk+1}, \; C^{wi}_{(0)rs,lk} = \delta_{p0} C^{wi+1}_{(0)rs+1,lk+1}\\
&\qquad\qquad \text{for} \quad i \leqq w-1,\\
&C^{ww}_{(1)rs,lk} = (-1)^{p-1}[C^{ww+1-p}_{(0)rs+1,lk+1}(1-\delta_{p0}) - C^{ww-p}_{(0)rs+1,lk}\\
&\qquad\qquad - C^{ww-p}_{(0)rs,lk+1} + C^{ww-p-1}_{(0)rs,lk}]
\end{aligned}
\tag{20}
$$

in all these equations we have no summation over p.

Now we are going to derive a few corollaries. Some of them will be basis independent, for this reason we need the following definitions. We define null spaces for u^{i}* by S_i (S_i are spanned by vectors I_{lk} for $l-i+1 \leqq k \leqq l$. By A_i we will signify vector spaces spanned by $I_{l\,l-i+1}$: $A_i = S_i \ominus S_{i-1}$, finally we remark that vector spaces $u^{i}S$ are spanned by all vectors I_s and I_{lk} for $k > i$.

Corollary 1. *All vector spaces $u^{i}S + S_p$,** for $i = 1 \ldots q$, have to form subalgebras in $R^{(0)}$.*

Corollary 2. *All vector spaces A_i and A_j have to satisfy the following commutation relations: $[A_i, A_j]^{(0)} \subset S_{i+j+p}$.*

Corollary 3. *For $p \neq 0$ one obtains condition $[S, S]^{(1)} \subset S_1$ what is Monique Nahas condition for (S, p).*

Corollary 4. *$R^{(1)}$ can be contracted by the same $A(\varepsilon)$ once more, if $p > 1$ one obtains an abelian algebra $R^{(2)}$; if $p = 0$ one obtains a series of contracted algebras $R^{(i)}$ where $R^{(i)}$ is obtained from $R^{(i-1)}$ by $A(\varepsilon)$ $1 \leqq i \leqq q$, the structure constants of $R^{(m)}$ are given by the following equalities:*

$$
\begin{aligned}
&C^{n}_{(m)s,t} = C^{n}_{(0)s,t}, \; C^{wi}_{(m)s,t} = 0,\\
&C^{n}_{(m)s,lk} = C^{n}_{(0)s,lk+m}, \; C^{wi}_{(m)s,lk} = C^{wi}_{(0)s,lk},\\
&C^{n}_{(m)rs,lk} = C^{n}_{(0)rs+m,lk+m}, \; C^{wi}_{(m)rs,lk} = C^{wi+m}_{(0)rs+m,lk+m} \quad \text{for} \quad i \leqq w-m,\\
&C^{ww-c}_{(m)rs,lk} = C^{ww}_{(0)rs+m,lk+c} + C^{ww}_{(0)rs+c,lk+m} - C^{ww-m+c}_{(0)rs+c,lk+c}\\
&\qquad\qquad \text{for} \quad 0 \leqq c < m;
\end{aligned}
\tag{21}
$$

if $p = 1$ one also obtains a series of contracted algebras $R_1^{(i)}$, for $i \leqq q$, which is different from the above mentioned one, the non vanishing structure constants of $R_1^{(m)}$ are following

$$
\begin{aligned}
C^{ww}_{1(m)s,t} &= C^{ww}_{(0)s,t},\\
C^{ww}_{1(m)s,lk} &= C^{ww}_{(0)s,lk+m},\\
C^{ww}_{1(m)rs,lk} &= C^{ww}_{(0)rs+m,lk+m}
\end{aligned}
\tag{21'}
$$

* u^{i} — means u taken to the i-th power.

** $+$ — here means vector sum of subspaces of S.

where from the following basis independent features of $R_1^{(m)}$ can be found:

$$[S_i, S]_1^{(m)} = 0 \quad \text{for} \quad m \geqq i\,,$$

$$[S_R, S_R]_1^{(m)} = [S_R, S_R]_1^{(1)} \subset S_1 \quad \text{for all} \quad m\,,$$

where $[,]_1^{(m)}$ denotes the bracket of $R_1^{(m)}$.

Corollary 5. *All vector spaces $u^i S$ form subalgebras in all $R^{(m)}$.*

Corollary 6. *All vector spaces A_i satisfy the following commutation relations $[A_i, A_j]^{(m)} \subset S_{i-m+j}$ for $m \geqq i$ and $m \geqq j$.*

Corollary 7. *All spaces S_k satisfy the following relations:*

$$[S_k, S_k]^{(k+l)} \subset S_{k-l} \text{ for } l = 0, 1 \ldots .$$

Corollary 8. *The vector spaces S_k form ideals in all $R^{(m)}$ for $m \geqq k$ and abelian ideals for $m \geqq 2k$.*

4. The Derivation of the Corollaries

At first we derive some simple corollaries of (17–19) expressed by the structure constants of $R^{(0)}$:

a) $C^{wi}_{(0)s,lk} = C^{wi-k+1}_{(0)s,l1}$ for $i \geqq k$,[5]

b) $C^{wi}_{(0)s,lk} = 0$ for $i < k$,

c) $C^{wi-1}_{(0)s,ll} = 0$,

d) $C^{wi-k-1}_{(0)s,ll-k} = 0$,

e) $C^{w1}_{(0)rk+1,lk+1} = 0$,

f) $C^{wi}_{(0)rs+1,lk+1} = C^{wi-k}_{(0)rs+1,l1}$ for $s \geqq i > k$,

g) $C^{wi}_{(0)rs+1,lk+1} = 0$ for $s, k \geqq i$,

h) $C^{wi}_{(0)rs+1,lk+1} = C^{wi-s}_{(0)r1,lk+1}$ for $k \geqq i > s$,

i) $C^{wi}_{(0)rs+1,lk+1} - C^{wi-1}_{(0)rs+1,lk} = C^{wi-s}_{(0)r1,lk+1} - C^{wi-s-1}_{(0)r1,lk}$ for $i > s$,

j) $C^{wi}_{(0)rs+1,lk+1} - C^{wi-1}_{(0)rs,lk+1} = C^{wi-k}_{(0)rs+1,l1} - C^{wi-k-1}_{(0)rs,l1}$ for $i > k$,

k) $C^{wi-1}_{(0)rr-k,ll} = C^{wi-2}_{(0)rr-k,ll-1}$, $C^{wi-2}_{(0)rr-k,ll} = 0$
for $i \leqq N_1 - k$ where $0 \leqq k \leqq N_1 - 1$,

l) $C^{wi-1}_{(0)rr,ll-s} = C^{wi-2}_{(0)rr-1,ll-s}$, $C^{wi-2}_{(0)rr,ll-s} = 0$
for $i \leqq N_1 - s$ where $0 \leqq s \leqq N_1 - 1$,

m) $C^{wN_1-k}_{(0)rr-i+1,ll-j+1} = 0$ for $k \geqq i + j$.

We give here only proofs of the more difficult points: **a**), **d**), **f**), **g**), and **i**):

We rewrite (19) in the following form:

$$C^{wi}_{(0)rs+1,lk+1} - C^{wi-1}_{(0)rs+1,lk} = C^{wi-1}_{(0)rs,lk+1} - C^{wi-2}_{(0)rs,lk}$$

treating this like a recurrence formula we obtain (**i**) for $i > s$, for $i \leqq s$ we obtain

$$C^{wi}_{(0)rs+1,lk+1} - C^{wi-1}_{(0)rs+1,lk} = C^{w1}_{(0)rs-i+2,lk+1}\,.$$

---

[5] In the following $1 \leqq i \leqq N_1$ if not assumed other limitation.

It vanishes from point (e) so we obtain new recurrence formula which gives us (f) for $i > k$ and (g) for $k \geqq i$.

ad h) and **j)** Proof is analogous to the above given.

ad k): We prove this by induction over k:

1. For $k = 0$, $C^{wi-1}_{(0)rr,ll} = C^{wi-2}_{(0)rr,ll-1}$, $C^{wi-2}_{(0)rr,ll} = 0$ we obtain this directly from (19) by putting

$$s = r,\ k = l - 1 \quad \text{and} \quad s = r,\ k = l\,.$$

2. Let us assume that for $k = m: C^{wi-1}_{(0)rr-m,ll} = C^{wi-2}_{(0)rr-m,ll-1}$ $C^{wi-2}_{(0)rr-m,ll} = 0$ for $i \leqq N_1 - m$, on rewriting (19) for $s = r - m - 1,\ k = l - 1$ we obtain

$$C^{wi-1}_{(0)rr-mll} - C^{wi-2}_{(0)rr-m,ll-1} = C^{wi-2}_{(0)rr-m-1,ll} - C^{wi-3}_{(0)rr-m-1,ll-1}$$

now if $i \leqq N_1 - m$ then from inductive assumption we obtain

$$C^{wi'-1}_{(0)rr-(m+1),ll} - C^{wi'-2}_{(0)rr-(m+1),ll-1} = 0$$

for $i' \leqq N_1 - (m + 1)$, on rewriting (19) for $s = r - m - 1$, $k = l$ we obtain $C^{wi'-2}_{(0)rr-(m+1)} = 0$ for $i' \leqq N_1 - (m + 1)$ what ends our proof.

ad l): The proof is analogous to the given one for the point (k).

ad m): We can prove this starting from (**k**) or (**l**). We start from (**k**). We take $C^{wi'-2}_{(0)rr-i+1,ll} = 0$ for $i' \leqq N_1 - (i - 1)$. Now using first part of (**k**) as a recurrence formula we have $C^{wi'-2-j+1}_{(0)rr-i+1,ll-j+1} = 0$ for $i' \leqq N - i + 1$. Now we give the proofs of the corollaries:

ad 1.: We can prove this in one particular basis because these features of Lie algebra are basis independent. We see that in our basis the proof follows from the Eq. (18) and from the points (**b**) and (g).

ad 2.: In our basis it is equivalent to the point (**m**).

ad 3.: Follows immediately from (20) which implies that only $C^{ww}_{(0)\alpha,\beta} \neq 0$.

ad 4.: From (20) it follows that the $C^{\gamma}_{(1)\alpha,\beta}$ being a linear combination of $C^{\gamma}_{(0)\alpha,\beta}$ satisfy the equations (17–19), then also from (20) one can read that for $p > 1$ $C^{\gamma}_{(2)\alpha,\beta} = 0$, for $p = 0$ one finds $C^{\gamma}_{(1)\alpha,\beta}$, the connection between $C^{\gamma}_{(1)\alpha,\beta}$ and $C^{\gamma}_{(0)\alpha,\beta}$ gives an inductive relation between $C^{\gamma}_{(i)\alpha,\beta}$ and $C^{\gamma}_{(i-1)\alpha,\beta}$ what allows to prove easily by induction relation (21). In the same way one proves (21′).

ad 5.: All $R^{(m)}$ can be contracted by $A(\varepsilon)$ so for each of them one can prove Corollary 1.

ad 6.: We have to prove the following equality $C^{ws}_{(m)rr-i+1,ll-j+1} = 0$, for $m \geqq i, j$ and $s \leqq w - (i + j - m)$. To prove this we use (21) and the point ($\boldsymbol{m}$) for $p = 0$. From (21) we obtain $C^{wi}_{(m)rr-i+1,ll-j+1} = 0$ for $i \leqq w - m$, $C^{ww-c}_{(m)rr-i+1,ll-j+1} = -C^{ww-m+c}_{(0)rr-i+1+c,ll-j+1+c}$ for $m \geqq i$ and $m \geqq j$, now from point ($\boldsymbol{m}$) for $p = 0$ $C^{ww-m+c}_{(0)rr-i+1+c,ll-j+1+c} = 0$ for

$(m - c) \geqq i + j - 2c$ it means for $c \geqq i + j - m$ which ends our proof, because from (21) easily follows that $C^n_{(m)rr-i+1, ll-j+1} = 0$.

ad 7.: Follows from (6) because $S_k = \bigoplus_{i=1}^{k} A_i$.

ad 8.: Follows from (7) and from (21) (where from using (**b**) one obtains $[S_R, S_k]^{(m)} \subset S_k$ for $m \geqq k$).

5. The (S, p) Contractions for $\varepsilon_0 = \infty$

Now one can try to investigate a contraction by the same $A(\varepsilon)$ (15) but assuming $\varepsilon_0 = \infty$, we will signify this contraction $(S, p)_\infty$. To find the necessary and sufficient conditions for $(S, p)_\infty$ one has to investigate convergence of the equalities (16) as $\varepsilon \to \infty$. We see that limit of (2), as $\varepsilon \to \infty$, exists if and only if:

$$\begin{aligned} C^n_{(0)s,t} &= \delta_{p0} C^n_{(0)s,t}, \quad C^{wi}_{(0)s,lk} = \delta_{p0} C^{wi}_{(0)s,lk} \\ C^{wi}_{(0)s,t} &= (\delta_{p0} + \delta_{p1}) C^{wi}_{(0)s,t}, \\ C^n_{(0)rs,lk} &= C^{wi}_{(0)rs,lk} = C^n_{(0)s,lk} = 0 . \end{aligned} \tag{22}$$

New structure constants of $R^{(1)}$ can be found from (16) by the use of (22) in the form:

$$\begin{aligned} C^n_{(1)s,t} &= C^n_{(0)s,t} \delta_{p0}, \quad C^{wi}_{(1)s,t} = \delta_{p1} C^{wi}_{(0)s,t}, \\ C^{wi}_{(1)s,lk} &= \delta_{p0} C^{wi}_{(0)s,lk}, \\ C^n_{(1)rs,lk} &= C^{wi}_{(1)rs,lk} = C^n_{(1)s,lk} = 0 . \end{aligned} \tag{23}$$

All these conditions can be expressed in a basis independent way:

Theorem 3. *The algebra $R^{(0)}$ can be contracted by $(S, p)_\infty$ if and only if:*

a) *for $p > 1$ – $R^{(0)}$ is abelian,*

b) *for $p = 1$ – $[S_R, S_R]^{(0)} \subset S_N$ and $[S_N, S]^{(0)} = 0$,*

c) *for $p = 0$ – $[S_N, S_N]^{(0)} = 0$ and $[S_N, S_R]^{(0)} \subset S_N$.*

Theorem 4. *If we contract the algebra $R^{(0)}$ by $(S, p)_\infty$ we obtain a nonisomorphic algebra $R^{(1)}$ only for $p = 0$. For $p = 0$, $R^{(1)}$ satisfies the following commutation relations*

$$[S_R, S_R]^{(1)} \subset S_R, \ [S_N, S_N]^{(1)} = 0, \ [S_N, S_R]^{(1)} \subset S_N .$$

So $R^{(1)}$ is a semidirect product of subalgebra G_R (generated by S_R) by an abelian ideal A (generated by S_N) $R^{(1)} = A \ni\!\!\!+ G_R$.

The proofs of the preceding theorems are trivial.

Corollary 9. *To obtain $R^{(1)} = A \oplus G_R$* by $(S, 0)_\infty$ we have to demand $R^{(0)}$ to satisfy the following condition $[S_N, S]^{(0)} = 0$.*

As shown by Doebner and Melsheimer [1] this special case is interesting from the physical point of view.

---

* $\oplus$ means here a direct product of algebras.

6. Discussion

So we obtained the necessary and sufficient conditions for (S, p) and the brackets of $R^{(1)}$ expressed only by the structure constants of $R^{(0)}$. We hope that our equations (17–20) are more handy than the equations (9–12) and they do not involve the unknown matrix "u". We obtained 9 basis independent corollaries and 13 small corollaries expressed by the structure constants of $R^{(0)}$. Some of our corollaries were obtained for $p = 0$ by Saletan. Saletan didn't obtain the corollaries 2, 3, 4, 6, 9.

We hope that using our basis (13) one can try to investigate more general contraction for example by $A(\varepsilon) = (I \oplus u_N) + B(\varepsilon)$ where $B(\varepsilon)$ is a p-contraction.

We also investigated $(S, p)_\infty$ contractions but they turned out to be very restrictive and one obtains nontrivial results only for $p = 0$.

Acknowledgement. The author is very indebted to Professor J. WERLE and to Dr. R. RACZKA for their valuable discussions.

References

1. DOEBNER,, H. D., and O. MELSHEIMER: Nuov. Cimento **49**A, 306 (1967).
2. SEGAL, S. I.: Duke Math. J. 18, 256 (1951).
3. INONÜ, E., and E. P. WIGNER: Proc. Natl. Acad. Sci. U.S. **39**, 510 (1953).
4. SALETAN, E. J.: J. Math. Phys. **2**, 1 (1961).
5. HERMAN, R.: Commun. Math. Phys. **3**, 53 (1966).
6. LEVY-NAHAS, M.: J. Math. Phys. 8, 1211 (1967).
7. —, and R. SENEOR: Commun. Math. Phys. **9**, 242 (1968).
8. HEGERFELDT, G. C.: Nuovo Cimento, **51**A, 439 (1967).
9. HALMOS, P. R.: Finite-dimensional vector space Princeton 1958.

MARIAN KUPCZYŃSKI
Institute of Theoretical Physics
Warsaw University,
ul. Hoza 69
Warsaw, Poland