ON THE GENERATION OF INFINITE DIMENSIONAL BILINEAR SYSTEMS AND VOLTERRA SERIES.

by

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ABSTRACT

The bilinearisation of infinite dimensional nonlinear systems defined on a Hilbert space H is examined and Volterra series on ℓ^2 are also considered. This will be achieved by introducing a new Hilbert space $L^2_{\underline{w}}[H; \mathbb{R}]$.

(1) Introduction

In this paper we shall be concerned with nonlinear systems and their linear and bilinear representations by nonlinear state transformations. Several methods have been proposed to introduce transformations which will linearise a nonlinear system; see for example, Hunt et al (1983) and Takata (1979). These depend to a large extent on a global inverse function theorem, as discussed in Sandberg (1981).

In order to generalise the ideas of Takata (1979) to infinite dimensional systems we shall introduce the Hilbert space $L^2_{\underline{w}}[H; \mathbb{R}]$ which consists of functions from (a subset) of H into $\mathbb{R}$ such that

$$\int_{H \cap D(f)} \underline{w}(h) \mid f(h) \mid^2 dh < \infty$$

where D(f) is the domain of f and h is an infinite dimensional measure defined on H. The function $\underline{w}$ is a weighting function and will be introduced in the paper. The space $L^2_{\underline{w}}[H;H]$ will be defined as the space of functions $f = D(f) \subseteq H \to H$ such that $\langle f, e_i \rangle \in L^2_{\underline{w}}[H; \mathbb{R}]$ for all i, where $\{e_i\}$ is a basis of (the assumed separable) space H. We shall prove that $L^2_{\underline{w}}[H; \mathbb{R}]$ is, in fact, a Hilbert space and exhibit an explicit basis in terms of a basis of $L^2_w[\mathbb{R}; \mathbb{R}]$ where w is a scalar weighting function. A typical basis of the latter space can be expressed in terms of Hermite polynomials.

If

$$x = f(x) \quad , \quad x \in H \tag{1.1}$$

is a differential equation defined on H, then, denoting a basis of $L^2_{\underline{w}}[H;\mathbb{R}]$ by $\{\psi_i\}$, the idea is to replace (1.1) with the system

$$\dot{\psi} = F_x \psi \dot{x} = F_x f(x)$$ (F denotes Fréchet derivative)

Then, if $F_x f(x) \in L^2_{\underline{w}}[H; H]$, we can write

$$(F_x f(x))_i = \sum_{i=1}^{\infty} \psi_j a_{ij}$$

and so

$$\dot{\psi} = A\psi.$$

We shall then consider a certain class of nonlinear control systems and show that, by using the above technique, these can be 'reduced' to bilinear systems. This generalises Brockett's (1976) Volterra series expansion of linear analytic systems. We shall end with a simple example and show how to calculate explicitly the bilinearisation.

In this paper we shall use the standard notation of Hilbert space theory. In particular, $L^2[-\infty,\infty]$ will denote the space of real valued functions f such that

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty$$

and ℓ^2 will denote the space of sequences $\{x_i\}$ such that

$$\Sigma x_i^2 < \infty.$$

Note that any separable Hilbert space is isomorphic to ℓ^2, and so we shall use the latter space as a standard model whenever necessary.

<u>(2) Infinite-Dimensional Integration</u>.

The space $L^2_w[-\infty, \infty]$ is well known to have a basis consisting of a sequence of polynomials ϕ_n with respect to some weighting function w ; i.e. any function $f \varepsilon L^2_w[-\infty, \infty]$ may be written in the form

$$f = \sum_{i=1}^{\infty} < \phi_i, f > \phi_i ,$$

where

$$< f,g > = \int_{-\infty}^{\infty} f(t)g(t)w(t)dt.$$

If the sequence ϕ_n is chosen to be orthonormal with respect to w (for example, the Hermite functions He_n, with weighting function $w = e^{-x^2/2}$), then any function $f : \mathbb{R} \to \mathbb{R}$, which belongs to $L^2_w [-\infty, \infty]$ has an orthogonal series expansion with coefficients $\langle \phi_i, f \rangle$.

In this section we shall be interested in generalising this result to the case of functions $f : H \to \mathbb{R}$ for some separable Hilbert space H, and hence to define a space

$$L^2_{\underline{w}} [H ; \mathbb{R}]$$

of functions $f : H \to \mathbb{R}$ such that

$$\int_H f^2(h)\, \underline{w}(h)\, dh < \infty$$

for some weighting function $\underline{w}$ and some measure dh on H.

In order to define such a space we first note that if $\{e_i\}_{i=1}^{\infty}$ is a basis of H, we may regard H as the space ℓ^2 under the standard isomorphism $\mu : H \to \ell^2$ given by

$$\mu(h) = \{h_i\}_{i=1}^{\infty}$$

where

$$h = \sum_{i=1}^{\infty} \langle h, e_i \rangle e_i$$

and

$$h_i = \langle h, e_i \rangle .$$

Any function $f : H \to \mathbb{R}$ can then be represented equivalently as a function $f : \ell^2 \to \mathbb{R}$ (we shall use the same letter f for either function, since no confusion is likely) given by

$$f(x_1, x_2, x_3, \ldots) = f(x)$$

where $x \in H$ corresponds to $(x_1, x_2, x_3, \ldots) \in \ell^2$ under the above isomorphism.

<u>Definition 2.1.</u> We shall call a function $f : H \to \mathbb{R}$ <u>eventually constant</u> if there exists, for any $\varepsilon > 0$, a function $g_\varepsilon : \mathbb{R}^{n(\varepsilon)} \to \mathbb{R}$ such that

$$|f(x_1, x_2, \ldots) - g_\varepsilon(x_1, x_2, \ldots, x_{n(\varepsilon)})| \leq \varepsilon \qquad (2.1)$$

for all $x = (x_1, x_2, \ldots) \in H \; (=\ell^2)$. Note that both the function g and the dimension of its domain depend on ε.

Remark 2.2. If $f = D(f) \subseteq H \to \mathbb{R}$ is not defined on the whole of H then we require (2.1) to hold only on D(f).

Example 2.3. Let $w \in L^2[-\infty,\infty]$ be such that

(i) $w(0) = 1, \quad w(t) \geq 0$

(ii) $w(t) < 1, \quad t \neq 0$

(iii) if $\{t_i\} \in \ell^2$ then $\prod_{i=n}^{\infty} w(t_i)$ converges for each $n \geq 1$ and $\lim_{n\to\infty} \prod_{i=n}^{\infty} w(t_i)$ exists and is independent of $\{t_i\}$.

Then the function $\underline{w}: H \to \mathbb{R}$ defined by

$$\underline{w}(x_1, x_2, \ldots) = \prod_{i=1}^{\infty} w(x_i)$$

is well defined and eventually constant.

This follows easily for if $x \in H$, $\prod_{i=1}^{m} w(x_i)$ is clearly uniformly bounded by M, say, in m and x and if $\alpha = \lim_{n\to\infty} \prod_{i=n}^{\infty} w(x_i)$ (which is independent of x) then, if $\varepsilon > 0$, choose n so that $|\alpha - \prod_{i=n+1}^{\infty} w(x_i)| < \frac{\varepsilon}{M}$ and we have

$$\left| \prod_{i=1}^{\infty} w(x_i) - \alpha \prod_{i=1}^{n} w(x_i) \right| \leq \left| \prod_{i=1}^{n} w(x_i) \right| \left| \prod_{i=n+1}^{\infty} w(x_i) - \alpha \right| \leq \varepsilon .$$

The function $e^{-t^2/2} \in L^2[-\infty,\infty]$ clearly satisfies (i) -(iii) and so the function $\underline{e} : H \to \mathbb{R}$ defined by

$$\underline{e}(x_1, x_2, \ldots) = \prod_{i=1}^{\infty} e^{-x_i^2/2}$$

is well-defined on H and eventually constant (with constant $\alpha = 1$).

Definition 2.4 Let $f : H \to \mathbb{R}$. Then we say that f is integrable over H if, for any $\varepsilon > 0$ there exists a finite set $I_1 \subseteq N$ of natural numbers such that for any finite set I with $I_1 \subseteq I \subseteq N$ we have

$$\Big| \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{\#(I)} f(x) \prod_{i \in I} \left(\frac{dx_i}{\omega}\right) - c \Big| \leq \varepsilon, \ \forall (x_i)_{i \notin I} \in \ell^2 . \tag{2.2}$$

for some constant c. ($\#(I)$ denotes the candinality of I).

Intuitively, we are requiring that the integral of f over any finite-dimensional subspace of H should exist and be eventually constant, with the constant function c taking the place of g_ε (the purpose of the factor $1/\omega$ will become clear presently.) c is called the <u>integral of f over H</u> or just the integral of f and we write

$$c = \int_H f(x)\,dh$$

where $$dh = \prod_{i=1}^{\infty}\left(\frac{dx_i}{\omega}\right).$$

<u>Definition 2.5</u> The space $L^2_{\underline{w}}[H;\mathbb{R}]$ is the space of functions $f : H\to\mathbb{R}$ such that

$$\int_H f^2(x)\,\underline{w}(x)\,dh<\infty$$

where $\underline{w}$ satisfies the conditions of example 2.3 and the measure dh is given by $dh = \prod_{i=1}^{\infty}\left(\frac{dx_1}{\omega}\right)$, where

$$\omega = \int_{-\infty}^{\infty} w(t)\,dt$$

In particular we shall be interested in the space $L^2_{\underline{e}}[H;\mathbb{R}]$ where $\underline{e}$ is defined as above and $dh = \prod_{i=1}^{\infty}\left(\frac{dx_i}{\sqrt{2\pi}}\right)$.

<u>Example 2.6</u> Let us first note that $\int_H \underline{e}(x)\,dh$ exists and has the value 1. In fact, if $I = \{1,\dots,n\}$, then

$$\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n}\{e^{-x_1^2/2}\,e^{-x_2^2/2}\cdots e^{-x_n^2/2}\}\,e^{-(x_{n+1}^2/2+\cdots)}\times$$

$$\frac{dx_1}{\sqrt{2\pi}}\cdots\frac{dx_n}{\sqrt{2\pi}}$$

$$= e^{-(x_{n+1}^2/2+\cdots)}$$

and so condition (2.2) is clearly satisfied if n is large enough, since $\{x_i\} \in \ell^2$.

We would now like to prove that $L^2_{\underline{w}}[H;\mathbb{R}]$ is separable which we shall do by specifying a basis. Let us first introduce some notation. If $\mathbb{N}_+ = \{1,2,\ldots\}$ denotes the set of positive natural numbers, then $\mathbb{N}^{\omega}_+$ will denote the subset of $\overset{\infty}{\underset{i=1}{X}}\mathbb{N}_+$ (i.e. the countable Cartesian product of copies of $\mathbb{N}_+$) consisting of those $\underline{n} \in \overset{\infty}{\underset{i=1}{X}}\mathbb{N}_+$ $(\underline{n}=(n_1,n_2\ldots))$ for which $n_i=1$ for all but finitely many i. Now let $\{\phi_i\}_{1\le i<\infty}$ be a basis of $L^2_w[-\infty,\infty]$, with weighting function w (for example, the Hermite functions as above). Then we shall show that

$$\mathcal{B} = \{\prod_{i=1}^{\infty} \phi_{n_i}(x_i)\}_{\underline{n}\in\mathbb{N}^{\omega}_+} \quad , \underline{n} = (n_1,n_2\ldots)$$

is a basis of $L^2_{\underline{w}}[H;\mathbb{R}]$. Note first that, since $\mathbb{N}^{\omega}_+$ is countable, we can order $\mathcal{B}$ in the form $\{\psi_i(x)\}_{i\in\mathbb{N}_+}$, where

$$\psi_i(x) = \prod_{j=1}^{\infty} \phi_{n_j}(x_j) \quad \text{for some } \underline{n}\in\mathbb{N}^{\omega}_+. \tag{2.3}$$

We shall use either representation of $\mathcal{B}$ without further comment.

<u>Theorem 2.7</u> $L^2_{\underline{w}}[H;\mathbb{R}]$ is a separable Hilbert space with basis $\mathcal{B}$.

<u>Proof</u> The only nontrivial part to prove is completeness; i.e. Parseval's relation

$$\|f\|^2 = \sum_{i=1}^{\infty} |\langle f,\psi_i\rangle|^2, \quad f \in L^2_{\underline{w}}[H;\mathbb{R}].$$

Let $\varepsilon > 0$ and choose n so that

$$\Big| \underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n} f^2(x_1,\ldots x_n,x_{n+1},\ldots)\underline{w}(x)dx_1\ldots dx_n - c\Big| \le \frac{\varepsilon}{3}$$

for any $(x_{n+1},x_{n+2},\ldots)\in\ell^2$, as in the definition (2.2) of the integral, where $c = \|f\|^2$. Also, if $g(x_1,\ldots x_n)$ is any function of n variables such that $g \in L^2_w[\mathbb{R}^n;\mathbb{R}]$ then let $\{\psi_i\}_{i\in I_g}$, for some finite set $I_g \subset \mathbb{N}_+$, be a subset of the basis so that

$$\left| \, \|g\|^2_{L^2_w[\mathbb{R}^n;\mathbb{R}]} - \sum_{i \in I_g} |\langle g, \psi_i \rangle|^2 \right| \leq \frac{\varepsilon}{3}$$

(Note that the set I_g is chosen so that the ψ_i's depend only on $x_1, \ldots, x_n$; this is clearly possible. Of course, I_g depends on g.) Hence we have

$$\left| \, \|f\|^2 - \sum_{i \in I_{f_n}} |\langle f_n, \psi_i \rangle|^2 \right|$$

$$\leq \left| c - \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f^2(x_1, \ldots, x_n, x_{n+1}, \ldots) \underline{w}(x) dx_1 \ldots dx_n \right|$$

$$+ \left| \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f^2(x_1, \ldots x_n, x_{n+1}, \ldots) \underline{w}(x) dx_1 \ldots dx_n \right.$$

$$\left. - \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f^2(x_1, \ldots, x_n, 0, 0, \ldots) \prod_{i=1}^{n} w(x_i) dx_1 \ldots dx_n \right|$$

$$+ \left| \, \|f_n\|^2 - \sum_{i \in I_{f_n}} |\langle f_n, \psi_i \rangle|^2 \right|$$

$$\leq \varepsilon \quad ,$$

where we have written

$$f_n(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, 0, 0, \ldots) \quad , \quad (x_1, \ldots, x_n) \in \mathbb{R}^n .$$

The result now follows. □

<u>Definition 2.8</u> Let $f: H \to H$ and let $f^i(x) = \langle f(x), e_i \rangle$, $x \in H$, where $\{e_i\}$ is a basis of H. Then, if $f^i \in L^2_{\underline{w}}[H;\mathbb{R}]$ for each i we say that $f \in L^2_{\underline{w}}[H;H]$.

In the next section we shall consider nonlinear differential equations and linear equivalent systems.

(3) Differential Equations on a Hilbert Space

In this section we shall consider the differential equation

$$\dot{x}(t) = f(x(t)) \qquad x(t_o)=x_o \varepsilon H \tag{3.1}$$

where H is a Hilbert space and suppose that $f(x)e^{\|x\|^2/\varepsilon}$ $\varepsilon\, L^2_{\underline{w}}[H;H]$ for some $\varepsilon>o$. Let $\mathcal{B}=\{\psi_i(x)\}$ be the basis of $L^2_{\underline{w}}[H;\mathbb{R}]$ introduced above. Then, for each i, $\psi_i : H\to\mathbb{R}$ and so the Fréchet derivative $F\psi_i$ of ψ_i is a map from H into $\mathcal{L}(H,\mathbb{R})\cong H$. Now, along a solution of (3.1), we have

$$\frac{d\psi_i}{dt}(x) = F\psi_i(x)\,\frac{dx}{dt} = F\psi_i(x)\ f(x(t)) \overset{\Delta}{=} f_i(x),$$

where $f_i : H\to\mathbb{R}$. (This is a standard trick. See for example, Takata, 1979). Note that this makes sense, since $F\psi_i(x)$: $H\to\mathcal{L}(H,\mathbb{R})$ and f : $H\to H$.

Lemma 3.1. $f_i \ \varepsilon\ L^2_{\underline{w}}\,[H;\mathbb{R}]$.

Proof. Let $\{e_i\}$ be a basis of H. Then, by definition $f^i(x) = \langle f(x), e_i\rangle \ \varepsilon L^2_{\underline{w}}\,[H;\ \mathbb{R}]$. Also, if $x = \Sigma x_i e_i$, then $\psi_i(x)$ depends only on a finite number of the x_i's, say $x_1, \ldots, x_n$. Hence we can write

$$F\psi_i(x) = \left(\frac{\partial\psi_i}{\partial x_1}(x), \frac{\partial\psi_i(x)}{\partial x_2}, \ldots , \frac{\partial\ _i(x)}{\partial x_n}, 0,0,\ldots\right) \varepsilon\, \ell^2 \cong H$$

and so

$$f_i(x) = \sum_{j=1}^{n} \frac{\partial\psi_i}{\partial x_j}(x)\ f^j(x) . \tag{3.2}$$

Now each $\partial\psi_i/\partial x_j$ is a polynomial and so is dominated by $e^{-\|x\|^2/\varepsilon}$. Hence each term in the sum (3.2) is in $L^2_{\underline{w}}\,[H,\mathbb{R}]$ and therefore, so is $f_i(x)$. □

Since $f_i\,\varepsilon\, L^2_{\underline{w}}\ [H;\ \mathbb{R}\ ]$, we can write

$$f_i(x) = \sum_{j=1}^{\infty} \langle f_i(x),\ \psi_j(x)\rangle\ \psi_j(x)$$

$$= \sum_{j=1}^{\infty} a_{ij}\ \psi_j(x), \text{ say}$$

where $a_{ij} = \langle f_i,\ \psi_j\rangle_{L^2_{\underline{w}}[H;\mathbb{R}]}$.

Hence, equation (3.1) is equivalent to the system of linear equations

$$\frac{d\psi_i(x)}{dt} = \sum_{j=1}^{\infty} a_{ij}\psi_j(x) \quad , \quad \psi_i(x(o)) = \psi_i(x_o) \qquad (3.3a)$$

or,

$$\frac{d\Xi}{dt} = A\Xi \quad , \quad \Xi_i(o) = \psi_i(x_o) \qquad (3.3b)$$

where

$$A = (a_{ij})_{1 \le i,j < \infty}$$

and

$$\Xi(t) = (\psi_1(x(t)) \ , \ \psi_2(x(t)),\ldots)^T \triangleq \psi(x(t)). \qquad (3.4)$$

We have therefore 'reduced' the original nonlinear differential equation (3.1) to a linear countable system of differential equations (3.3) and the algebraic problem of solving (3.4). Then,

$$x(t) = \psi^{-1}(\Xi(t)) \ ,$$

provided that ψ is invertible on Range ψ . (Note that ψ maps H into $\mathbb{R}^{\infty}$* and that Range ψ may not equal $\mathbb{R}^{\infty}$.) In order to consider (3.3) on a Hilbert space we must ensure that $\psi(x_o)$ belongs to this space. Consider the space ℓ_c^2 defined as the set of sequences $\{x_i\}$ such that $\sum_{i=1}^{\infty} c_i x_i^2 < \infty$, where $\{c_i\}$ is chosen to satisfy $\sum_{i=1}^{\infty} c_i \psi_i^2(x_o) < \infty$. Clearly c_i depends on x_o, which is the initial condition of the original equation. (For example, we could take $c_i = \left[1/(\psi_i(x_o))^2 i^2\right]$.) We can then consider equation (3.3b) on ℓ_c^2, for a given initial condition x_o of (3.1). Alternatively we can consider the equation

$$\frac{d(c_i\psi_i(x))}{dt} = \sum_{j=1}^{\infty} \frac{c_i}{c_j} a_{ij}(c_j\psi_j(x)) \ ,$$

$$c_i\psi_i(x(o)) = c_i\psi_i(x_o) \qquad (3.5)$$

on ℓ^2. Hence the matrix A generates a semigroup on ℓ_c^2 if and

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* We denote the space of sequences $\{x_i\}_{i=1}^{\infty}$, where $x_i \in \mathbb{R}$, by $\mathbb{R}^{\infty}$.

only if the matrix ($\frac{c_i}{c_j} a_{ij}$) generates a semigroup on ℓ^2. Note, however, that not all solutions of the linear equation (3.5) will correspond to solutions of (3.1). Only those with initial conditions in Range ψ will be related to a corresponding solution of the nonlinear equation.

Relating existence results for equations (3.1) and (3.3) is not particularly easy, but we can use the following result of Shinderman (1968), in order to prove existence theorems for the nonlinear equation.

Lemma 3.2 Consider the countable system

$$\dot{x}_i = - a_{ii} x_i - \sum_{j \neq i} a_{ij} x_j \quad , \; i \geq 1,$$

where Re $a_{ii} \geq \sum_{j=i} \max \{|a_{ij}|, |a_{ji}|\}$ and $\sum_{j \neq i} |a_{ij} a_{jj}^{-1}| < 1$ for $i \geq 1$.

Then $A = (a_{ij})$ generates a contraction semigroup. □

This lemma then implies

Theorem 3.3. Suppose that $\psi : H \rightarrow \text{Range } \psi \subseteq \mathbb{R}^\infty$ is invertible and assume that the conditions of lemma 3.2 are satisfied with

$$a_{ij} = \langle f_i , \psi_j \rangle_{L^2_w [H;\mathbb{R}]} \cdot \frac{\psi_j (x_o)^2 j^2}{\psi_i (x_o)^2 i^2} \quad (\text{for } \psi_i (x_o) \neq o, \forall i)$$

Then equation (3.1) has a solution $x(t)$ with $x(O) = x_o$ defined on $[o,T)$ provided the solution $\Xi(t)$ of $\frac{d\Xi}{dt} = A\Xi$, $\Xi(o) = \psi(x_o)$ belongs to Range ψ on $[o,T)$.

Proof This follows directly from lemma 3.2 since the conditions imply that the equation (3.3b) has a solution on $[o,\infty)$, since A generates a contraction semigroup. This will then imply the existence of a solution of (3.1) on an interval $[o,T)$ for as long as $\Xi(t)$ is in Range ψ.

Alternatively, suppose we know that the nonlinear equation (3.1) has unique local solutions, for $t \in [o,T)$ and for $x_o \in U \subseteq H$ for some set U, which are continuous in t and x_o, and assume again that $\psi : H \to \mathbb{R}^\infty$ is invertible on Range ψ. Then let

$$\overline{U} = \psi^{-1}(\ell^2) \cap U$$

and define

$$m = \overline{sp}\{\psi(\overline{U})\} \subseteq \ell^2 ,$$

where $\overline{sp}$ denotes the closed linear span of a subset of ℓ^2. Then we have

Theorem 3.4 If the system (3.3) with $\Xi(o) = o$ has the unique solution $\Xi(t) = o$, and $\overline{U}$ is invariant, then the matrix A in (3.3) generates a strongly continuous semigroup on m.

Proof Using the above notation, let $x_o \in \overline{U}$. Then (3.1) has a solution on $[o,T)$ by assumption and, therefore, so does (3.3). Denote the solution of (3.1) by $x(t;x_o)$ and that of (3.3) by $T(t)\psi(x_o)$. Then $x(t;x_o) = \psi^{-1}(T(t)\psi(x_o))$. However, by uniqueness of the solutions of (3.1), $x(t;x_o)$ satisfies the dynamical system property

$$x(t+\tau;x_o) = x(t;x(\tau;x_o)).$$

Hence

$$\psi^{-1}(T(t+\tau)\psi(x_o)) = \psi^{-1}(T(t)\psi\psi^{-1}(T(\tau)\psi(x_o)).$$
$$= \psi^{-1}T(t)T(\tau)\psi(x_o)$$

and so, for each $x_o \in \overline{U}$, $T(t)\psi(x_o)$ satisfies the semigroup property. Moreover, by continuity of $x(t;x_o)$ in t and x_o, it is clear that T(t) is strongly continuous. By uniqueness of (3.3) we can extend T(t) to m (using the strong continuity and linearity), and the result follows, since T(t) is clearly invariant on m by the invariance of $\overline{U}$ under the dynamics of (3.1). □

(4) Reducing Nonlinear systems to Bilinear Systems

In this section we shall consider the nonlinear control problem

$$\dot{x} = f(x) + g(x)u \quad , \quad x\varepsilon H \tag{4.1}$$

where

$$f(x)e^{\|x\|^2/\varepsilon} \quad , \quad g(x)\ e^{\|x\|^2/\varepsilon} \quad \varepsilon L^2_{\underline{w}}\ [H\ ;\ H]$$

for some $\varepsilon > o$. Then we may write

$$\frac{d\psi}{dt} = F\psi(x)\ \frac{dx}{dt} = F\psi(x)(f(x) + g(x)u)$$

$$= \quad A\psi \quad + uB\psi \ , \tag{4.2}$$

where

$$(A\psi)_i = \langle F\psi(x)f(x),\ \psi_i(x)\rangle_{L^2_{\underline{w}}\ [H;\mathbb{R}]}$$

and

$$(B\psi)_i = \langle F\psi(x)g(x)\ ,\ \psi_i(x)\rangle_{L^2_{\underline{w}}\ [H;\mathbb{R}]}$$

Hence we can reduce the nonlinear problem (4.1) to the bilinear system defined by (4.2). The system (4.2) defines, in the usual way (Brockett, 1976), the Volterra series

$$\psi(t) = w_o(t) + \sum_{n=1}^{\infty} \int_o^t \cdots \int_o^t w_n(t,\sigma_1,\ldots,\sigma_n)u(\sigma_1)\ldots u(\sigma_n)\cdot d\sigma_1\ldots d\sigma_n \tag{4.3}$$

where

$$w_n(t,\sigma_1,\ldots\sigma_n) = T(t-\sigma_1)BT(\sigma_1-\sigma_2)B\ldots BT(\sigma_n)\psi_o$$

for $t \geq \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$ and $w_n = O$ otherwise. This series is defined for all $u\varepsilon L^\infty[o,t]$, if B is bounded, for then,

$$\|\psi(t)\| \leq \|w_o(t)\| + \sum_{n=1}^{\infty} \|u\|_\infty^n \|B\|^n \int_o^t \cdots \int_o^{\sigma_{n-2}} \int_o^{\sigma_{n-1}} Me^{\omega t} \cdot d\sigma_1 \cdots d\sigma_n$$

$$= \|w_o(t)\| + \sum_{n=1}^{\infty} \|u\|_\infty^n \|B\|^n \cdot Me^{\omega t} \cdot t^n/_{n!}$$

$$= \|w_o(t)\| + M \exp\{\omega t + \|B\| \, \|u\|_\infty t\}$$

where

$$\|u\|_\infty = \underset{s\varepsilon[o,t]}{\text{ess sup}} \, |u(s)| \, .$$

Alternatively, if B is not bounded on ℓ^2 but is a bounded operator as a map from ℓ^2 into some space W and T(t) is a semigroup which satisfies $\|T(t)\|_{\mathcal{L}(W,\ell^2)} \leq g(t)$, where $g\varepsilon L^1_{\ell oc}[o,\infty]$, then, by estimating (4.3) again, we have

$$\|\psi(t)\| \leq \|w_o(t)\| + \sum_{n=1}^{\infty} \|u\|_\infty^n \|B\|^n_{\mathcal{L}(W,\ell^2)} \|g*g*\ldots*g\|_{L^1[o,t]}$$

$$\leq \|w_o(t)\| + \sum_{n=1}^{\infty} \|u\|_\infty^n \|B\|^n \frac{\|g\|}{n!}{}_{L^1[o,t]} \, C^n$$

We have assumed that g also satisfies

$$\|g*g*\ldots*g\|_{L^1[o,t]} \leq \frac{\|g\|}{n!}{}_{L^1[o,t]} \cdot C^n$$

where * denotes convolution and C is some constant. (An example of such a function g(t) is $1/t^\alpha$, $\alpha<1$, and then $C = 1+\alpha$.

The system

$$\frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial x^2} + u\phi\delta(x-x_1)$$

is a bilinear system whose semigroup generated by $\partial^2/\partial x^2$ has this property.)

The series (4.3) is obtained, of course from the mild form

$$\psi(t) = T(t)\psi_o + \int_o^t T(t-s)u(s)B\psi(s)ds \qquad (4.4)$$

by Picard iteration. If we let P_n denote the projection

$$P_n x = (x_1, x_2, \ldots x_n, 0, 0, \ldots)$$

for $x = (x_1, x_2, \ldots) \varepsilon \ell^2$, then we can also consider the finite dimensional system

$$\psi_m(t) = T_m(t)\psi_{om} + \int_o^t T_m(t-s)u_m(s)B_m\psi_m(s)\, ds \qquad (4.5)$$

where

$$T_m(t) = P_m T(t) P_m \;,\; \psi_{om} = P_m \psi_o \;,\; B_m = P_m B P_m$$

Then we obtain the Volterra series

$$\psi_m(t) = w_o^m(t) + \sum_{n=1}^{\infty} \int_o^t \ldots \int_o^t w_n^m(t, \sigma_1, \ldots, \sigma_n) u_m(\sigma_1) \ldots u_m(\sigma_n) \cdot d\sigma_1 \ldots d\sigma_n \qquad (4.6)$$

where

$$w_n^m(t, \sigma_1, \ldots, \sigma_n) \begin{cases} = T_m(t-\sigma_1) B_m T_m(\sigma_1 - \sigma_2) B_m \ldots B_m T_m(\sigma_n)\psi_{om}, & t \geq \sigma_1 \geq \ldots \geq \sigma_n \\ = 0 & \text{otherwise} \end{cases}$$

Note that, in general,

$$T_m(t) \neq \exp(A_m t).$$

where

$$A_m = P_m A P_m \;.$$

<u>Lemma 4.1</u> The operator $T_m(t-\sigma_1)B_mT_m(\sigma_1-\sigma_2)B_m\ldots B_mT_m(\sigma_n)$ converges strongly to $T(t-\sigma_1)BT(\sigma_1-\sigma_2)B\ldots BT(\sigma_n)$ uniformly on compact sets in $\mathbb{R}^{n+1}$.

<u>Proof</u>. We shall prove that $T_m(t) \xrightarrow{s} T(t)$ (strong convergence) uniformly on compact intervals. The lemma then follows easily by induction. Since $T(t)$ is a semigroup we have $\|T(t)\| \leq Me^{\omega t}$ for some M,ω and so $\|T(t)\|$ is bounded on compact intervals. However, if $x\varepsilon\ell^2$, we have

$$\|T(t)x - P_mT(t)P_mx\| \leq \|T(t)x - P_mT(t)x\| + \|P_mT(t)x - P_mT(t)P_mx\|$$
$$\leq \|T(t)x - P_mT(t)x\| + \|P_m\| \, \|T(t)\| \, \|x-P_mx\| .$$

The second term on the right hand side clearly converges to zero, uniformly in t. The first term also converges to zero; however, if the convergence is not uniform for all x on the compact interval $[0,\tau]$, say, then there exists an x and $\varepsilon>0$ such that, for each m, there is a $t_m\varepsilon\ [0,\tau]$ such that

$$\|T(t_m)x - P_mT(t_m)x\| \geq \varepsilon .$$

The sequence $\{t_m\}$ has a cluster point, say $\bar{t}$, in $[0,\tau]$ and so

$$\varepsilon \leq \|T(t_m)x - P_mT(t_m)x\| \leq \|T(t_m)x-T(\bar{t})x\| + \|T(\bar{t})x-P_mT(\bar{t})x\|$$
$$+ \|P_mT(t_m)x-P_mT(\bar{t})x\|$$
$$\leq (1+\|P_m\|)(\|T(t_m)x-T(\bar{t})x\|) + \|T(\bar{t})x-P_mT(\bar{t})x\|$$

for any m. The result now follows by the strong continuity of $T(t)$. □

An elementary argument now shows that the Volterra series (4.6) of the finite dimensional approximations to equation (4.4) converges uniformly on compact intervals to the Volterra series (4.3) of equation (4.4). We can therefore approximate the infinite dimensional system with a finite dimensional Volterra series.

The above methods are important theoretically, but when evaluating a bilinearisation of a nonlinear system in practice we will usually not know T(t) explicitly and hence will not be able to determine $P_n T(t) P_n$. Hence we can revert to the (poorer) approximation

$$\dot{\psi}_n = P_n A P_n \psi_n + u_n P_n B P_n \psi_n$$

and obtain, for example, suboptimal controls as in Takata (1979). In the next section we shall evaluate a simple example and show that even in simple cases B is likely to be unbounded. We shall then appeal to the existence theory of the nonlinear system to guarantee that the bilinearisation also has solutions, since finding a space W such that the function g satisfies the above conditions will be difficult.

(5) Example

In this section we shall illustrate the bilinearisation of the system

$$\frac{\partial T}{\partial t} = (1+\alpha T)\frac{\partial^2 T}{\partial x^2} + uT \qquad (5.1)$$

on the interval $x \in [0, \pi,]$ with $T(o) = T(\pi) = 0$. We have taken B equal to the identity for simplicity of exposition; this is not necessary, of course. This equation represents heat flow with temperature dependent thermal conductivity. This equation is well known to have global unique solutions (in $L^2[0,\pi]$ for example). First, we recall the explicit definition and standard formula for the Hermite polynomials (cf. Abramowitz and Stegun, 1965). These are:

(i) Definition. $He_n(x) = n! \sum_{m=o}^{[n/2]} (-1)^m \dfrac{1}{m!\,2^m (n-2m)!} x^{n-2m}$

(ii) Orthogonality $\int_{-\infty}^{\infty} He_n(x) He_m(x) e^{-x^2/2} \dfrac{dx}{\sqrt{2\pi}} = n!\,\delta_{nm}$

(iii) Recurrence relation $He_{n+1}(x) = x\,He_n(x) - n\,He_{n-1}(x)$.

Hence we obtain, for the first four polynomials:

$He_o(x) = 1$

$He_1(x) = x$

$He_2(x) = x^2-1$

$He_3(x) = x^3 - (n+1)x.$

We shall write

$$f_{n+1}(x) = (n!)^{-\frac{1}{2}} He_n(x),$$

so that the sequence $\{f_n\}$ is orthonormal.

Next, we must specify the basis of $L^2_{\underline{w}}[H;\mathbb{R}]$ where

$$\underline{w}(T) = \underline{w}(T_1,T_2, \ldots) = \prod_{i=1}^{\infty} \exp(-T_i^2/2)$$

and $H = L^2[0,\pi]$, with

$$T = \sum_{i=1}^{\infty} T_i\{\sqrt{\tfrac{2}{\pi}} \sin ix\}.$$

A basis of $L^2_{\underline{w}}[H;\mathbb{R}]$ is specified by (2.3). The only thing remaining is to decide on the order of the functions $\psi_i(T)$. (Note that we shall now use T rather than x for the elements of $H = L^2[0,\pi]$.)

The order which we choose is the following:

Define

$$\psi_1(T) = f_1(T_1)\ f_1(T_2)f_1(T_3) \ldots \equiv 1$$

and proceeding by induction suppose that, for $n>2$ given, the basis vectors $\psi_i(T)$ have been defined for $1\le i\le(n-1)^{(n-1)}$. Then we shall define $\psi_i(T)$ for $(n-1)^{(n-1)}<i\le n^n$. Let $\overline{S}_n$ denote the set

$$\overline{S}_n = \{(i_1,i_2\ldots,i_n) : 1\le i_j\le n,\ 1\le j\le n\}$$

and let S_n be given by

$$S_n = \overline{S}_n\backslash\{(i_1,i_2,\ldots i_n): i_j\neq n,\ 1\le j\le n-1 \text{ and } i_n = 1\}.$$

It is easy to see that S_n has $n^n-(n-1)^{(n-1)}$ elements. We order $\overline{S}_n$ in the obvious way, namely $\{\{\ldots\{\{(i_1, i_2, \ldots, i_n), 1\le i_1\le n\}, 1\le i_2\le n\} \ldots \}, 1\le i_{n-}\le n\}$

i.e. i_1 varies first, then i_2, etc. and S_n is given the induced order from $\overline{S}_n$. Then we define

$$\psi_{i+(n-1)^{(n-1)}}(T) = f_{i_1}(T_1)\ldots f_{i_n}(T_n)\, f_1(T_{n+1})\, f_i(T_{n+2})\ldots$$
$$= f_{i_1}(T_1)\ldots f_{i_n}(T_n) \quad , \quad 1 \le i \le n^n-(n-1)^{(n-1)}$$

where $(i_1, \ldots, i_n)$ is the ith element of S_n in the above ordering. Note that in the above ordering we include terms (at the nth stage) of the form

$$f_{i_1}(T_1) \ldots f_{i_n}(T_n)$$

which have not already appeared. These correspond to the indices in the set S_n. We shall now illustrate this ordering up to n=3, which will give 27 terms, as follows:

n=1 $\{$ $\psi_1 = f_1f_1f_1$

n=2:
$\psi_2 = f_2f_1f_1 \leftarrow f_1f_1f_1$
$\psi_3 = f_1f_2f_1$
$\psi_4 = f_2f_2f_1$

n=3:
$\psi_5 = f_3f_1f_1 \leftarrow f_1f_1f_1,\ f_2f_1f_1$
$\psi_6 = f_3f_2f_1 \leftarrow f_1f_2f_1,\ f_2f_2f_1$
$\psi_7 = f_1f_3f_1$
$\psi_8 = f_2f_3f_1$
$\psi_9 = f_3f_3f_1$
$\psi_{10} = f_1f_1f_2$
$\psi_{11} = f_2f_1f_2$
$\psi_{12} = f_3f_1f_2$
$\psi_{13} = f_1f_2f_2$
$\psi_{14} = f_2f_2f_2$
$\psi_{15} = f_3f_2f_2$

cont'd

$$
n=3 \quad \left|\begin{aligned}
\Psi_{16} &= f_1 f_3 f_2 \\
\Psi_{17} &= f_2 f_3 f_2 \\
\Psi_{18} &= f_3 f_3 f_2 \\
\Psi_{19} &= f_1 f_1 f_3 \\
\Psi_{20} &= f_2 f_1 f_3 \\
\Psi_{21} &= f_3 f_1 f_3 \\
\Psi_{22} &= f_1 f_2 f_3 \\
\Psi_{23} &= f_2 f_2 f_3 \\
\Psi_{24} &= f_3 f_2 f_3 \\
\Psi_{25} &= f_1 f_3 f_3 \\
\Psi_{26} &= f_2 f_3 f_3 \\
\Psi_{27} &= f_3 f_3 f_3
\end{aligned}\right.
$$

Note that, in the above sequence we have omitted the arguments of the functions and so, for example, ψ_{12} should be interpreted as

$$
\begin{aligned}
\psi_{12}(T) &= f_3(T_1) f_1(T_2) f_2(T_3) f_1(T_4) f_1(T_5) \ldots\ldots \\
&= f_3(T_1) f_2(T_3),
\end{aligned}
$$

since $f_1 \equiv 1$. We have also indicated on the right those basis vectors in group n which have been removed because they have appeared before in the sequence.

Returning now to the system (5.1), consider first the term $g(T) = T$. Then, by (3.2), we must find the coefficients of

$$
B_i \triangleq \sum_{j=1}^{n} \frac{\partial \psi_i(T)}{\partial T_j} T_j
$$

with respect to the basis $\{\psi_i\}$, or

$$
B_{ij} \triangleq \left\langle \sum_{k=1}^{n} \frac{\partial \psi_i}{\partial T_k}(T) T_k \; , \; \psi_j(T) \right\rangle \tag{5.2}
$$

Let us examine the first few terms of (5.2). Now $\psi_1 \equiv 1$ so that $B_{ij} = 0$ for all j. Also, $\psi_2(T) = f_2(T_1)$ and so

$$B_{2j} = \left\langle \frac{df_2(T_1)}{dT_1} T_1, \psi_j \right\rangle = \left\langle T_1, \psi_j(T) \right\rangle$$

$$= \left\langle \psi_2(T), \psi_j(T) \right\rangle = \delta_{2j},$$

$$B_{3j} = \left\langle \frac{df_2(T_2)}{dT_2} T_2, \psi_j(T) \right\rangle = \left\langle \psi_3(T), \psi_j(T) \right\rangle = \delta_{3j}$$

$$B_{4j} = \left\langle f_2(T_2) \frac{df_2}{dT_1}(T_1)T_1 + \frac{df_2(T_2)}{dT_2} T_2 f_2(T_1), \psi_j(T) \right\rangle$$

$$= \langle 2f_2(T_1)f_2(T_2), \psi_j(T) \rangle = 2 \langle \psi_4(T), \psi_j(T) \rangle = 2\delta_{4j} .$$

The remaining B_{ij}'s can be found in a similar manner, but it should be noted that the resulting linear operator B is not bounded. Hence we see that the above linearising method does not map bounded operators to bounded operators. We must now determine the term

$$A_{ij} \triangleq \left\langle \sum_{k=1}^{n} \frac{\partial \psi_i(T)}{\partial T_k} \left((1+ \alpha T) \frac{\partial^2 T}{\partial x^2} \right)_k , \psi_j(T) \right\rangle$$

where n is again the maximal number of nonzero derivatives of ψ_i. Now, in terms of the basis $\sqrt{2/\pi} \sin i x$ of $L^2 [0,\pi]$ introduced above, we have

$$\frac{T \partial^2 T}{\partial x^2} = - \left(\Sigma T_i \sqrt{2/\pi} \sin i x \right) \left(\Sigma T_1 (i)^2 \sqrt{2/\pi} \sin i x \right)$$

$$= - \Sigma_i \Sigma_j T_i T_j \frac{2}{\pi} (j)^2 \sin(i x) \sin(j x).$$

However,

$$\sin i x \sin j x = \sum_{\ell=1}^{\infty} S(i,j,\ell) \sin \ell x,$$

where

$$4S(i,j,\ell) = \frac{1-(-1)^{i+j+\ell}}{i+j+\ell} + \frac{(-1)^{i+j-\ell}-1}{i+j-\ell}$$

$$+ \frac{1-(-1)^{i-j-\ell}}{i-j+\ell} + \frac{(-1)^{i-j-\ell}-1}{i-j-\ell}$$

and these terms where the denominator is zero are to be interpreted as 0. Hence,

$$\frac{T\partial^2 T}{\partial x^2} = -\frac{2}{\pi}\sum_{\ell=1}^{\infty}\left(\Sigma_i \Sigma_j\ T_i T_j (j)^2 S(i,j,\ell)\right) \sin \ell x \quad ,$$

and so

$$(1+\alpha T)\frac{\partial^2 T}{\partial x^2} = -\sum_{\ell=1}^{\infty}\left(T_\ell \ell^2 + \alpha\sqrt{\frac{2}{\pi}}\ \Sigma_i \Sigma_j\ T_i T_j\ (j)^2 S(i,j,\ell)\right)\sqrt{\frac{2}{\pi}}\ \sin \ell x.$$

Hence

$$\left((1+\alpha T)\ \frac{\partial^2 T}{\partial x^2}\right)_k = -\left(T_k k^2 + \alpha\sqrt{\frac{2}{\pi}}\ \Sigma_i \Sigma_j\ T_i T_j (j)^2 S(i,j,k\ )\right)$$

and we may then evaluate the terms A_{ij} above in just the same way as before.

(6) Conclusions

In this paper we have defined the space $L^2_{\underline{w}}[H;\mathbb{R}]$ and have determined a basis of this space in terms of Hermite polynomials. We have then used the space to determine infinite dimensional bilinearisations of nonlinear distributed systems and conditions under which bilinear systems defined on ℓ^2 define Volterra series have been examined. Finally a simple example to illustrate the bilinearisation of a nonlinear system has been given.

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