Progress of Theoretical Physics, Vol. 51, No. 5, May 1974

On the Generation of Sound from the Primordial Turbulence in an Expanding Universe. II

——*Refinement of the Formalism*——

Hidekazu NARIAI

Research Institute for Theoretical Physics
Hiroshima University, Takehara, Hiroshima-ken

(Received October 17, 1973)

The aim of this paper is to remedy two defects of our formalism for dealing with the generation of sound from the primordial cosmic turbulence. One is the fact that the true retarded Green's function in the Friedmann universe to describe the propagation of sound waves to be generated has not explicitly been settled. Another is the fact that the cosmological counterpart of Crighton's formula for the intensity of sound at the center of a static turbulent fluid sphere has been derived by invoking the principle of reciprocity to that of Lighthill's intensity formula whose underlying condition cannot be satisfied in the real universe. It is shown that the refinement leads to a very simple formula for the intensity of sound at the center of the expanding turbulent fluid sphere within the acoustic event horizon, which shows further that the main contribution to the intensity comes from the acoustic past "time-like" region rather than from the acoustic past "light-cone".

§ 1. Introduction

In a previous paper[1] (which is referred to as [I] in what follows), the generation of sound from the primordial cosmic turbulence has been dealt with by construction of the appropriate retarded Green's function into which all significant influences of the cosmic expansion upon the propagation of sound waves to be generated are woven. For doing so, we have rewritten an inhomogeneous wave equation for the density contrast in a Friedmann universe as that for a massless scalar wave in an auxiliary universe, the retarded Green's function for the latter wave being constructable by our procedure.[2] In the case when the observation point is far from the source region, it has been shown that the retarded solution for the latter wave is reduced to a generalized version (cf. Eq. (4·10) of [I]) of the Lighthill solution[3] for the density contrast in the case of a static turbulent medium, from which the intensity of sound at the observation point (cf. Eq. (5·4) of [I]) is derivable. In order to derive a generalized version (cf. Eq. (7·1) of [I]) of Crighton's formula[4] for the intensity of sound at the center of a static turbulent fluid sphere from that of Lighthill's, we have appealed to the principle of reciprocity between a source point and an observation point. Since the effect of cosmic expansion is very significant, we have concluded that the problem whether the primordial cosmic turbulence can survive against its acoustic decay must be studied on the basis of our formalism rather than Crighton's.

However, our formalism is rather devoid of transparency in the sense that we have used the retarded Green's function for a massless scalar wave in the auxiliary universe, but not the one for the density contrast in the Friedmann universe.*) In addition, like Lighthill's formula, our formula for the intensity of sound at the center of the expanding turbulent fluid sphere within the acoustic event horizon has been derived by invoking the principle of reciprocity (whose definition is not necessarily unique) to Eq. (5·4) of [I] whose underlying condition specified by the phrase "in the case when the observation point is far from the source region" cannot be satisfied in the real universe. Such a procedure has been taken for the preservation of parallelism with the Lighthill-Crighton theory, but it is physically unreasonable in the cosmological problem. The aim of this paper is to remedy the above two defects of our formalism.

In § 2 various mathematical expressions derived in [I] are summarized. In § 3, by the use of the mathematical expressions in § 2, the retarded solution of the density contrast in the Friedmenn universe and its Fourier component are obtained in such a way that the retarded Green's function for the sound wave and its Fourier component appear explicitly in the respective solutions. Section 4 is devoted to the rederivation of Eq. (4·10) in [I] for illustrating the substantial equivalence between the refined formalism and the original one. In § 5 the formula for the intensity of sound at the center of the expanding turbulent fluid sphere within the acoustic event horizon in the Friedmann universe is derived directly from the retarded solution obtained in § 3 for the density contrast, in contrast with the situation in the derivation of the old formula, i.e., Eq. (7·1) of [I]. In addition, the new formula is much simpler than the old one for numerical analysis. Accordingly we shall prefer the new formula to the old one.

§ 2. Mathematical expressions in [I]

Let us consider a Friedmann universe with flat 3-space which consists of matter and radiation whose background densities are represented by $\rho_{Bm}=\rho^*_{Br}z^{-3}$ and $\rho_{Br}=\rho^*_{Br}z^{-4}$, respectively, where $z\equiv a(t)/a(t_*)$ stands for the scale factor normalized in such a way that $\rho_{Bm}=\rho_{Br}$ at $t=t_*$. Then its metric and the one for the auxiliary universe mentioned already are of the form

$$\begin{cases}(\text{Friedmann universe}):\ ds^2=-c^2dt^2+a^2(t)\,\delta_{ij}dx_idx_j\,,\\ (\text{Auxiliary universe})\ :\ d\tilde{s}^2=-d\tau^2+R^2(\tau)\,\delta_{ij}dx_idx_j\,,\end{cases}\tag{2·1}$$

*) The relation (A·10) in the Appendix of [I] between two Green's functions $\tilde{D}_{\text{ret}}(\tau, \boldsymbol{x}\,;\,\tau', \boldsymbol{x}')$ and $G_{\text{ret}}(t, \boldsymbol{x}\,;\,t', \boldsymbol{x}')$ must be abandoned, because the latter Green's function is shown to be incompatible with Eq. (A·8), as will be shown in § 3 (e.g., Eqs. (3·2) and (3·5)).

where

$$\begin{cases} dt=(3T_*/2)\dfrac{zdz}{\sqrt{1+z}}, & (T_*\equiv 1/\sqrt{6\pi G\rho^*_{Br}}) \\ v_s=c(\zeta/3)^{1/2}, & \zeta\equiv(1+3z/4)^{-1}, \end{cases} \tag{2·2}$$

and

$$d\tau=(Rv_s/a)dt\,,\qquad R=R_*(1+3z/4)^{1/4}\propto v_s^{-1/2}\,, \tag{2·3}$$

in which v_s stands for the sound velocity and $R_*=\text{const}>0$. It has been shown in [I] that Eqs. (2·2) and (2·3) provide us with the following approximate expression:

$$R=(\tau+\tau_0)/r_0\,,\qquad \{a(t_*)r_0\equiv 4cT_*/\beta\} \tag{2·4}$$

where $\beta=1$ for $z\gg 1$ or $\zeta\simeq 0$ (the matter dominant stage) or $\beta=\sqrt{3}/2$ for $z\ll 1$ or $\zeta\simeq 1$ (the radiation dominant stage) and $\tau_0\equiv(1-\beta)r_0$.

On the other hand, it has been shown in [I] that the density contrast $K\equiv\rho_m(t,\boldsymbol{x})/\rho_{Bm}(t)-1$ in the Friedmann universe must satisfy the following inhomogeneous wave equation:

$$\overset{(s)}{\Box}K\equiv\{-\partial_t^2-(2-\zeta)(\dot{a}/a)\partial_t+(v_s/a)^2\nabla^2\}K=-(v_s/ar_0)^2S\,, \tag{2·5}$$

where S is the dimensionless quantity defined by

$$S\equiv(r_0/v_s)^2(v_iv_j)_{,ij} \tag{2·6}$$

in which $v_i=v_i(t,\boldsymbol{x})$ stands for the physical velocity field. By the use of Eqs. (2·2) and (2·3), we can reduce Eq. (2·5) to

$$\tilde{\Box}K\equiv\{-\partial_\tau^2-3(R'/R)\partial_\tau+R^{-2}\nabla^2\}K=-(Rr_0)^{-2}S\,, \tag{2·7}$$

which is the inhomogeneous wave equation in the auxiliary universe for a massless scalar wave, as mentioned already.

To obtain the retarded solution of Eq. (2·7), we must know the retarded Green's function satisfying

$$\tilde{\Box}\tilde{D}_{\text{ret}}(\tau,\boldsymbol{x};\tau',\boldsymbol{x}')=-\{R(\tau)R(\tau')\}^{-3/2}\delta(\tau-\tau')\delta^3(\boldsymbol{x}-\boldsymbol{x}'). \tag{2·8}$$

It has been shown in [I] that, when $R(\tau)$ is given by Eq. (2·4), the requisite Green's function is of the form

$$\tilde{D}_{\text{ret}}(\tau,\boldsymbol{x};\tau',\boldsymbol{x}')=\frac{\theta(\eta)e^\eta}{4\pi(\tau+\tau_0)^2}\left\{\frac{\delta(\eta-\xi)}{\xi}+\theta(\eta-\xi)i_1(\sqrt{\eta^2-\xi^2})\right\} \tag{2·9}$$

with

$$\xi\equiv r/r_0\equiv|\boldsymbol{x}-\boldsymbol{x}'|/r_0\,,\quad \eta\equiv\ln\left(\frac{\tau+\tau_0}{\tau'+\tau_0}\right)=\frac{1}{2}\ln\{v_s(t')/v_s(t)\}\,, \tag{2·10}$$

where $\theta(\eta-\xi)$ is the step function such that $d\theta(x)/dx=\delta(x)$ and $I_1(z)\equiv zi_1(z)$ is the modified Bessel function of first order. As pointed out in [I], the above

Green's function shows that the space-time point $(\tau, \boldsymbol{x})$ is influenced not only from points on the past light-cone $\eta=\xi$, but also from points in the past time-like region $\eta>\xi$. It is to be noticed that Eq. (2·9) is reduced to

$$\tilde{D}_{\rm ret}(\tau, \boldsymbol{x}; \tau', \boldsymbol{x}') = \frac{\sqrt{3}/c}{4\pi(R_* r_0)^2}\theta(\eta)\{v_s(t)v_s(t')\}^{1/2}\{\frac{\delta(\eta-\xi)}{\xi} + \theta(\eta-\xi)i_1(\sqrt{\eta^2-\xi^2})\}, \tag{2·9'}$$

in the right-hand side of which the auxiliary time variables (τ, τ') have been eliminated.

§ 3. Retarded solution for the density contrast and its Fourier component

By the use of the retarded Green's function given by Eq. (2·9), we can integrate Eq. (2·7) as

$$K_{\rm ret}(\tau, \boldsymbol{x}) = \int_{R(\tau')\geq R_*} R^3(\tau')d\tau' \int d\boldsymbol{x}'\tilde{D}_{\rm ret}(\tau, \boldsymbol{x}; \tau', \boldsymbol{x}')\{R(\tau')r_0\}^{-2}S(\tau', \boldsymbol{x}').$$

On inserting Eq. (2·9′) into the above expression and making use of Eq. (2·3), we obtain

$$K_{\rm ret}(t, \boldsymbol{x}) = \int_0^\infty du(t') \int d\boldsymbol{x}' G_{\rm ret}(t, \boldsymbol{x}; t', \boldsymbol{x}')v_s^{-2}(t')S(t', \boldsymbol{x}') \tag{3·1}$$

with

$$G_{\rm ret}(t, \boldsymbol{x}; t', \boldsymbol{x}') \equiv \frac{\theta(\eta)}{4\pi r_0^3}\{v_s(t)v_s(t')\}^{1/2}\left\{\frac{\delta(\eta-\xi)}{\xi} + \theta(\eta-\xi)i_1(\sqrt{\eta^2-\xi^2})\right\} \tag{3·2}$$

where

$$du(t) \equiv \frac{v_s^2(t)}{a(t)r_0}dt \tag{3·3}$$

and $K_{\rm ret}(t, \boldsymbol{x})$ and $S(t', \boldsymbol{x}')$ are abbreviations of $K_{\rm ret}\{\tau(t), \boldsymbol{x}\}$ and $S\{\tau'(t'), \boldsymbol{x}'\}$, respectively. Similarly, by the use of Eq. (3·3), we can reduce Eq. (2·5) to

$$\overset{(u)}{\Box} K \equiv \{-\partial^2_{u(t)} + (r_0/v_s)^2\nabla^2\}K = -v_s^{-2}S. \tag{3·4}$$

Since $K_{\rm ret}(t, \boldsymbol{x})$ given by Eq. (3·1) satisfies the rewritten wave equation (3·4), it is the required retarded solution, provided that we can prove that $G_{\rm ret}(t, \boldsymbol{x}; t', \boldsymbol{x}')$ defined by Eq. (3·2) is the retarded Green's function satisfying

$$\overset{(u)}{\Box} G_{\rm ret}(t, \boldsymbol{x}; t', \boldsymbol{x}') = -\delta\{u(t)-u(t')\}\delta^3(\boldsymbol{x}-\boldsymbol{x}'). \tag{3·5}$$

In place of the direct proof of Eq. (3·5), let us consider the Fourier components of $K_{\rm ret}(t, \boldsymbol{x})$, $S(t, \boldsymbol{x})$ and $G_{\rm ret}(t, \boldsymbol{x}; t', \boldsymbol{x}')$, i.e.,

$$\begin{cases} A_{\boldsymbol{k}}(t) \equiv \int e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} K_{\mathrm{ret}}(t, \boldsymbol{x})\, d\boldsymbol{x}\,, \\ B_{\boldsymbol{k}}(t) \equiv \int e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} S(t, \boldsymbol{x})\, d\boldsymbol{x} \end{cases} \tag{3·6}$$

and

$$G_{\boldsymbol{k}}(t, t') \equiv \int e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} G_{\mathrm{ret}}(t, \boldsymbol{x}; t', \boldsymbol{x}')\, d\boldsymbol{r}\,, \quad (\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}')$$

$$= \theta(\eta)\, \{v_s(t)\, v_s(t')\}^{1/2} \sin(\gamma_k \eta)/\gamma_k\,, \tag{3·7}$$

where $\gamma_k \equiv \{(kr_0)^2 - 1\}^{1/2}$. Then we can reduce Eqs. (3·1) and (3·4) to

$$A_{\boldsymbol{k}}(t) = \int_0^\infty du(t')\, G_{\boldsymbol{k}}(t, t')\, v_s^{-2}(t')\, B_{\boldsymbol{k}}(t') \tag{3·8}$$

and

$$L_u A_{\boldsymbol{k}}(t) \equiv \{d^2_{u(t)} + (kr_0/v_s)^2\}\, A_{\boldsymbol{k}}(t) = v_s^{-2} B_{\boldsymbol{k}}(t)\,, \tag{3·9}$$

respectively. Similarly, from Eq. (3·5), we obtain

$$L_u G_{\boldsymbol{k}}(t, t') = \delta\{u(t) - u(t')\}\,. \tag{3·10}$$

Thus it may be said that, if $G_{\boldsymbol{k}}(t, t')$ is shown to be the one-dimensional retarded Green's function satisfying Eq. (3·10), $G_{\mathrm{ret}}(t, \boldsymbol{x}; t', \boldsymbol{x}')$ defined by Eq. (3·2) is the required 4-dimensional retarded Green's function.

Since $G_{\boldsymbol{k}}(t, t')$ given by Eq. (3·7) has the requisite symmetric property, we have only to examine the validity of Eq. (3·10). For that purpose, let us rewrite Eq. (3·7) as follows:

$$G_{\boldsymbol{k}}(t, t') = v_s(t')\, \theta(\eta)\, e^{-\eta} \sin(\gamma_k \eta)/\gamma_k\,, \tag{3·7'}$$

where we have used the second of Eq. (2·10) or $v_s(t) = v_s(t') \exp(-2\eta)$. On the other hand, it follows from Eqs. (2·2) and (3·3) that

$$v_s(t)\, \partial_{u(t)} \eta = (1/\beta)\, (1 - \zeta/4)^{1/2} = 1\,, \tag{3·11}$$

in which the second equality becomes rough unless $\zeta \simeq 0$ or $\zeta \simeq 1$, just as the expression (2·4) for $R(\tau)$. In addition, we may put $\delta\{u(t) - u(t')\} = (dt/du) \times \delta(t - t') = (\partial_u \eta)\, \delta(\eta)$. Now it is an easy matter to check that $G_{\boldsymbol{k}}(t, t')$ given by Eq. (3·7′) satisfies Eq. (3·10). Thus it has been shown that $G_{\mathrm{ret}}(t, \boldsymbol{x}; t', \boldsymbol{x}')$ defined by Eq. (3·2) is the required retarded Green's function for dealing with the propagation of sound waves in the Friedmann universe.

Remark In the treatment of the formation of galaxies from the primordial turbulence, Silk and Ames[5] too have touched upon the Green's function to solve an inhomogeneous equation corresponding to Eq. (3·9). However, their one-dimensional Green's function (cf. $G(t, t')$ defined by Eq. (45)*)) is merely a formal device, contrary to $G_{\boldsymbol{k}}(t, t')$ given by Eq. (3·7) which may play a

*) The factor $\theta(t - t')$ is devoid of their expression.

significant role, as will be shown in § 5. It would not be useless to point out that the fixation of a new time variable $u(t)$ defined by Eq. (3·3) has been essential in the derivation of our Green's function $G_{\rm ret}(t, \boldsymbol{x}; t', \boldsymbol{x}')$ and its Fourier component $G_{\boldsymbol{k}}(t, t')$.

§ 4. Rederivation of Eq. (4·10) in [I] for $K_{\rm ret}(t, x)$

In order to show the substantial equivalence between the refined formalism obtained in § 3 and the original one in [I], we shall rederive Eq. (4·10) of [I] from Eq. (3·1).

On inserting Eqs. (2·6) and (3·2) into Eq. (3·1) and making use of Eq. (3·3), we obtain

$$K_{\rm ret}(t, \boldsymbol{x}) = \frac{1}{4\pi v_s^2(t)} \frac{\partial^2}{\partial x_i \partial x_j} \int \frac{d\boldsymbol{r}}{r} \int_0^{\eta_m} d\eta e^{-5\eta} \{\delta(\eta-\xi) + \theta(\eta-\xi)\xi i_1(\sqrt{\eta^2-\xi^2})\}$$
$$\times v_i(t', \boldsymbol{x}+\boldsymbol{r}) v_j(t', \boldsymbol{x}+\boldsymbol{r}), \tag{4·1}$$

where η_m is the maximum value of η defined by

$$\eta_m \equiv \tfrac{1}{2} \ln\{c/\sqrt{3} v_s(t)\}. \tag{4·2}$$

In the above $d\eta$-integral, the part $\delta(\eta-\xi)$ represents the contribution from the acoustic past "light-cone" and another part $\propto \theta(\eta-\xi)$ that from the acoustic past "time-like" region. As in [I], we shall assume here that the second contribution can be approximated by a form of the first contribution. For doing so, let us consider the following integral:

$$X \equiv \int_0^{\eta_m} d\eta e^{-5\eta} \{\delta(\eta-\xi) + \theta(\eta-\xi)\xi i_1(\sqrt{\eta^2-\xi^2})\} = e^{-5\xi}\{1 + \xi \sum_{n=0} Y_n(\xi)\}, \tag{4·3}$$

where

$$Y_0 = 10^{-1}\{1 - e^{5(\xi-\eta_m)}\},$$
$$Y_1 = 10^{-3}[1 + 5\xi - e^{5(\xi-\eta_m)}\{1 + 5\eta_m + (25/2)(\eta_m^2 - \xi^2)\}], \tag{4·4}$$
$$\cdots\cdots$$

which satisfy the condition $|Y_n| \lesssim |Y_0|^n \sim 10^{-n}$ $(n=1, 2, \cdots)$. Accordingly we may approximately put

$$X = \int_0^{\eta_m} d\eta e^{-5\eta} [1 + (\eta/10)\{1 - e^{5(\eta-\eta_m)}\}] \delta(\eta-\xi). \tag{4·5}$$

Taking the above situation in view, let us assume that Eq. (4·1) can be approximated by

$$K_{\rm ret}(t, \boldsymbol{x}) = \frac{1}{4\pi v_s^2(t)} \frac{\partial^2}{\partial x_i \partial x_j} \int \frac{d\boldsymbol{r}}{r} e^{-5\xi} [1 + (\xi/10)\{1 - e^{5(\xi-\eta_m)}\}]$$
$$\times \{v_i(t', \boldsymbol{x}+\boldsymbol{r}) v_j(t', \boldsymbol{x}+\boldsymbol{r})\}_{\eta=\xi}, \tag{4·6}$$

where $\eta=\xi$ or $v_s(t')=v_s(t)\exp(2\xi)$ stands for the acoustic past "light-cone".

In spite of its unphysical nature (cf. § 1), we shall impose at this stage a condition that the space-time point $(t, \boldsymbol{x})$ is far from any point in the source (turbulent) region with non-vanishing $\boldsymbol{v}(t', \boldsymbol{x}+\boldsymbol{r})$. Then we can reduce Eq. (4·6) to

$$K_{\text{ret}}(t, \boldsymbol{x})=\frac{1}{4\pi v_s^4(t)}\cdot\frac{a^2(t)x_i x_j}{|\boldsymbol{x}|^3}\int d\boldsymbol{r}J(\xi, \eta_m)[\{a(t')/a(t)\}^2 \times\frac{\partial^2}{\partial t'^2}\{v_i(t', \boldsymbol{x}+\boldsymbol{r})v_j(t', \boldsymbol{x}+\boldsymbol{r})\}]_{t'=t-h(r,t)} \tag{4·7}$$

with

$$J(\xi, \eta_m)\equiv e^{-9\xi}[1+(\xi/10)\{1-e^{5(\xi-\eta_m)}\}], \qquad (\xi\equiv r/r_0) \tag{4·8}$$

where $h(r, t)$ is the retardation function defined by Eq. (4·12) in [I]; e.g., when $z(t)\gtrsim 10$, we have

$$h(r, t)=a(t)\sigma(r)/v_s(t), \quad \sigma(r)\equiv(r_0/6)(1-e^{-6\xi}). \tag{4·9}$$

Equation (4·7) is equivalent to Eq. (4·10) of [I], while $J(\xi, \eta_m)$ given by Eq. (4·8) is somewhat different from the original one derived in a more rough approximation.

§ 5. The intensity of sound at the center of the expanding turbulent fluid sphere within the acoustic event horizon

In the following, by the use of Eq. (3·1), but not its reduced form given by Eq. (4·7), for $K_{\text{ret}}(t, \boldsymbol{x})$, we shall derive the formula for the intensity of sound at the center of the expanding turbulent fluid sphere within the acoustic event horizon defined by

$$l_s(t)\equiv\eta_s(t)a(t)r_0, \quad \eta_s(t)\equiv\tfrac{1}{2}\ln\{v_s(t_i)/v_s(t)\}, \tag{5·1}$$

where t_i stands for an initial epoch, the acoustic wave emitted at $\boldsymbol{x}'$ prior to which instant cannot arrive at $\boldsymbol{x}$ at the time t.

Now let us define the required intensity by

$$I(t)\equiv\rho_{Bm}(t)v_s^3\overline{\{K_{\text{ret}}(t, \boldsymbol{x})-\overline{K}\}^2}, \tag{5·2}$$

where the over-bar stands for the mean value of a turbulent quantity. By the use of Eqs. (3·6) and (3·8), we have

$$\overline{\{K_{\text{ret}}(t, \boldsymbol{x})-\overline{K}\}^2}=(2\pi)^{-6}\int_0^\infty\frac{du(t')}{v_s^2(t')}\iint d\boldsymbol{k}d\boldsymbol{k}'e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}} \times\{G_{\boldsymbol{k}}(t, t')G_{-\boldsymbol{k}'}(t, t')\}^{1/2}\{\overline{B_{\boldsymbol{k}}(t')B_{-\boldsymbol{k}'}(t')}-\overline{B_{\boldsymbol{k}}(t')}\cdot\overline{B_{-\boldsymbol{k}'}(t')}\}, \tag{5·3}$$

where we have taken account of the situation that the random character of the velocity field $\boldsymbol{v}(t, \boldsymbol{x})$ and its associated quantity $B_{\boldsymbol{k}}(t)$ refers to their spatial orientations rather than their time-dependences.[6] On the other hand, it follows from Eqs. (2·6) and (3·6) that

$$B_{\boldsymbol{k}}(t) = -\{r_0/v_s(t)\}^2 k_i k_j \int d\boldsymbol{x} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} v_i(t,\boldsymbol{x}) v_j(t,\boldsymbol{x}). \qquad (5\cdot4)$$

As in [I], let us assume that the following decomposition for velocity correlations is permissible:

$$\overline{v_i v_j v_m' v_n'} - \overline{v_i v_j} \cdot \overline{v_m' v_n'} = R_{im}(t,\boldsymbol{r}) R_{jn}(t,\boldsymbol{r}) + R_{in}(t,\boldsymbol{r}) R_{jm}(t,\boldsymbol{r}) \qquad (5\cdot5)$$

with

$$\overline{v_i v_j'} \equiv R_{ij}(t,\boldsymbol{r}) = v^2(t)\{-(f'/2r) r_i r_j + (f + rf'/2)\delta_{ij}\}, \qquad (5\cdot6)$$

which holds in an isotropic and incompressible turbulence, where $v_i \equiv v_i(t,\boldsymbol{x})$, $v_j' \equiv v_j(t,\boldsymbol{x}+\boldsymbol{r})$ and $f' \equiv \partial f(t,r)/\partial r$. By making use of Eqs. (5·4) $\sim$ (5·6), we have

$$\overline{B_{\boldsymbol{k}}(t') B_{-\boldsymbol{k}'}(t')} - \overline{B_{\boldsymbol{k}}(t')} \cdot \overline{B_{-\boldsymbol{k}'}(t')} = (8\pi^2)^2 \{v(t')/v_s(t')\}^4$$
$$\times \delta^3(\boldsymbol{k}'-\boldsymbol{k})(kr_0)^4 \int_0^{r_s(t')} r^2 dr \left[\left\{f^2 + \frac{2rf'(f-rf')}{(kr)^2} + \frac{6(rf')^2}{(kr)^4}\right\}\frac{\sin(kr)}{kr}\right.$$
$$\left. - \left\{2rff' + \frac{6(rf')^2}{(kr)^2}\right\}\frac{\cos(kr)}{(kr)^2}\right], \qquad (5\cdot7)$$

where $r_s(t') \equiv l_s(t')/a(t')$. On inserting Eqs. (3·7) and (5·7) into Eq. (5·3) and performing a lengthy but straightforward computation, we obtain

$$\overline{\{K_{\rm ret}(t,\boldsymbol{x}) - \overline{K}\}^2} = 2\int_0^{\eta_s(t)} M^4(t') C(t',\eta) e^{-\eta} d\eta(t') \qquad (5\cdot8)$$

with

$$M(t') \equiv v(t')/v_s(t') \quad (:\text{Mach number}) \qquad (5\cdot9)$$

and

$$C(t',\eta) \equiv \int_0^\eta [(\xi^2 \hat{f})^2 i_5(\sqrt{\eta^2-\xi^2}) - \xi^2(\xi\partial_\xi \hat{f}^2 + 10\hat{f}^2) i_4(\sqrt{\eta^2-\xi^2})$$
$$+ \{2(\xi\partial_\xi \hat{f})^2 + 5\xi\partial_\xi \hat{f}^2 + 15\hat{f}^2\} i_3(\sqrt{\eta^2-\xi^2})]\xi^2 d\xi, \qquad (5\cdot10)$$

where $\eta_s(t) \equiv l_s(t)/a(t) r_0$, $d\eta(t') \equiv -\{v_s(t')/a(t') r_0\} dt'$ as before, $\hat{f} = \hat{f}(t',\xi) \equiv f(t',r)$ and $I_\nu(z) \equiv z i_\nu(z)$ is the modified Bessel function of the ν-th order. It is to be noticed that the right-hand side of Eq. (5·8) depends only on the time variable t, by virtue of our reliance on Eq. (3·1).

On inserting Eq. (5·8) into Eq. (5·2), we obtain the required formula

$$I(t) = 2\rho_{Bm}(t) v_s^3(t) \int_0^{\eta_s(t)} M^4(t') C(t',\eta) e^{-\eta} d\eta(t'). \qquad (5\cdot11)$$

The above intensity formula is far simpler than the original one, i.e., Eq. (7·1) of [I]. In addition, the appearance of $i_\nu(\sqrt{\eta^2-\xi^2})$ ($\nu = 3,4,5$) in the expression

(5·10) for $C(t', \eta)$ shows that the main contribution to the intensity $I(t)$ comes from the acoustic past "time-like" region rather than from the acoustic past "light-cone", in a striking contrast with the common situation in the original formalism and in the Lighthill-Crighton theory for a static turbulent medium.

On the basis of our theory[7] of the cosmic subsonic turbulence, we can in principle determine the two turbulent quantities $\hat{f}(t', \xi)$ and $v(t') \equiv M(t') v_S(t')$. Then, from Eqs. (5·10) and (5·11), the intensity can be evaluated, but the required analysis will be done in a separate paper.

References

1) H. Nariai, Prog. Theor. Phys. **50** (1973), 472.
2) H. Nariai and T. Kimura, Prog. Theor. Phys. **29** (1963), 296.
3) M. J. Lighthill, Proc. Roy. Soc. **A211** (1952), 564; *Proc. I.A.U. Symposium No. 2, Cambridge, 1953* (North-Holland Publ. Co., Amsterdam, 1955).
4) D. G. Crighton, Proc. Cambridge Phil. Soc. **65** (1969), 557.
5) J. Silk and S. Ames, Astrophys. J. **178** (1973), 77.
6) I. Proudman, Proc. Roy. Soc. **A214** (1952), 119.
7) H. Nariai, Sci. Rep. Tohoku Univ. **39** (1956), 213; **40** (1956), 40.
H. Nariai and K. Tomita, Prog. Theor. Phys. Suppl. No. 49 (1971), 83.