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ON THE GENERATION OF WATER WAVES AT AN INERTIAL SURFACE

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Abstract

In this paper we develop the Laplace-transform method to solve initial-value problems for the velocity potential describing the generation of infinitesimal capillary-gravity waves in a motionless liquid with an inertial surface composed of uniformly distributed floating particles. The two principal problems considered are the forced motions due to a submerged wave source and an immersed vertical plane wave-maker, which begin to operate in a time-dependent manner at a given instant. The transformed potentials are calculated using techniques similar to those which are effective in traditional time-harmonic problems with a free surface. The steady-state development in the time-harmonic example taken demonstrates the existence of outgoing progressive waves under any inertial surface, in contrast to the case of no surface tension when such waves cannot propagate under an inertial surface that is too heavy. The solution is also noted of the Cauchy-Poisson problem for the free motion following an initial elevation of the inertial surface, which is obtained by the same method.

1. Introduction

It is a notable fact that small time-harmonic progressive gravity waves of given angular frequency cannot exist at the surface of an ideal liquid (water) covered by a thin uniform distribution of floating matter (broken ice, unstretched mat) if the layer, or *inertial surface*, is too heavy. This result is in marked contrast to that for a sufficiently light layer which can, like the familiar free surface, support such waves. These two possibilities for an inertial surface were remarked on and exemplified in the investigations of waves on a free surface incident upon an adjoining inertial surface made by Peters [5] for infinite depth and Weitz and Keller [12] for finite constant depth. It has also been shown, however, that the

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presence of surface tension ensures the existence of progressive capillary-gravity waves at any inertial surface; see Rhodes-Robinson [9]. Both types of result are exemplified herein for problems which involve small forced two-dimensional wave motion generated from rest, perhaps impulsively, by specific action within a liquid of infinite depth with an inertial surface.

The first problem considered involves motion due to a submerged line *wave source* which starts to operate at a given time; the strength of the source is in general variable, being time-dependent, and its position is fixed. The second problem considered involves motion due to a flexible vertical plane *wave-maker* after this starts to move with a time-dependent normal velocity, which is also depth-dependent and must correspond to a small displacement of the wave-maker from its original position (like some periodic oscillation, say).

The obvious approach in such initial-value problems, where the velocity potential of the irrotational flow satisfies a linearized boundary-value problem with additional initial conditions, is to use the Laplace-transform method to eliminate the time variable. This results in a reduced boundary-value problem involving the transform parameter which is similar to that occuring in the analogous time-harmonic problem with a free surface and can be solved by parallel techniques. The required velocity potential is then expressed as the inverse of the transformed potential. This method was used to investigate a moving point wave source of variable strength and position in the survey of Wehausen and Laitone [11], but was not developed any further. A direct approach is in general difficult and more so for the wave-maker problem. An exception, however, is the special case of a fixed line wave source of constant strength whose potential was obtained by Finkelstein [2] after trying a likely form. An interesting but involved method for the wave-maker was used by Kennard [4], who first calculated the easier classical solution assuming absence of gravity and then adjusted this using intuitive ideas from Cauchy-Poisson theory for motion under gravity following an initial elevation of the free surface. (These studies are for a free surface and ignore surface tension; the latter two are also for two-dimensional motion.)

The Laplace-transform method is developed herein for an inertial surface to solve the two problems described earlier, assuming first that there is no surface tension. It is found that the wave-source potential makes its usual appearance in the wave-maker solution so is fundamental; this is a consequence of a basic formula obtained using Green's theorem with transformed potentials. Particular wave-source potentials are discussed as examples for the case of time-harmonic strength, where the two possible steady-state forms with and without outgoing progressive waves are obtained by appropriate asymptotic procedures; also for the interesting case where the strength is impulsive at the initial instant but otherwise zero, and for the classical case of constant strength.

The same two problems are then solved in the presence of surface tension, when the layer can also be thought of as a heavy stretched membrane. Now, however, a second fundamental potential is needed describing motion having a small time-dependent inertial-surface slope at a fixed vertical plane wall. The same methods, suitably generalized, give the solutions and there are always outgoing progressive waves in the time-harmonic case now. Simplified results for a free surface can be deduced, which have not been obtained previously either.

Extension of the results as obtained for two-dimensional motion in a single liquid of infinite depth may be made to three-dimensional motion, finite constant depth and two superposed liquids; other generating mechanisms can also be taken. A useful application of the result for a fixed point wave source of variable strength would appear to be in modelling a submerged vapour bubble under an inertial surface, done recently in a simplified form by Blake and Cerone [1].

The solution is also obtained herein for the initial-value problem describing free two-dimensional motion in a liquid of infinite depth following an initial displacement of its inertial surface from rest (the classical Cauchy-Poisson problem). The Laplace-transform method is convenient in the present context and Green's theorem provides a suitable basic formula again. This result also may be extended to other situations.

2. Basic formulation for gravity waves at an inertial surface

We consider the motion under gravity g of an ideal liquid of volume density ρ whose surface is completely covered by an inertial surface composed of a thin uniform distribution of disconnected heavy floating matter of area density $\rho\varepsilon$, say. The special case of a free surface is well known and corresponds to $\varepsilon = 0$. The depth is taken as infinite, and the effect of surface tension is temporarily omitted.

The motion is small and commences at time $t = 0$ from a state of rest in which the inertial surface has position $y = 0$ and the liquid occupies the region $y > 0$. There is no effect from the constant atmospheric pressure that is assumed above the inertial surface. Two-dimensional motion is envisaged, and in addition to the vertical coordinate y we use the horizontal coordinate x. The motion is irrotational since it is generated from rest and may be described in linearized theory by a velocity potential $\phi(x, y, t)$ for $t > 0$ which satisfies Laplace's equation

$$\nabla^2\phi = 0 \quad \text{in} \quad y > 0 \tag{2.1}$$

(continuity of mass in fluid region). If the inertial surface has depression $\eta(x, t)$ from the equilibrium position, the joint *boundary* conditions relating ϕ, η at this are the kinematic condition

$$\phi_y = \eta_t \quad \text{on} \quad y = 0 \tag{2.2}$$

(inertial-surface particles remain there) and the dynamic condition

$$\phi_t = g\eta + \varepsilon\eta_{tt} \quad \text{on} \quad y = 0 \tag{2.3}$$

(equation of motion of inertial surface). For convenience, (2.3) may be expressed as

$$\Phi_t = g\eta \quad \text{on} \quad y = 0, \tag{2.4}$$

if we use (2.2) and define

$$\Phi = \phi - \varepsilon\phi_y. \tag{2.5}$$

Elimination of η in (2.2) and (2.4) then gives the single inertial-surface condition

$$\Phi_{tt} - g\phi_y = 0 \quad \text{on} \quad y = 0 \tag{2.6}$$

for ϕ. There are also *initial* conditions on the inertial surface; since η must be continuous at all times (even if the motion starts impulsively), it is seen from (2.4) that Φ_t is continuous; thus Φ is continuous (differentiable). In particular this applies at the start of the motion, so initial values correspond to equilibrium values obtainable by noting that $\phi = \eta = 0$ $(t < 0)$. Thus we obtain the pair of initial conditions

$$\Phi = \eta = 0 \quad \text{on} \quad y = 0 \quad \text{at} \quad t = 0; \tag{2.7}$$

or equivalently, using (2.4), we have instead

$$\Phi = \Phi_t = 0 \quad \text{on} \quad y = 0 \quad \text{at} \quad t = 0 \tag{2.8}$$

in terms of Φ alone. Both the sets (2.7) and (2.8) will be used.

To summarize then, a problem will be formulated mathematically as an initial-value problem for Laplace's equation (2.1), subject to the basic boundary and initial conditions (2.6) and (2.8) at the inertial surface and additional specific boundary conditions.

We note that for the particular case of established time-harmonic motion of angular frequency σ, when

$$\phi_{tt} + \sigma^2\phi = \Phi_{tt} + \sigma^2\Phi = 0, \tag{2.9}$$

the inertial-surface condition (2.6) becomes

$$K\Phi + \phi_y = K\phi + (1 - K\varepsilon)\phi_y = 0 \quad \text{on} \quad y = 0, \tag{2.10}$$

where $K = \sigma^2/g$ and (2.5) has been used. For $0 \leqslant K\varepsilon < 1$ the form of (2.10) is $K^*\phi + \phi_y = 0$ on $y = 0$ so it is merely a modification of the usual free-surface condition corresponding to $\varepsilon = 0$, where $K^* = K/(1 - K\varepsilon)$; in particular, it allows progressive waves with wave number K^*. However, for $K\varepsilon \geqslant 1$ the form of (2.10) is different and does not allow progressive waves. These two possibilities were noted by Peters [5], who investigated the transmission of incident waves into an inertial surface (unstretched floating mat).

3. Laplace-transform method

The obvious method of solution of the initial-value problem now formulated in principle is to take the Laplace transform in time t (keeping other variables fixed), defined as

$$\bar{f}(p) = \int_0^\infty e^{-pt} f(t)\, dt \qquad (p > 0)$$

for any time-dependent quantity f. It follows in the usual manner that the transforms of the time derivatives $\dot{f}$, $\ddot{f}$ are $p\bar{f} - f(0)$, $p^2\bar{f} - pf(0) - \dot{f}(0)$ respectively in terms of $\bar{f}$ and the initial values f, $\dot{f}(0)$. Note that the transforms of ϕ, η will have the dependence $\bar{\phi}(x, y;\, p)$, $\bar{\eta}(x;\, p)$ for two-dimensional motion.

Thus the initial-value problem for ϕ can be transformed into a boundary-value problem for $\bar{\phi}$ which also, from (2.1), satisfies Laplace's equation

$$\nabla^2\bar{\phi} = 0 \quad \text{in} \quad y > 0. \tag{3.1}$$

The transform of the inertial-surface condition (2.6) is

$$p^2\bar{\Phi} - g\bar{\phi}_y = p^2\bar{\phi} - (g + \varepsilon p^2)\bar{\phi}_y = 0 \quad \text{on} \quad y = 0, \tag{3.2}$$

from (2.8) and (2.5). The form of (3.1) and (3.2) suggests that the determination of $\bar{\phi}$ might be accomplished using methods similar to those that are successful for analogous time-harmonic problems with $\varepsilon = 0$; ϕ is then found as an inverse transform.

We note that the transformed joint conditions on the inertial surface are

$$\bar{\phi}_y = p\bar{\eta} \quad \text{on} \quad y = 0 \tag{3.3}$$

and

$$p\bar{\Phi} = g\bar{\eta} \quad \text{on} \quad y = 0, \tag{3.4}$$

from (2.2), (2.4) and (2.7); thus we find that

$$\bar{\eta} = \left[\bar{\phi}_y\right]_{y=0}/p \tag{3.5}$$

$$= p\left[\bar{\phi}\right]_{y=0}/(g + \varepsilon p^2) \tag{3.6}$$

in terms of $\bar{\phi}$, from (3.3), (3.4) and (2.5). Either of (3.5) or (3.6) may be used to find η as an inverse once $\bar{\phi}$ is known. Note also that elimination of $\bar{\eta}$ in (3.5) and (3.6) again gives (3.2).

4. Variable wave source under an inertial surface

The first problem is to find the potential denoted by $G(x, y; X, Y; t)$, say, describing the symmetric motion due to a submerged wave source of given arbitrary strength $m(t)$ located at the fixed position (X, Y), $Y > 0$ in a liquid of infinite horizontal expanse that starts to operate at $t = 0$. Then for $t > 0$ we find that G is the solution in the region $y > 0$ of the

initial-value problem

$$\left.\begin{array}{c} \nabla^2 G = 0, \quad \text{except at} \quad (X, Y), \\ H_{tt} - gG_y = 0 \quad \text{on} \quad y = 0, \\ G \sim m(t)\log\rho \quad \text{as} \quad \rho \to 0, \\ |\nabla G| \to 0 \quad \text{as} \quad \rho \to \infty, \\ H(0) = H_t(0) = 0 \quad \text{on} \quad y = 0, \end{array}\right\} \tag{4.1}$$

where $H = G - \varepsilon G_y$ and $\rho = [(x - X)^2 + (y - Y)^2]^{1/2}$; this includes (2.1), (2.6) and (2.8), to which are added the singularity condition near a source and the condition for no motion at infinity (an abbreviated notation is used for the initial values).

Taking the transform, it is seen that $\bar{G}(x, y; X, Y; p)$ is the solution in the same region $y > 0$ of the

boundary-value problem

$$\left.\begin{array}{c} \nabla^2 \bar{G} = 0, \quad \text{except at} \quad (X, Y), \\ p^2\bar{G} - (g + \varepsilon p^2)\bar{G}_y = 0 \quad \text{on} \quad y = 0, \\ \bar{G} \sim \bar{m}(p)\log\rho \quad \text{as} \quad \rho \to 0, \\ |\nabla \bar{G}| \to 0 \quad \text{as} \quad \rho \to \infty \end{array}\right\} \tag{4.2}$$

$(p > 0)$; this includes (3.1), (3.2) and the transforms of the additional conditions.

The solution of (4.2) is clearly a multiple of $\bar{m}$ and may be readily obtained by several methods (similar to those used in the familiar problem for the time-harmonic wave source with $\varepsilon = 0$) and in different forms, the only suitable one here being

$$\bar{G} = \bar{m}\left[\log\frac{\rho}{\rho'} - 2(g + \varepsilon p^2)\int_0^\infty \frac{e^{-k(y+Y)}}{(g + \varepsilon p^2)k + p^2}\cos k(x - X)\,dk\right], \tag{4.3}$$

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where also $\rho' = [(x-X)^2 + (y+Y)^2]^{1/2}$; or, on rearranging,

$$\bar{G} = \bar{m}\left[\log\frac{\rho}{\rho'} - 2\varepsilon\int_0^\infty \frac{e^{-k(y+Y)}}{1+k\varepsilon}\cos k(x-X)\,dk\right.$$
$$\left. -2\int_0^\infty \frac{e^{-k(y+Y)}}{k(1+k\varepsilon)}\cos k(x-X)\frac{\Omega^2}{\Omega^2+p^2}\,dk\right], \tag{4.4}$$

where $\Omega = [gk/(1+k\varepsilon)]^{1/2}$.

The inverse transform of (4.4) presents no difficulty. Thus, noting that the inverse of $\Omega/(\Omega^2+p^2)$ is $\sin\Omega t$, we have by the convolution theorem that

$$G = m\left[\log\frac{\rho}{\rho'} - 2\varepsilon\int_0^\infty \frac{e^{-k(y+Y)}}{1+k\varepsilon}\cos k(x-X)\,dk\right]$$
$$-2\int_0^\infty \frac{\Omega e^{-k(y+Y)}}{k(1+k\varepsilon)}\cos k(x-X)\int_0^t m(\tau)\sin\Omega(t-\tau)\,d\tau\,dk, \tag{4.5}$$

the required wave-source potential. (This is not nor could be a multiple of m.)

Note that $G(0) = 0$ (smooth start to motion) if $m(0) = 0$, but $G(0) \neq 0$ (impulsive start to motion) if $m(0) \neq 0$; however,

$$H(0) = m(0)\left[\log\frac{\rho}{\rho'} - \varepsilon\left(\frac{y-Y}{\rho^2} + \frac{y+Y}{\rho'^2}\right)\right] = 0$$

on $y = 0$ for any value of $m(0)$ as required.

The shape of the inertial surface can be calculated directly from (4.5) using (2.2); noting (2.7), this gives on integration (using an abbreviated notation)

$$\eta = \int_0^t [G_y(\tau)]_{y=0}\,d\tau = [G_y^*]_{y=0},$$

where

$$G^*(t) = \int_0^t G(\tau)\,d\tau.$$

To facilitate this calculation, we note that G^* is the potential for a wave source of strength m^* given by

$$m^*(t) = \int_0^t m(\tau)\,d\tau;$$

thus we find that

$$\eta = -2\int_0^\infty \frac{e^{-kY}}{1+k\varepsilon}\cos k(x-X)\left[m^*(t) - \Omega\int_0^t m^*(\tau)\sin\Omega(t-\tau)\,d\tau\right]dk$$
$$= -2\int_0^\infty \frac{e^{-kY}}{1+k\varepsilon}\cos k(x-X)\int_0^t m(\tau)\cos\Omega(t-\tau)\,d\tau\,dk, \tag{4.6}$$

on integration by parts. This calculation is equivalent to using (3.5) to obtain $\bar{\eta}$ and taking the inverse; however, the simplified result (4.6) can be obtained directly using instead (3.6) which gives

$$\bar{\eta} = -2\bar{m}\int_0^\infty \frac{e^{-kY}}{1+k\varepsilon}\cos k(x-X)\frac{p}{\Omega^2+p^2}\,dk, \tag{4.7}$$

using (4.3) also.

Note that η as given by (4.6) must satisfy the mass-input equation

$$-\int_X^\infty \eta(x,t)\,dx = \pi\int_0^t m(\tau)\,d\tau, \tag{4.8}$$

since the amount of liquid above the equilibrium level of the inertial surface must correspond to the net amount emitted into the region by the source since it started operating. In terms of transformed quantities (4.8) becomes

$$-\int_X^\infty \bar{\eta}(x,p)\,dx = \pi\bar{m}(p)/p, \tag{4.9}$$

and may be verified in this form using $\bar{\eta}$ as given by (4.7) and Fourier cosine-integral theory.

It will be found helpful later to use the special wave source $G = G^{\text{spec}}$ that has *impulsive* strength $m = \delta(t)$; then from (4.5) this has potential

$$G^{\text{spec}} = \delta(t)\left[\log\frac{\rho}{\rho'} - 2\varepsilon\int_0^\infty \frac{e^{-k(y+Y)}}{1+k\varepsilon}\cos k(x-X)\,dk\right]$$
$$-2\int_0^\infty \frac{\Omega e^{-k(y+Y)}}{k(1+k\varepsilon)}\cos k(x-X)\sin\Omega t\,dk, \tag{4.10}$$

which is regular for $t > 0$. Further, note that for the general wave source

$$G(t) = \int_0^t m(\tau)G^{\text{spec}}(t-\tau)\,d\tau \tag{4.11}$$

in terms of G^{spec} (since $\bar{G} = \bar{m}\bar{G}^{\text{spec}}$).

The classical wave source of *constant* strength $m = 1$, say, has potential

$$\int_0^t G^{\text{spec}}(\tau)\,d\tau = \log\frac{\rho}{\rho'} - 2\varepsilon\int_0^\infty \frac{e^{-k(y+Y)}}{1+k\varepsilon}\cos k(x-X)\,dk$$
$$-2\int_0^\infty \frac{e^{-k(y+Y)}}{1+k\varepsilon}\cos k(x-X)\frac{1-\cos\Omega t}{k}\,dk, \tag{4.12}$$

from (4.10) and (4.11). This potential was found for $\varepsilon = 0$ by Finkelstein [2] using a different method (trial form of solution).

An important concluding example now follows.

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5. Time-harmonic wave source and steady-state development

Taking $m = \sin \sigma t$, we find from (4.5) that

$$G = \sin \sigma t \left[\log \frac{\rho}{\rho'} - 2\varepsilon \int_0^\infty \frac{e^{-k(y+Y)}}{1+k\varepsilon} \cos k(x-X)\, dk \right]$$
$$-2\int_0^\infty \frac{\Omega e^{-k(y+Y)}}{k(1+k\varepsilon)} \cos k(x-X) \frac{\Omega \sin \sigma t - \sigma \sin \Omega t}{\Omega^2 - \sigma^2}\, dk. \qquad (5.1)$$

The established form of this potential is now sought as $t \to \infty$ and from (2.9) will be $G \sim A \cos \sigma t + B \sin \sigma t$, where A, B will satisfy (2.1) and (2.10). To find the steady-state terms in (5.1), we must isolate and eliminate the transient terms. Two quite different outcomes are expected, from the earlier discussion on progressive waves. These depend on whether or not the denominator of the second integrand in (5.1) *vanishes* in the range of integration $k > 0$: precisely, $\Omega^2 - \sigma^2$ or equivalently $(1 - K\varepsilon)k - K$ has such a root when $0 \leqslant K\varepsilon < 1$ but none when $K\varepsilon \geqslant 1$.

Taking first the case $0 \leqslant K\varepsilon < 1$, this root is $k = K/(1 - K\varepsilon) = K^*$ as defined earlier. Introduce a Cauchy principal value at $k = K^*$ $(\Omega = \sigma)$ and split the second integral in (5.1) into two. Then clearly the transient terms of G are wholly contained in the part

$$2\sigma \fint_0^\infty \frac{\Omega e^{-k(y+Y)}}{k(1+k\varepsilon)} \cos k(x-X) \frac{\sin \Omega t}{\Omega^2 - \sigma^2}\, dk$$
$$= 4\sigma \int_0^{(g/\varepsilon)^{1/2}} \left[\frac{e^{-k'(y+Y)}}{\Omega' + \sigma} \cos k'(x-X) \right]_{k'=K^*}^{k} \frac{\sin \Omega t}{\Omega - \sigma}\, d\Omega$$
$$+ 2e^{-K^*(y+Y)} \cos K^*(x-X) \fint_0^{(g/\varepsilon)^{1/2}} \frac{\sin \Omega t}{\Omega - \sigma}\, d\Omega \qquad \left[k = \Omega^2/(g - \Omega^2 \varepsilon)\right],$$

on substituting Ω as the variable of integration and modifying the integrand—putting $\Omega' = \Omega(k')$. (We assume $\varepsilon \neq 0$, but the conclusion is the same for $\varepsilon = 0$ also.) It is then seen that the first of these integral terms is transient, by the Riemann-Lebesgue lemma; also, the integral in the second term may be expanded as

$$\cos \sigma t \int_{-\sigma t}^{[(g/\varepsilon)^{1/2} - \sigma]t} \frac{\sin u}{u}\, du + \sin \sigma t \fint_{-\sigma t}^{[(g/\varepsilon)^{1/2} - \sigma]t} \frac{\cos u}{u}\, du,$$

after substituting $u = (\Omega - \sigma)t$, so is partly transient since the two integrals have the limiting values π and 0 respectively as $t \to \infty$—noting that $(g/\varepsilon)^{1/2} > \sigma$ here.

Thus collecting all steady-state terms together and putting $\Omega^2 = gk/(1 + k\varepsilon)$ and $\sigma^2 = gK$, we find that

$$G \sim \sin\sigma t\left[\log\frac{\rho}{\rho'} - 2\varepsilon\int_0^\infty \frac{e^{-k(y+Y)}}{1+k\varepsilon}\cos k(x-X)\,dk\right]$$
$$-2\sin\sigma t \fint_0^\infty \frac{e^{-k(y+Y)}\cos k(x-X)}{(1+k\varepsilon)[(1-K\varepsilon)k-K]}\,dk$$
$$+2\pi\cos\sigma t\, e^{-K^*(y+Y)}\cos K^*(x-X)$$

as $t \to \infty$; or, when the two integrals are combined, this is

$$G \sim \sin\sigma t\left[\log\frac{\rho}{\rho'} - 2\fint_0^\infty \frac{e^{-k(y+Y)}}{k-K^*}\cos k(x-X)\,dk\right]$$
$$+2\pi\cos\sigma t\, e^{-K^*(y+Y)}\cos K^*(x-X) \tag{5.2}$$

as $t \to \infty$. This potential satisfies (2.10) in the form $K^*G + G_y = 0$ on $y = 0$ and represents outgoing progressive waves as $|x - X| \to \infty$; these are not fully explicit in (5.2), but may be determined as having potential

$$2\pi e^{-K^*(y+Y)}\cos(K^*|x-X| - \sigma t).$$

Note that the result (5.2) may also be obtained directly by comparison with the known time-harmonic result for $\varepsilon = 0$ in Thorne [10].

Taking now the case $K\varepsilon \geqslant 1$, there is no root and the calculation is quite straightforward since a direct application of the Riemann-Lebesgue lemma in (5.1) suffices to remove the transient terms immediately. Thus instead

$$G \sim \sin\sigma t\left[\log\frac{\rho}{\rho'} - 2\varepsilon\int_0^\infty \frac{e^{-k(y+Y)}}{1+k\varepsilon}\cos k(x-X)\,dk\right]$$
$$+2\sin\sigma t\int_0^\infty \frac{e^{-k(y+Y)}\cos k(x-X)}{(1+k\varepsilon)[(K\varepsilon-1)k+K]}\,dk$$
$$= \sin\sigma t\left[\log\frac{\rho}{\rho'} - 2(K\varepsilon-1)\int_0^\infty \frac{e^{-k(y+Y)}}{(K\varepsilon-1)k+K}\cos k(x-X)\,dk\right] \tag{5.3}$$

as $t \to \infty$. This potential satisfies (2.10) in the form $KG - (K\varepsilon - 1)G_y = 0$ on $y = 0$, and has a particularly simple form for $K\varepsilon = 1$. Note the absence of outgoing progressive waves as $|x - X| \to \infty$, which means that the separation of the time factor $\sin\sigma t$ could be expected. The potential (5.3), like its predecessor

(5.2) in fact, has a concise alternative form; this is found by contour integration and the singular term may then be incorporated in the integral to give

$$G \sim -2\sin\sigma t \int_0^\infty \frac{e^{-k|x-X|}}{k\left[(K\varepsilon-1)^2k^2+K^2\right]} F(y;k)F(Y;k)\,dk \tag{5.4}$$

as $t \to \infty$, where $F(s; k) = (K\varepsilon - 1)k\cos ks + K\sin ks$.

6. Vertical wave-maker problem

Next we turn to a practical problem and find the potential $\phi(x, y, t)$ describing motion in a liquid of semi-infinite horizontal expanse ($x > 0$) due to a suitably prescribed outward normal velocity distribution $U(y, t)$ on the vertical boundary $x = 0$, where $U \to 0$ as $y \to \infty$. This begins at $t = 0$ and must correspond to a small horizontal displacement of a flexible boundary from $x = 0$, on which the boundary condition is applied in the linearization. Then for $t > 0$ we now have that ϕ is the solution in the region $x > 0, y > 0$ of the

initial-value problem

$$\left.\begin{aligned} \nabla^2\phi &= 0, \\ \Phi_{tt} - g\phi_y &= 0 \quad \text{on} \quad y = 0, \\ \phi_x &= U(y,t) \quad \text{on} \quad x = 0, \\ |\nabla\phi| &\to 0 \quad \text{as} \quad r \to \infty, \\ \Phi(0) = \Phi_t(0) &= 0 \quad \text{on} \quad y = 0, \end{aligned}\right\} \tag{6.1}$$

where $\Phi = \phi - \varepsilon\phi_y$ and $r = (x^2 + y^2)^{1/2}$.

Then $\bar\phi$ is the solution in the same region $x > 0, y > 0$ of the

boundary-value problem

$$\left.\begin{aligned} \nabla^2\bar\phi &= 0, \\ p^2\bar\phi - (g + \varepsilon p^2)\bar\phi_y &= 0 \quad \text{on} \quad y = 0, \\ \bar\phi_x &= \bar U(y; p) \quad \text{on} \quad x = 0, \\ |\nabla\bar\phi| &\to 0 \quad \text{as} \quad r \to \infty \end{aligned}\right\} \tag{6.2}$$

($p > 0$), in which $\bar U \to 0$ as $y \to \infty$.

The solution of (6.2) is most easily obtained (following one method used in the familiar problem for the time-harmonic wave-maker with $\varepsilon = 0$) by use of Green's theorem with the auxiliary exact Green's function

$$\bar G^{\mathrm{spec}}(x, y; X, Y; p) + \bar G^{\mathrm{spec}}(x, y; -X, Y; p);$$

this satisfies the same boundary-value problem as $\bar{G}^{\text{spec}}(x, y;\ X, Y;\ p)$ referred to earlier in the above region if $X > 0$, that is (4.2) with $\bar{m} = 1$, and in addition gives *zero* normal velocity on the wave-maker boundary $x = 0$. Taking the contour for Green's theorem as the boundary of the fluid region with a small circular indentation excluding (X, Y), we obtain the integral formula over the wave-maker boundary

$$\bar{\phi}(X, Y;\ p) = \frac{1}{2\pi}\int_0^\infty \left[\bar{G}^{\text{spec}}(0, y;\ X, Y;\ p) + \bar{G}^{\text{spec}}(0, y; -X, Y;\ p)\right]\bar{U}(y;\ p)\, dy,$$

since the contributions from the inertial surface and infinity both vanish (as does the unknown part of the wave-maker contribution involving $\bar{\phi}$), so that

$$\bar{\phi}(x, y;\ p) = \frac{1}{\pi}\int_0^\infty \bar{G}^{\text{spec}}(x, y; 0, Y;\ p)\bar{U}(Y;\ p)\, dY \tag{6.3}$$

on simplification and using the obvious reciprocity property (before changing back to original variables).

Thus, quite simply, the required inverse potential $\phi(x, y, t)$ is equivalent to a distribution on the wave-maker boundary $Y > 0$ of the variable *wave sources* $G(x, y; 0, Y; t)$ with strength density $(1/\pi)U(Y, t)$, whose potentials are identifiable as

$$\frac{1}{\pi}\int_0^t G^{\text{spec}}(x, y; 0, Y; t - \tau)U(Y, \tau)\, d\tau$$

from the earlier result (4.11). This representation of the solution, which might have been expected, is a generalization of the known steady-state time-harmonic result with $\varepsilon = 0$. The explicit form can be found using the result (4.10) for G^{spec}.

If $U(y, t)$ is proportional to $\sin \sigma t$, the result for $t > 0$ involves the time-harmonic wave source (5.1); in the steady state, there are outgoing progressive waves as $x \to \infty$ for $0 \leqslant K\varepsilon < 1$ only of course.

The general wave-maker solution was obtained in explicit form for $\varepsilon = 0$ by Kennard [4] using a somewhat complicated method; a detailed discussion of the time-harmonic case was also given, and the classical result of Havelock [3] obtained as the steady-state development.

7. Effect of surface tension

If in the basic formulation we now allow for the presence of surface tension $T = \rho T'$, say, and consider capillary-gravity waves, the kinematic condition (2.2) at the inertial surface remains as

$$\phi_y = \eta_t \quad \text{on} \quad y = 0, \tag{7.1}$$

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but the dynamic condition (2.3) at the inertial surface is modified to

$$\phi_t = g\eta + \varepsilon\eta_{tt} - T'\eta_{xx} \quad \text{on} \quad y = 0; \tag{7.2}$$

thus, on eliminating η in (7.1) and (7.2), we now obtain

$$\Phi_{tt} - g\phi_y - T'\phi_{yyy} = 0 \quad \text{on} \quad y = 0 \tag{7.3}$$

as the new inertial-surface condition replacing (2.6), where again $\Phi = \phi - \varepsilon\phi_y$ as in (2.5). The initial conditions (2.7) and (2.8) on the inertial surface are also unchanged, and are

$$\Phi(0) = \eta(0) = 0, \quad \text{or} \quad \Phi(0) = \Phi_t(0) = 0 \quad \text{on} \quad y = 0. \tag{7.4}$$

For established time-harmonic motion of angular frequency σ, the potentials satisfy (2.9) and the inertial-surface condition replacing (2.10) is obtained from (7.3) as

$$K\phi + (1 - K\varepsilon)\phi_y + M\phi_{yyy} = 0 \quad \text{on} \quad y = 0, \tag{7.5}$$

where $K = \sigma^2/g$ as before and $M = T'/g$. The form of (7.5) allows progressive waves for *all* $K\varepsilon \geqslant 0$ as shown in Rhodes-Robinson [9], provided $M > 0$; these have wave number κ^* given by $\kappa^*(1 - K\varepsilon + M\kappa^{*2}) = K$.

The transform of the inertial-surface condition (7.3) is now

$$p^2\bar{\phi} - (g + \varepsilon p^2)\bar{\phi}_y - T'\bar{\phi}_{yyy} = 0 \quad \text{on} \quad y = 0, \tag{7.6}$$

obtained in the same manner as (3.2) before, and the same problems may now be attempted using modified techniques appropriate to the presence of surface tension (similar again to those in problems for time-harmonic motion with $\varepsilon = 0$) to determine the transform potential $\bar{\phi}$. The shape of the inertial surface is then found as an inverse using (3.5) as before, or else from the solution of the simple differential equation

$$\left(g + \varepsilon p^2 - T'\frac{d^2}{dx^2}\right)\bar{\eta} = p[\bar{\phi}]_{y=0},$$

subject to certain end conditions, which replaces (3.6).

Thus the variable *wave-source* potential G of strength m now has transform

$$\bar{G} = \bar{m}\left[\log\frac{\rho}{\rho'} - 2\int_0^\infty \frac{(g + \varepsilon p^2 + T'k^2)e^{-k(y+Y)}}{k(g + \varepsilon p^2 + T'k^2) + p^2}\cos k(x - X)\,dk\right] \tag{7.7}$$

which generalises (4.3), but may still be put in the *same* form (4.4) as before ($T' = 0$) if now we take instead $\Omega = [k(g + T'k^2)/(1 + k\varepsilon)]^{1/2}$; thus G has its previous form (4.5) also. The corresponding η in (4.6) therefore remains the same, and continues to satisfy the mass-input equation (4.8).

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The examples of particular wave-source potentials like G^{spec} in (4.10) and also (4.12) are formally unchanged, as is the result (4.11). The time-harmonic wave-source potential (5.1) has only one possible steady-state development, however, since $\Omega^2 - \sigma^2$ or equivalently $k(1 - K\varepsilon + Mk^2) - K$ vanishes at $k = \kappa^*$ $(\Omega = \sigma)$ for all $K\varepsilon \geqslant 0$; see Rhodes-Robinson [9]. This is found to be

$$G \sim \sin \sigma t \left[\log \frac{\rho}{\rho'} - 2 \int_0^\infty \frac{(1 - K\varepsilon + Mk^2) e^{-k(y+Y)}}{k(1 - K\varepsilon + Mk^2) - K} \cos k(x - X)\, dk \right]$$
$$+ 2\pi \cos \sigma t \frac{1 - K\varepsilon + M\kappa^{*2}}{1 - K\varepsilon + 3M\kappa^{*2}} e^{-\kappa^*(y+Y)} \cos \kappa^*(x - X) \tag{7.8}$$

as $t \to \infty$ $(M > 0)$ and represents outgoing progressive waves at infinity. Again the result (7.8) may be obtained directly by comparison with the known time-harmonic result for $\varepsilon = 0$ in Rhodes-Robinson [6].

8. Variable-slope potential

Before proceeding to the wave-maker problem with surface tension, we first look for a potential $G_0(x, y, t)$ describing hypothetical motion in a liquid of semi-infinite horizontal expanse $(x > 0)$ corresponding to a suitably prescribed downward inertial-surface slope $\pi\Lambda(t)$ at a fixed wall along $x = 0$, where we require $\Lambda(0) = 0$. (Such a potential is known to exist for established time-harmonic motion in the presence of surface tension when $\varepsilon = 0$; see Rhodes-Robinson [6].) Putting $\eta_x(0, t) = \pi\Lambda$ in (7.1) after taking the x derivative, we thus obtain the edge condition $G_{0xy}(0+, 0, t) = \pi\dot{\Lambda}$; also $G_{0x} = 0$ on $x = 0$, since there is no normal velocity on the wall. The initial-value problem for G_0 could now be written down for $t > 0$. Thus it is found that the transformed potential $\bar{G}_0(x, y; p)$ is the solution in the region $x > 0, y > 0$ of the

boundary-value problem

$$\left.\begin{array}{c} \nabla^2 \bar{G}_0 = 0, \\ p^2 \bar{G}_0 - (g + \varepsilon p^2) \bar{G}_{0y} - T' \bar{G}_{0yyy} = 0 \quad \text{on} \quad y = 0, \\ \bar{G}_{0x} = 0 \quad \text{on} \quad x = 0, \\ \bar{G}_{0xy}(0+, 0) = p\pi\bar{\Lambda}(p), \\ |\nabla \bar{G}_0| \to 0 \quad \text{as} \quad r \to \infty \end{array}\right\} \tag{8.1}$$

$(p > 0)$.

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The easiest way to solve (8.1) is by Green's theorem with the exact Green's function $\bar{G}^{\text{spec}}(x, y; X, Y; p) + \bar{G}^{\text{spec}}(x, y; -X, Y; p)$ again. Thus we obtain the result

$$\bar{G}_0(X, Y; p) = -T'\bar{\Lambda}(p)\bar{G}_y^{\text{spec}}(0,0; X, Y; p)/p, \tag{8.2}$$

with only the inertial surface contributing to the contour integral around the boundary of the liquid region; the details omitted are similar to some time-harmonic calculations for $\varepsilon = 0$ in Rhodes-Robinson [6]. Evaluating (8.2) using the formally unchanged $\bar{G}^{\text{spec}}$ from (4.4) with $\bar{m} = 1$, we find that (changing variables)

$$\bar{G}_0 = 2T'\bar{\Lambda}\int_0^\infty \frac{e^{-ky}}{1 + k\varepsilon}\cos kx \frac{p}{\Omega^2 + p^2}\,dk. \tag{8.3}$$

Thus, noting that the inverse of $p/(\Omega^2 + p^2)$ is $\cos \Omega t$, we have by the convolution theorem that

$$G_0 = 2T'\int_0^\infty \frac{e^{-ky}}{1 + k\varepsilon}\cos kx \int_0^t \Lambda(\tau)\cos \Omega(t - \tau)\,d\tau\,dk, \tag{8.4}$$

the required slope potential which exists only for $T' > 0$.

Taking $\Lambda = \sin \sigma t$ for which $\Lambda(0) = 0$ as required, we find from (8.4) that

$$G_0 = 2T'\sigma\int_0^\infty \frac{e^{-ky}}{1 + k\varepsilon}\cos kx \frac{\cos \sigma t - \cos \Omega t}{\Omega^2 - \sigma^2}\,dk \tag{8.5}$$

$$\sim 2M\sigma\left[\cos \sigma t \fint_0^\infty \frac{e^{-ky}\cos kx}{k(1 - K\varepsilon + Mk^2) - K}\,dk + \frac{\pi \sin \sigma t}{1 - K\varepsilon + 3M\kappa^{*2}} e^{-\kappa^* y}\cos \kappa^* x\right] \tag{8.6}$$

as $t \to \infty$ $(M > 0)$, in agreement with the known time-harmonic result for $\varepsilon = 0$ in Rhodes-Robinson [6].

The *wave-maker* problem for ϕ, in which are prescribed both the normal velocity U on the wave-maker and also—in view of the existence of the variable-slope potential G_0—the inertial-surface slope $\pi\Lambda$ at the wave-maker, can now be solved by considering $\phi - G_0$; this has the same normal velocity on, but zero inertial-surface slope at, the wave-maker. Then $\phi - G_0$ is found by the method just used to be given by a distribution of the variable wave sources G on the wave-maker boundary of strength density $(1/\pi)U$ as in the absence of surface tension. Thus ϕ is found in the same form as the known time-harmonic result for $\varepsilon = 0$ obtained in Rhodes-Robinson [6]. (The problem remains to assign the function Λ in practice; but see Rhodes-Robinson [8].)

9. Extension to other situations

The potentials found herein for infinite depth may also be extended to finite constant depth, both sets of results being for a single liquid with an inertial surface. Further, both can then be extended to two superposed liquids of either infinite or equal finite constant depth and height (the latter having a horizontal bottom and lid) that are separated by an inertial interface, since each of the problems may be reduced essentially to one for a corresponding single liquid with an inertial surface exactly as for free-interface problems in Rhodes-Robinson [7]. There are always outgoing waves, except in the absence of surface or interfacial tension when the inertial surface or interface is too heavy; see Rhodes-Robinson [9] for details. The two possibilities for a single liquid of finite depth with $T' = 0$ were noted by Weitz and Keller [12], who investigated the transmission of incident waves into an inertial surface (floating ice).

In conclusion, we note that generalizations to three-dimensional problems can also be made, like that for the variable submerged point wave source discussed for a single liquid of infinite depth with $\varepsilon = T' = 0$ in Wehausen and Laitone [11]. Other types of singularity can of course be examined in any situation.

10. Cauchy-Poisson problem for an inertial surface with an initial elevation

Here we find the potential ϕ for free two-dimensional wave motion in the presence of surface tension for a liquid of infinite horizontal expanse and depth which is initially at rest and has given depression $\Delta(x)$ of its inertial surface, where $\Delta \to 0$ as $|x| \to \infty$. This leads to an easier initial-value problem which may be solved fairly readily by superposition of elementary solutions, as for $\varepsilon = 0$. However, using some of the preceding results the Laplace-transform method is also quite effective in the present context. Initial conditions on the inertial surface are

$$\Phi(0) = 0, \quad \Phi_t(0) = g\Delta - T'\Delta'' \quad \text{on} \quad y = 0,$$

where $\Phi = \phi - \varepsilon\phi_y$ again. The transformed inertial-surface condition is then

$$p^2\bar{\phi} - (g + \varepsilon p^2)\bar{\phi}_y - T'\bar{\phi}_{yyy} = g\Delta - T'\Delta'' \quad \text{on} \quad y = 0$$

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and is non-homogeneous. The transformed boundary-value problem is solved by Green's theorem with the *unmodified* exact Green's function $\bar{G}^{\text{spec}}$, which gives the integral formulas

$$\bar{\phi}(X, Y; p) = -\frac{1}{2\pi p^2}\int_{-\infty}^{\infty}\left[\bar{G}_y^{\text{spec}}\right]_{y=0}(g\Delta - T'\Delta'')\,dx$$
$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\bar{G}^{\text{spec}} - \varepsilon\bar{G}_y^{\text{spec}}\right]_{y=0}\Delta\,dx,$$

on integration by parts twice, with only the inertial surface contributing. Thus, on substituting for $\bar{G}^{\text{spec}}$ from (4.4) with $\bar{m} = 1$, we have (changing variables)

$$\bar{\phi} = \frac{1}{\pi}\int_0^{\infty}\frac{e^{-ky}}{k}\frac{\Omega^2}{\Omega^2 + p^2}\int_{-\infty}^{\infty}\Delta(X)\cos k(x - X)\,dX\,dk$$

in explicit form. Therefore

$$\phi = \frac{1}{\pi}\int_0^{\infty}\frac{e^{-ky}}{k}\Omega\sin\Omega t\int_{-\infty}^{\infty}\Delta(X)\cos k(x - X)\,dX\,dk,$$

the required Cauchy-Poisson solution in which $\Omega = [k(g + T'k^2)/(1 + k\varepsilon)]^{1/2}$ as before; the classical result is recovered by putting $\varepsilon = T' = 0$. The shape of the inertial surface is given by

$$\eta = \frac{1}{\pi}\int_0^{\infty}\cos\Omega t\int_{-\infty}^{\infty}\Delta(X)\cos k(x - X)\,dX\,dk.$$

Note that by conservation of mass we must have

$$\int_{-\infty}^{\infty}\Delta(x)\,dx = \int_{-\infty}^{\infty}\eta(x, t)\,dx = 0.$$

This result may also be extended to other situations.

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