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On the geometric structure of spatio-temporal patterns

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Abstract. The structure of hypersurfaces corresponding to different spatio-temporal patterns is considered, and in particular representations based on geometrical invariants, such as the Riemann and Einstein tensors and the scalar curvature are analyzed. The spatio-temporal patterns result from translations, Lie-group transformations, accelerated and discontinuous motions and modulations. Novel methods are obtained for the computation of motion parameters and the optical flow. Moreover, results obtained for accelerated and discontinuous motions are useful for the detection of motion boundaries.

Keywords: dynamic features, motion, flow field, differential geometry, curvature tensor, Lie transformation groups.

1 Introduction

The input to the human and most technical vision systems is light intensity f as a function of space and time. This function defines a hypersurface

$$S = \{x, y, t, f(x, y, t)\} \tag{1}$$

which has the form of a 3-dimensional Monge patch. From a geometric point of view the curvature is the most important property of the surface in that it determines the intrinsic structure of the manifold [10], so it is of interest to investigate how different types of visual inputs are represented by the curvature tensor of (1). Further, two other geometric invariants, namely the scalar curvature and the Einstein tensor, will be also considered. The goal is, to gain a better understanding of multidimensional signals and visual processing. In vision-science terms, nonlinear representations of dynamic visual inputs are considered. Such representations are generic but of interest to the perception-action cycle. For example, the points on (1) with significant curvature can track moving patterns and the curvature tensor can be used to compute motion parameters [6, 3, 5]. In this paper, however, we consider the theoretical aspects only. Applications have been presented elsewhere, including models of biological visual processing [6, 3, 5].

Geometric methods in computer vision most often deal with the extrinsic geometry of objects in 3D space and how these objects and their motions project on the image plane. However, the geometry of the hypersurface (1) has been used for motion detection [8] with an algorithm based on the gradient of (1). It has also been shown that the Gaussian curvature of (1) can be used to detect motion discontinuities [14]. Our approach is related to the so-called structure-tensor method - see [7] for a review - and this relationship will be discussed in a forthcoming paper [4].

2 Translation with constant velocity

If the image sequence $f(x, y, t)$ results from any spatial pattern moving with constant velocity $\boldsymbol{v} = \{v_x, v_y\}$, f is assumed to satisfy the constraint [2]

$$f(x, y, t) = f(x + dx, y + dy, t + dt), \tag{2}$$

that leads to [2]

$$-\frac{\partial f}{\partial t} = \boldsymbol{\nabla} f \cdot \boldsymbol{v}, \tag{3}$$

with $\boldsymbol{\nabla} f$ being the spatial gradient of f. Finally the solution of (3) is

$$f(x, y, t) = f(x - v_x t, y - v_y t), \tag{4}$$

showing that the image can be thought of as a "solitary wave" which moves, without distortion, with constant velocity along a straight line and whose shape is determined at any given time t by $f(\cdot, t)$.

2.1 Riemann curvature tensor

In this section we first summarize results that have been obtained previously [6, 3] and that will be compared to the results in the following sections.

If we compute the components of the curvature tensor (see Eq. 31 in the Appendix) for the specific function f in Eq. 4, and then simplify all possible ratios of components, we obtain the following results: [1]

$$\begin{aligned} \boldsymbol{v}_1 &= \{R_{3221}, -R_{3121}\}/R_{2121} \\ \boldsymbol{v}_2 &= \{R_{3231}, -R_{3131}\}/R_{3121} \\ \boldsymbol{v}_3 &= \{R_{3232}, -R_{3231}\}/R_{3221}. \end{aligned} \tag{5}$$

Here indices simply denote the fact that we obtain different expressions for $\boldsymbol{v}$. All representations $\boldsymbol{v}_i$ were obtained by assuming the constant brightness constraint.

[1] These and the following simplifications have been performed with the aid of the software *Mathematica* [13].

Note that $\boldsymbol{v}_1$ is the classical solution obtained for the optical flow under the assumption of constant spatial gradient [12] (this is not surprising since this assumption is more general and includes the constraint in Eq. 4).

From Eqs. 5 we can obtain a further motion vector $\boldsymbol{v}_4 = \{v_{4x}, v_{4y}\}$ with

$$v_{4x} = \text{sign}(v_{1x})\sqrt{R_{3232}/R_{2121}}, v_{4y} = \text{sign}(v_{1y})\sqrt{R_{3131}/R_{2121}}. \tag{6}$$

It seems an interesting result that the sectional curvatures (cf. Eq. 31) determine the direction of motion (but for the sign which is here taken from the vector $\boldsymbol{v}_1$).

To summarize, we found four different combinations of $\boldsymbol{R}$ components that are equal and equal to the motion vector in case that Eq. 4 holds ($\boldsymbol{v} = \boldsymbol{v}_1 = \boldsymbol{v}_2 = \boldsymbol{v}_3 = \boldsymbol{v}_4$). [2] We shall see in later sections, how these expressions might differ for patterns other than (4).

2.2 Einstein tensor

As for the curvature tensor, we can obtain four expressions for the motion vector by simplifying the components of the Einstein tensor $\mathbf{G}$ that is obtained from the Riemann tensor through a contraction of the indices (see [11] for definition and properties):

$$\begin{aligned} \boldsymbol{v}_1 &= \{G_{11}, G_{21}\}/G_{31} \\ \boldsymbol{v}_2 &= \{G_{21}, G_{22}\}/G_{32} \\ \boldsymbol{v}_3 &= \{G_{31}, G_{32}\}/G_{33}. \end{aligned} \tag{7}$$

The expressions for the components of $\boldsymbol{G}$ contain first and second order derivatives. Unfortunately, these expressions are too large to be printed here but are available on this paper's website [1].

As in Section 2.1 we can obtain a further motion vector $\boldsymbol{v}_4$ from the relation $\{v_{4x}^2, v_{4y}^2\} = \{G_{11}, G_{22}\}/G_{33}$.

2.3 Scalar curvature

So far we have considered tensor-based representations of spatio-temporal patterns. It can be useful, however, to consider also scalar quantities that can be derived from S. The scalar curvature C is a contraction of $\mathbf{R}$ [10, 11]. Under the constraint (4) C simplifies to

$$C = \frac{2\ (1 + \boldsymbol{v} \cdot \boldsymbol{v})\ \left(f_{\xi\xi} f_{\chi\chi} - {f_{\chi\xi}}^2\right)}{\left(1 + \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} f + (\boldsymbol{\nabla} f \cdot \boldsymbol{v})^2\right)^2} \tag{8}$$

[2] Note that if we simplify the indices in Eqs. 5 and 6, i.e., we just set 3221/2121 = 3/1, 3121/2121 = 3/2, ... , we obtain {3/1, 3/2} for the first three vectors and {33/11, 33/22} for $\{v_{4x}^2, v_{4y}^2\}$.

with f being a function of (χ, ξ) where $\chi = x - v_x t$, $\xi = y - v_y t$, and $\nabla f = \{f_\chi, f_\xi\}$. The dot "·" denotes the scalar product and indices in f_χ and $f_{\chi\chi}$ denote first- and second order partial derivatives respectively. Note that for zero velocity, the 3D scalar curvature is just the 2D Gaussian curvature in (x, y), as should be expected.

3 Lie transformation groups

So far we have considered spatio-temporal patterns that arise from a translation, however, spatio-temporal patterns can result from a variety of transformations. To investigate how the constant brightness constraint is modified in this case we shall make use of the theory of Lie transformation groups [9].

If the image is transformed by the action of a linear one-parameter Lie transformation group, whose infinitesimal operator $X_\lambda = a_1(x, y)\partial/\partial x + a_2(x, y)\partial/\partial y$, λ being the parameter of the transformation, then the fundamental flow constraint can be written as

$$f(\boldsymbol{r}, t) = f(\boldsymbol{r'}, t + dt), \tag{9}$$

where $\boldsymbol{r} = \{x, y\}$ and $\boldsymbol{r'} = \boldsymbol{r} + d\boldsymbol{r}$. The transformation $\boldsymbol{r} \to \boldsymbol{r'}$ results in

$$dx = x' - x = a_1(x, y)d\lambda \qquad dy = y' - y = a_2(x, y)d\lambda. \tag{10}$$

A straightforward application (omitted here for brevity) of Lie group theory shows that Eq. 9 leads to

$$-\frac{\partial f}{\partial t} = X_\lambda f \frac{d\lambda}{dt} = \nabla f \cdot \boldsymbol{a} \frac{d\lambda}{dt}, \tag{11}$$

$\boldsymbol{a}(x, y) = \{a_1(x, y), a_2(x, y)\}^T$ and here λ has been considered a function of t, as it must be in case of motion.

If several transformation groups are considered, with differential operators X_{λ_j} then Eq. 11 becomes

$$-\frac{\partial f}{\partial t} = \sum_j X_{\lambda_j} f \frac{d\lambda_j}{dt} = \nabla f \cdot \left(\sum_j \boldsymbol{a}_j \frac{d\lambda_j}{dt} \right). \tag{12}$$

Suppose $d\lambda_j/dt = \nu_j$ to be constant and that $a_{j1} = a_{j1}(y)$, $a_{j2} = a_{j1}(x)$, then the solution of Eq. 12 is

$$f(x, y, t) = f\left(x - \sum_j a_{j1}(y)\nu_j t, y - \sum_j a_{j2}(x)\nu_j t \right). \tag{13}$$

For instance consider the general rigid motion in $2D$, that is given by two translations along the coordinate axis and by a rotation; in this case $\boldsymbol{a}_1 = \{1, 0\}, \nu_1 = v_{Ox}$, $\boldsymbol{a}_2 = \{0, 1\}, \nu_2 = v_{Oy}$, where O is the center of rotation. Then the velocity of the center of rotation is just $\boldsymbol{v}_O = \nu_1 \boldsymbol{a}_1 + \nu_2 \boldsymbol{a}_2 = \{v_{Ox}, v_{Oy}\}$ that can also be obtained by usual kinematics. The rotation around O is given by $\boldsymbol{a}_3 = \{-y, x\}$, $\nu_3 = \omega$, where ω is the angular velocity. Suppose $\boldsymbol{v}_O$ and ω constant; Eq. 13 becomes

$$f(x, y, t) = f\left(x - v_{Ox}t + \omega yt, y - v_{Oy}t - \omega xt\right), \tag{14}$$

where x, y are coordinates with respect to O.

For this general case, however, it seems difficult to analyze the effect of such patterns on the spatio-temporal curvature without additional assumptions.

From Eq. 14 a rotation constraint is defined by

$$f(x, y, t) = f\left(x + \omega yt, y - \omega xt\right). \tag{15}$$

As Eq. 15 itself, the results for this transformation can be obtained by simply setting $v_{Ox} = 0$ and $v_{Oy} = 0$ in Eq. 14 and in the equations obtained below for the transformation (14).

3.1 Riemann curvature tensor

For this type of input the vectors v_i differ, and they depend on $x, y, t, v_{Ox}, v_{Oy}, \omega$ and the first and second order derivatives of $f(\chi, \xi)$ with $\chi = x - v_{Ox}t + \omega yt$ and $\xi = y - v_{Oy}t - \omega xt$.

However, we obtain interesting results if we further assume that the gradient of f vanishes. In this case the components of $\mathbf{R}$ are:

$$\begin{aligned}
R_{2121} &= D\big(1 + t^2\omega^2\big)^2 \\
R_{3131} &= D\big(v_{Oy} + tv_{Ox}\omega + x\omega - ty\omega^2\big)^2 \\
R_{3232} &= D\big(v_{Ox} - tv_{Oy}\omega - y\omega - tx\omega^2\big)^2 \\
R_{3121} &= D\left(1 + t^2\omega^2\right)\left(-v_{Oy} - tv_{Ox}\omega - x\omega + ty\omega^2\right) \\
R_{3221} &= -D\left(1 + t^2\omega^2\right)\left(-v_{Ox} + tv_{Oy}\omega + y\omega + tx\omega^2\right) \\
R_{3231} &= -D\left(v_{Ox} - tv_{Oy}\omega - y\omega - tx\omega^2\right)\left(v_{Oy} + tv_{Ox}\omega + x\omega - ty\omega^2\right) \\
with &: \\
D &= f_{\chi\chi} f_{\xi\xi} - {f_{\chi\xi}}^2
\end{aligned} \tag{16}$$

and f as a function of (χ, ξ) defined as above. It is straightforward to check that in this case all the vectors v_i (Eqs. 5 and 6) point in the same direction, which is the direction of the vector:

$$\{v_{Ox} - tv_{Oy}\omega - y\omega - tx\omega^2, -(v_{Oy} + tv_{Ox}\omega + x\omega - ty\omega^2)\} \tag{17}$$

3.2 Einstein tensor

For the Einstein tensor, again, we could not obtain useful simplifications but for the case of zero gradient. Surprisingly, the independent components of $\mathbf{G}$ are equal to those of **R** in this case (but for the signs):

$$
\begin{aligned}
G_{33} &= -\ R_{2121} \\
G_{22} &= -\ R_{3131} \\
G_{11} &= -\ R_{3232} \\
G_{32} &= \quad R_{3121} \\
G_{31} &= -\ R_{3221} \\
G_{21} &= \quad R_{3231}
\end{aligned}
\tag{18}
$$

3.3 Scalar curvature

For zero gradient the scalar curvature simplifies to

$$
C = -2D\left(1+t^2\omega^2\right)\left(1+{v_{Ox}}^2+{v_{Oy}}^2+2v_{Oy}x\omega-2v_{Ox}y\omega+\left(t^2+x^2+y^2\right)\omega^2\right) \tag{19}
$$

Note that for zero rotation and velocity, C is, in coordinates (χ, ξ), the 2D Gaussian curvature (with zero gradient).

4 Translation with time-dependent velocity

We now consider the more general case where the image shift contains higher-order terms, i.e., the motion can be accelerated, i.e.,

$$
f(x, y, t) = f(x - d_1(t), y - d_2(t)). \tag{20}
$$

4.1 Riemann curvature tensor

With the constraint in Eq. 20, we still obtain for the curvature tensor

$$
\{R_{3221}, -R_{3121}\}/R_{2121} = \{d_1'(t), d_2'(t)\}, \tag{21}
$$

but the other three expressions $\{R_{3231}, -R_{3131}\}/R_{3121}$, $\{R_{3232}, -R_{3231}\}/R_{3221}$, and $\{R_{3232}, R_{3131}\}/R_{2121}$ do not simplify to yield the velocity components.

However, if we assume that the gradient of $f(\chi, \xi)$ vanishes (${f_\chi}^2 + f_\xi^2 = 0$), we obtain the following relations:

$$
\begin{aligned}
\{R_{3231}, -R_{3131}\}/R_{3121} &= \{d_1'(t), d_2'(t)\} \\
\{R_{3232}, -R_{3231}\}/R_{3221} &= \{d_1'(t), d_2'(t)\} \\
\{R_{3232}, R_{3131}\}/R_{2121} &= \{d_1'(t)^2, d_2'(t)^2\}
\end{aligned}
\tag{22}
$$

i.e., the motion vectors $\boldsymbol{v}_2$, $\boldsymbol{v}_3$, and $\boldsymbol{v}_4$ are obtained only for local extrema of $f(\chi, \xi)$ (that are extrema of $f(x, y)$ also).

4.2 Einstein tensor

For the Einstein tensor under the constraint (20) we could not obtain any simplifications. However, under the additional constraint of zero gradient (see above) we obtain:

$$\begin{array}{l} \{G_{11}, G_{21}\}/G_{31} = \{d_1'(t), d_2'(t)\} \\ \{G_{21}, G_{22}\}/G_{32} = \{d_1'(t), d_2'(t)\} \\ \{G_{31}, G_{32}\}/G_{33} = \{d_1'(t), d_2'(t)\} \\ \{G_{11}, G_{22}\}/G_{33} = \{d_1'(t)^2, d_2'(t)^2\} \end{array} \tag{23}$$

4.3 Scalar curvature

In case of the additional assumption of zero gradient (see above), the scalar curvature simplifies to:

$$C = 2(1 + {d_1}'(t)^2 + {d_2}'(t)^2)(f_{\xi\xi} f_{\chi\chi} - {f_{\chi\xi}}^2) \tag{24}$$

with f being a function of (χ, ξ) where $\chi = x - d_1(t)$, $\xi = y - d_2(t)$.

5 Discontinuous motion

In this section we consider different types of motion discontinuities and how they are represented by the curvature tensor. In particular, we will show that the expressions for the vectors $\boldsymbol{v}_i$ in Eqs. 5 and 6 differ. Therefore the differences can be used as indicators of discontinuous motions [3, 4]. An exception are the locations where the gradient of f vanishes (local extrema).

5.1 Velocity step

We first consider the case where the velocity vector changes suddenly from zero to $\{v_x, v_y\}$, i.e. the image-sequence intensity $f(x, y, t)$ is defined by

$$f(x, y, t) = f(x - v_x\gamma(t), y - v_y\gamma(t)) \tag{25}$$

where $\gamma(t)$ is the unit step function. We obtain

$$\{R_{3221}, -R_{3121}\}/R_{2121} = \{\delta(t)v_x, -\delta(t)v_y\} \tag{26}$$

where $\delta(t)$ is the Dirac-Delta distribution.

Note that this vector is different from zero only at $t = 0$ when it points in the direction of the motion vector $\{v_x, v_y\}$, i.e., $-R_{3121}/R_{3221} = v_y/v_x$. This is not the case for the other three vectors in Eqs. 5 and 6. For example, for $\boldsymbol{v}_2$ we obtain:

$$-R_{3131}/R_{3231} = \frac{v_y^2\delta(t)^2 {f_{\chi\xi}}^2 + v_y\delta'(t) f_\xi f_{\chi\chi} - {v_y}^2\delta(t)^2 f_{\xi\xi} f_{\chi\chi} + v_x\delta'(t) f_\chi f_{\chi\chi}}{\delta(t)^2(v_x v_y f_{\xi\xi} f_{\chi\chi} - v_x v_y {f_{\chi\xi}}^2)} \tag{27}$$

with f as a function of (χ, ξ) and $\chi = x - v_x\gamma(t)$, $\xi = y - v_y\gamma(t)$. Similar but different expressions are obtained for R_{3231}/R_{3232} and R_{3131}/R_{3232}. For the extrema of $f(\chi, \xi)$ (assumption of zero gradient as above), however, all the four vectors (Eq. 5 and 6) point in the direction of $\{v_x, v_y\}$.

5.2 Onset of a spatial pattern

Here we consider the case:

$$f(x, y, t) \rightarrow f(x, y)\gamma(t) \tag{28}$$

i.e., the spatial pattern $f(x, y)$ is turned on at time $t = 0$.

We obtain the following results:

$$\begin{aligned}
\frac{\{R_{3221}, -R_{3121}\}}{R_{2121}} &= \delta(t)\{\frac{f_{yy}f_x - f_y f_{xy}}{{f_{xy}}^2 - f_{yy}f_{xx}}, \frac{f_x f_{xy} - f_y f_{xx}}{{f_{xy}}^2 - f_{yy}f_{xx}}\} \\
\frac{\{R_{3231}, -R_{3131}\}}{R_{3121}} &= \{\frac{\delta(t)^2 f_y f_x - f(x,y)\gamma(t)\delta'(t)f_{xy}}{\delta(t)(\gamma(t)f_x f_{xy} - \gamma(t)f_y f_{xx})}, \frac{-(\delta(t)^2 {f_x}^2) + f(x,y)\gamma(t)\delta'(t)f_{xx}}{-(\delta(t)\gamma(t)f_x f_{xy}) + \delta(t)\gamma(t)f_y f_{xx}}\} \\
\frac{\{R_{3232}, -R_{3231}\}}{R_{3221}} &= \{\frac{-\delta(t)^2 {f_y}^2 + f(x,y)\gamma(t)\delta'(t)f_{yy}}{-\delta(t)\gamma(t)f_{yy}f_x + \delta(t)\gamma(t)f_y f_{xy}}, \frac{-\delta(t)^2 f_y f_x + f(x,y)\gamma(t)\delta'(t)f_{xy}}{-\delta(t)\gamma(t)f_{yy}f_x + \delta(t)\gamma(t)f_y f_{xy}}\} \\
\frac{\{R_{3232}, R_{3131}\}}{R_{2121}} &= \{\frac{-\delta(t)^2 {f_y}^2 + f(x,y)\gamma(t)\delta'(t)f_{yy}}{-\gamma(t){f_{xy}}^2 + \gamma(t)f_{yy}f_{xx}}, \frac{-\delta(t)^2 {f_x}^2 + f(x,y)\gamma(t)\delta'(t)f_{xx}}{-\gamma(t){f_{xy}}^2 + \gamma(t)f_{yy}f_{xx}}\}
\end{aligned} \tag{29}$$

Note that the four expressions, which are equal for translations, differ for this specific dynamic pattern. For this type of input (Eq. 28) it is interesting to look at the components of $\boldsymbol{R}$ for the case of zero spatial gradient. We obtain the following results:

$$\begin{aligned}
R_{2121} &= \gamma(t)(-{f_{xy}}^2 + f_{yy}f_{xx})/N \\
R_{3131} &= (f(x,y)\gamma(t)\delta'(t)f_{xx})/N \\
R_{3232} &= (f(x,y)\gamma(t)\delta'(t)f_{yy})/N \\
R_{3121} &= 0 \\
R_{3221} &= 0 \\
R_{3231} &= (f(x,y)\gamma(t)\delta'(t)f_{xy})/N \\
&\text{with:} \\
N &= 1 + \delta(t)^2 f(x,y)^2
\end{aligned} \tag{30}$$

Note that two of the components are zero, such the the vector $\boldsymbol{v}_1$ is zero and the vectors $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are undefined due to a zero denominator.

By substituting δ for γ, δ' for δ, and δ'' for δ' we obtain the results for flashing pattern, i.e., $f(x, y, t) \rightarrow f(x, y)\delta(t)$. The above results are a special case of modulation, i.e. $f(x, y, t) \rightarrow f(x, y)a(t)$ with $a(t) = \gamma(t)$ and $a'(t) = \delta(t)$.

6 Discussion

Differential geometry provides powerful tools for analyzing the geometric structure of multidimensional manifolds. With these tools it is possible to construct invariants that capture the structure of the manifold [10, 11]. We have considered the visual input as a manifold with a specific metric that is defined by image intensity $f(x, y, t)$ (it is the metric of the hypersurface in Eq. 1), and we have looked at the curvature tensor $\mathbf{R}$ of that manifold as the most prominent geometric invariant and at two specific contractions of $\mathbf{R}$ (we had also looked at the Ricci tensor but had not obtained any meaningful result). In particular, we have shown how selected constraints on f, that are related to motion, affect these geometric invariants. By doing so we have found novel methods for the computation of motion parameters.

Thus, the reported results show that relevant information about spatio-temporal patterns can be gained by analyzing the above-mentioned curvature measures. We have first considered translations and have obtained new expressions for the flow fields in terms of the components of the Einstein tensor. We have then generalized the usual constraint Eqs. 2 and 3 to the more general case of transformations that form Lie transformation groups. For these transformations we have shown how the transformations giving rise to spatio-temporal patterns are encoded by the curvature measures. Meaningful results, however, have been obtained only for zero gradient, i.e. the local extrema of $f(\chi, \xi)$ (that are extrema of $f(x, y)$ as well) with coordinates (χ, ξ) depending on the transformation. Finally, we have also considered discontinuous motions that have been described by step functions and Dirac-Delta distributions. These functions have been analyzed analytically as global patterns, but, of course, the scope is to detect local discontinuities and motion boundaries. In practical applications, the size of the local neighborhood will be determined by the filters used to compute the derivatives, and these filters can be implemented on multiple scales.

Methods based on the four motion vectors derived from $\mathbf{R}$, i.e. Eqs. 5 and 6, have already been applied, both to obtain robust motion estimations and to model biological motion sensitivity [6, 3, 5]. The authors had assumed that the four motion vectors will differ in case of discontinuous motions and have used these differences as indicators of occlusions and noise. Here we have shown that the vectors do indeed differ for such patterns. It seems an important result that the vector $\boldsymbol{v}_1$ still yields the correct motion in case of accelerated motions (Eq. 21) but the other three vectors do not (except for zero gradient). For discontinuous motions the vector $\boldsymbol{v}_1$ again plays a distinct role and thus supports the idea of confidence measures based on the differences among the vectors $\boldsymbol{v}_i$. The question of how many vectors to use in which combinations still needs further investigation, but applications show that the use of all four vectors improves the results compared to using only two or three vectors.

In conclusion, we have shown that the intrinsic geometry of spatio-temporal patterns, generated by specific transformations, provides useful information on the parameters of the transformations and new insights for the coding of motion and dynamic features.

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A Components of the Riemann curvature tensor

$$
\begin{array}{ll}
R_{2121} & = (f_{yy}f_{xx} - {f_{xy}}^2)/(1+\boldsymbol{\nabla} f) \\
R_{3131} & = (f_{tt}f_{xx} - {f_{xt}}^2)/(1+\boldsymbol{\nabla} f) \\
R_{3232} & = (f_{tt}f_{yy} - {f_{yt}}^2)/(1+\boldsymbol{\nabla} f) \\
R_{3121} & = (f_{yt}f_{xx} - f_{xt}f_{xy})/(1+\boldsymbol{\nabla} f) \\
R_{3221} & = (f_{yt}f_{xy} - f_{yy}f_{xt})/(1+\boldsymbol{\nabla} f) \\
R_{3231} & = (f_{tt}f_{xy} - f_{xt}f_{yt})/(1+\boldsymbol{\nabla} f) \\
\text{with:} & \\
1+\boldsymbol{\nabla} f = 1 + {f_x}^2 + {f_y}^2 + {f_t}^2 &
\end{array}
\tag{31}
$$