On the Geometry of Piecewise Circular Curves

Thomas Banchoff and Peter Giblin

In this article we would like to promote a class of plane curves that have a number of special and attractive properties, the piecewise circular curves, or PC curves. (We feel constrained to point out that the term has nothing to do with Personal Computers, Privy Councils, or Political Correctness.) They are nearly as easy to define as polygons: a *PC curve* is given by a finite sequence of circular arcs or line segments, with the endpoint of one arc coinciding with the beginning point of the next. These curves are more versatile than polygons in that they can have a well-defined tangent line at every point: a PC curve is said to be *smooth* if the directed tangent line at the end of one arc coincides with the directed tangent line at the beginning of the next. (In particular, in a smooth PC curve, no arc degenerates to a single point.)

In the literature of descriptive geometry and more recently in computer graphics, PC curves have been used to approximate smooth curves so that the approximation is not only pointwise close, as in the case of an inscribed polygon, but also has the property that the tangent lines at the points of the smooth curve are approximated by the tangent lines of the PC curve. Given a pair of nearby points on a smooth curve together with their tangent directions, there will not in general be a single circular arc through the points with those directions at its endpoints, but there will be a family of *biarcs* meeting these boundary conditions, PC curves composed of two tangent circular arcs. (See **[M-N]** for a discussion of this construction.)

EXAMPLES OF PC CURVES. PC curves arise naturally as the solutions of a number of variational problems related to isoperimetric problems. A classical problem is to find the curve of shortest length enclosing a fixed area, and the solution is a circle. If the curve is required to surround a fixed pair of points, then the curve of shortest length enclosing a given area will be either a circle or a *lens* formed by two arcs of circles of the same radius meeting at the two points. More generally Besicovitch has shown that a curve of fixed length surrounding a given convex polygon and enclosing the maximum area must be a PC curve with all radii of arcs equal **[Be]**. One such curve is the Reuleaux "triangle", a three-arc PC curve enclosing an equilateral triangle, with each radius equal to the length of a side of the triangle. Such three-arc PC curves, and many far more elaborate examples can be found in the tracery of gothic windows **[A]**.

If we require that a curve of fixed length L surround a given pair of discs of the same radius, then, for a certain range of values of L, the curve that encloses the greatest area is a smooth convex PC curve consisting of two arcs on the boundary circles of the discs and two arcs of equal radius tangent to both discs. Such four-arc convex PC curves have long been used in engineering drawing for approximating ellipses, and we call such a curve a *PC ellipse* [FIGURE 1]. One special PC ellipse is

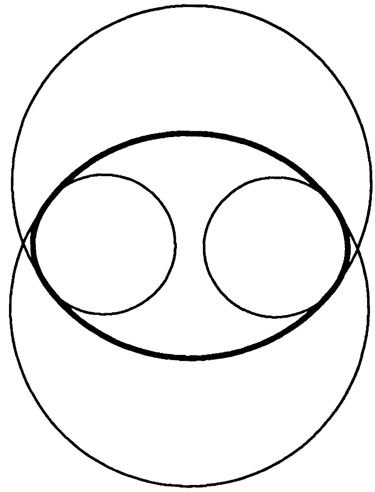


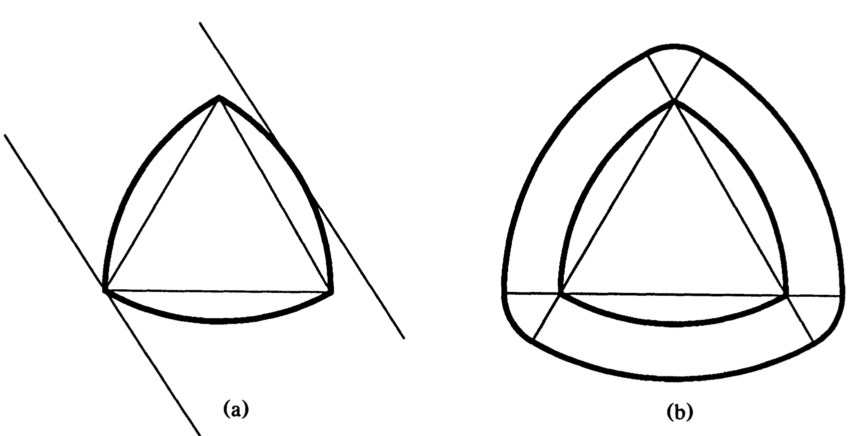
Figure 1

the boundary of the smallest convex set containing the two discs, called the *convex envelope* of the two discs, consisting of two semicircles and two line segments. (We thank Salvador Segura for pointing out the importance of PC curves in such isoperimetric problems.)

The collection of PC curves is invariant not only under Euclidean motions and scaling, but also under inversion with respect to a circle.

PARALLEL CURVES OF PC CURVES. The Reuleaux triangle is a non-smooth PC curve of *constant width*, so that every strip containing the curve and bounded by a pair of parallel lines through points of the curve has the same width. [FIGURE 2a]. We can obtain a smooth PC curve of constant width by taking an *outer parallel curve* of the Reuleaux triangle, i.e. the boundary of the parallel region, the locus of all points within a fixed distance of the points of Reuleaux triangle [FIGURE 2b].

This construction points out one of the main properties of PC curves: since the parallel curves of circular arcs are circular arcs, the parallel curves of a PC curve



(a) (b)

Figure 2

are PC curves. As an example, consider a convex PC ellipse. If we increase all of the radii by the same amount, keeping the same centers for the arcs, we obtain an outer parallel curve which is also a PC ellipse. This situation is in contrast with the case of an actual ellipse, for which the exterior parallel curves are not conic sections but rather algebraic curves of fourth degree.

Consideration of parallel curves is especially important in computer graphics, in particular in robotics, where it is necessary to find the centers of all discs of a fixed radius touching a given curve. In this subject, parallel curves are often called *offset* curves, obtained by moving away from the curve a given distance. For a curve defined by an algebraic equation, the offset curves are also algebraic, but the degrees of the offset curve is in general much higher [**R-R**].

If instead of increasing all radii of arcs of a PC ellipse, we decrease all radii by the same amount, keeping the same centers, we obtain the family of *inner parallel curves*. As in the case of the ordinary ellipse, for sufficiently small radius, the inner parallel curves remain smooth, and in the PC case, they remain PC ellipses. For an ellipse, after a certain distance the inner parallel curve develops four *cusps* where the directed tangent line reverses direction. Similarly, at a distance equal to the smaller radius, the parallel PC curve degenerates into a lens, and just after this we obtain a four-arc PC curve with cusps, where two arcs come together at the same tangent line but with different directions. Beyond a certain distance, the parallel curve of an ellipse is again a convex curve (but not a conic section). For a convex PC ellipse, the inner parallel curve at distance equal to the larger radius is a lens, and after that, it is again a convex PC ellipse.

EVOLUTE POLYGONS OF PC CURVES. In the case of an ellipse, the cusps of parallel curves trace out the *evolute curve*, consisting of the locus of centers of curvature of the ellipse. For a PC ellipse, the cusps of the parallel curves trace out the edges of a polygon with vertices at the centers of the arcs, called the *evolute polygon* of the PC curve [FIGURE 3]. If the radius of the inner parallel curve equals

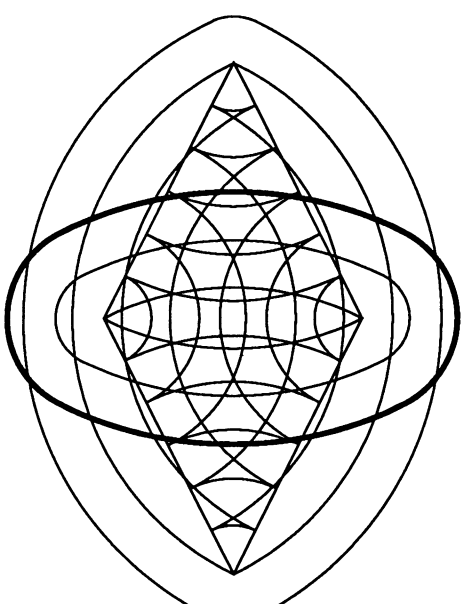


Figure 3

the radius of one of the arcs of a PC curve, then that arc degenerates to a single point. We say that a PC curve is *non-degenerate* if all arcs have non-zero length. Almost all parallel curves of a PC curve are non-degenerate.

We can obtain smooth PC curves by starting with a sequence of circles, each one tangent to its successor. The points of tangency divide each circle into two arcs, and choosing one arc from each circle gives a PC curve with a well-defined tangent line at each *node* where two successive arcs meet. If we wish the resulting PC curve to be smooth, then once we have chosen an arc from the first circle, the arcs on all subsequent circles are uniquely determined. If the last circle is tangent to the first, then this construction gives a *closed* PC curve. If each of the circles is externally tangent to the next, then the resulting PC curve which will be smooth if the number of arcs is even [FIGURE 4], while if the number of arcs is odd, then we inevitably obtain a cusp when we return to the starting point.

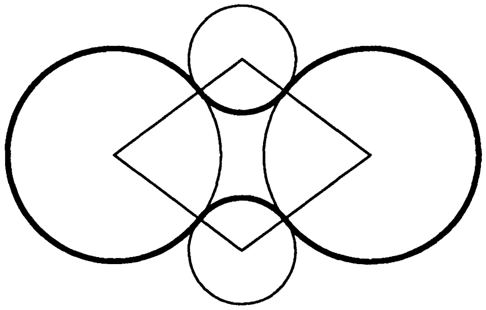
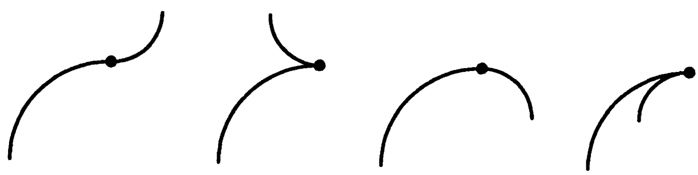


Figure 4

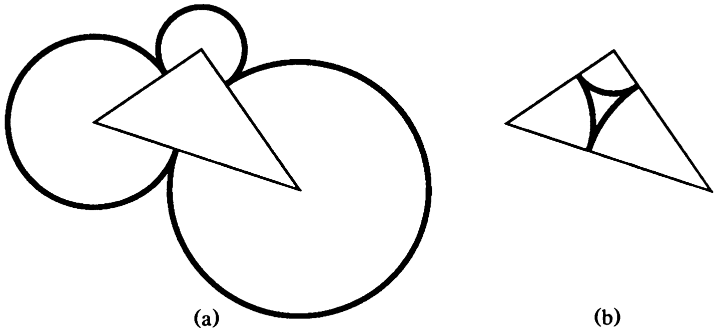
If two successive circles are externally tangent, then the node of the PC curve will either be a smooth *inflection point* [FIGURE 5a] if the tangent lines have the same direction or an *ordinary cusp* [FIGURE 5b] if the directions are different. In each of these cases, the two arcs lie on opposite sides of their common tangent line at the node. If two circles are internally tangent, then the two arcs at the node lie on the same side of their common tangent line, and we obtain either a *smooth locally convex point* [Figure 5c] if the tangent lines have the same direction or a *rhamphoid cusp* [FIGURE 5d] if the directions are different.



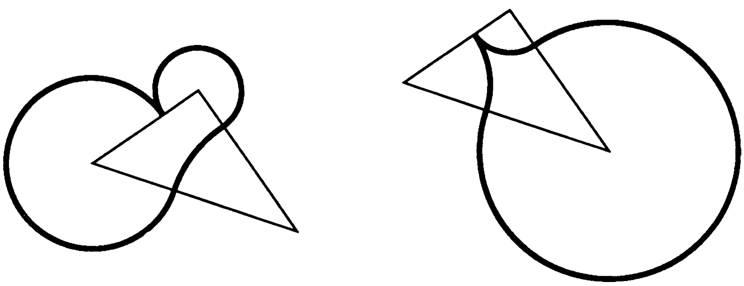
(a) (b) (c) (d)

Figure 5

Once we begin to consider curves with cusps, we can obtain many of them from the same collection of successively tangent circles. Selecting one of the two possible arcs of each of the circles, we get 2^n such curves if there are n circles. If n is odd, we obtain two such curves with cusps at all nodes [FIGURE 6a–d].



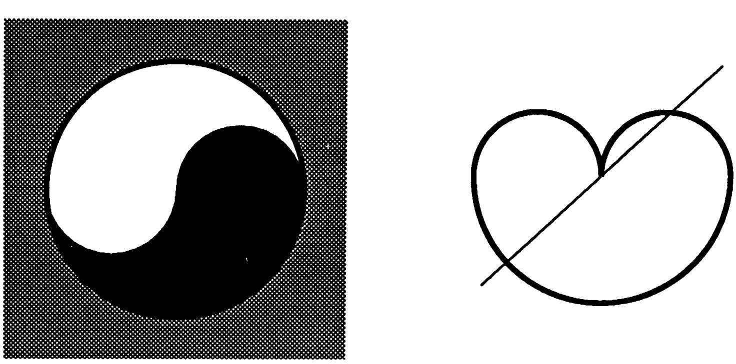
(a) (b)



(c) (d)

Figure 6

THREE-ARC PC CURVES. If two of the circles are inside a third, we can form eight PC curves in this way, leading to three distinct types, each of which appears in a classical guise. If the inner circles have half the radius of the outer one, then one such curve is the Yin-Yang curve, with one convex smooth node, one inflection node, and one node which is a rhamphoid cusp [FIGURE 7a]. From the same set of circles we can form the *PC cardioid* with two smooth convex points and one ordinary cusp. This curve appears in the work of the eighteenth century Jesuit geometer Roger Boscovich as an example of a non-centrally symmetric curve with



(a) (b)

Figure 7

a *center of length*, so that every line through this center cuts the curve into two pieces of equal length [FIGURE 7b]. A third type of PC curve determined by these three circles is the *arbelos*, or shoemaker's knife, with two rhamphoid cusps and one ordinary cusp. This curve was originally studied by Archimedes and Pappus, and it has inspired numerous articles in recreational mathematics, for example [**Ba**], [**Ga**], and [**H**].

EVOLUTES, INVOLUTES, AND OSCULATING CIRCLES. The sequence of centers of the circular arcs of a PC curve determines the *evolute polygon* of the curve. For a PC ellipse, the evolute polygon is a rhombus, and we can find non-convex PC curves with the same rhombus as its evolute polygon [FIGURE 4 and FIGURE 8]. Any parallel curve of either of these PC curves will be a PC curve with the same evolute polygon.

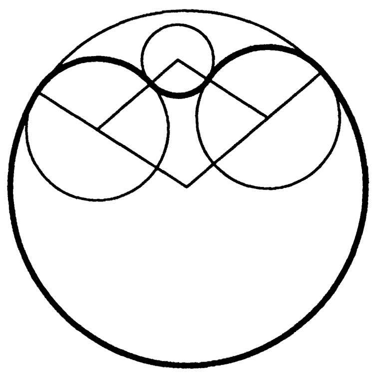


Figure 8

If we start with a sequence of circles, each one internally tangent to its successor and contained within it, then we obtain a *PC spiral*. The evolute polygon of the spiral will be a locally convex polygonal arc, and we may recover the spiral by a "string construction". We think of a string attached at one end of the polygonal arc and pulled tightly along it. As we unwind the string, keeping it tightly along the polygon at all times, the endpoint of the string traces out a PC curve with the polygon as evolute polygon [FIGURE 9]. By using a longer string, we may construct a parallel PC curve with the same evolute polygon. Such PC spirals have been used

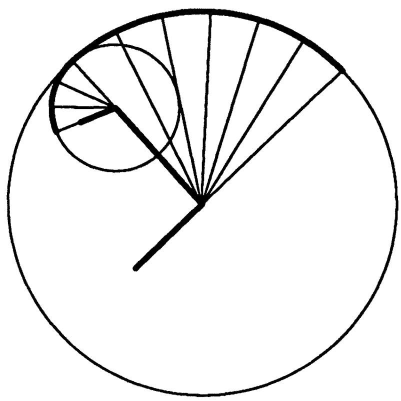


Figure 9

by several authors in computer-aided design as a means of approximating curves with increasing curvature [**M-P**].

For a smooth curve with continually increasing curvature, the best approximating circle at a point, called the *osculating circle* at the point, is defined by the properties that it is tangent to the curve at the point and it crosses from one side of the curve to the other near that point. The evolute curve is then the locus of centers of osculating circles at the points of the curve. At a node of a convex PC curve, the circles which are tangent to the curve at the node and which cross from one side of the curve to the other near the point have their centers on the segment joining the centers of the two arcs that meet at the node [FIGURE 10]. For this reason we may consider the evolute polygon as the locus of centers of "osculating circles" of the PC curve. At an inflection node, the circles tangent to the curve that cross from one side of the curve to the other have their centers on the line containing the centers of the arcs meeting at the node, but on the two rays that are the complement of the segment joining the two centers. In this case, the focal polygon is said to go to infinity.

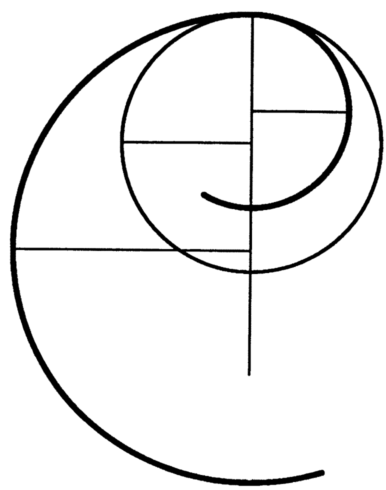


Figure 10

For a smooth spiral with continually increasing curvature, the radii of the osculating circles continually decrease, and conversely. A point where the curvature stops increasing and begins decreasing or conversely is called a *vertex*. For a PC curve, a *vertex arc* is an arc such that both adjacent arcs either are inside the circle of the arc or outside it. For a convex PC ellipse, each arc is a vertex arc.

FOUR-ARC PC CURVES. In the remainder of this article, we will discuss some results about closed four-arc PC curves, and point out an interesting connection between these and four-bar linkages in the plane. In effect, we show that all closed four-arc PC curves can be generated in a simple way from a very special class of "collapsed" quadrilaterals.

Let us establish some notational conventions. Let C_i be a circle with center c_i, $i = 1, 2, 3, 4$, each C_i being tangent to C_{i+1}. (We adopt the convention that all subscripts are to be reduced modulo 4, so for example C_4 is the same as C_0). Our

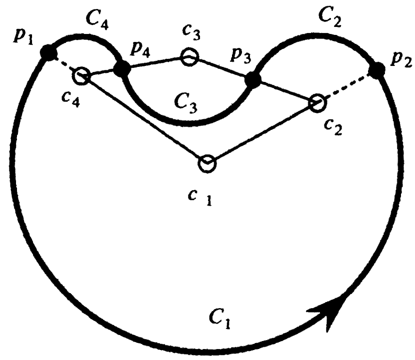


Figure 11

PC curve C will be made from successive arcs of the C_i, so that the quadrilateral $c_1c_2c_3c_4$ is the evolute polygon of C. The nodes of C will be denoted p_i, with p_i as the node where C_{i-1} meets C_i [FIGURE 11]. Finally the side-lengths of the evolute polygon will be denoted by l_i: this is the distance between c_{i-1} and c_i. In the example of FIGURE 11 it is clear, by splitting each l_i into a sum of two radii, that $l_1 + l_4 = l_2 + l_3$. Whenever we have a PC curve based on the above quadrilateral, we must have some relation of the form

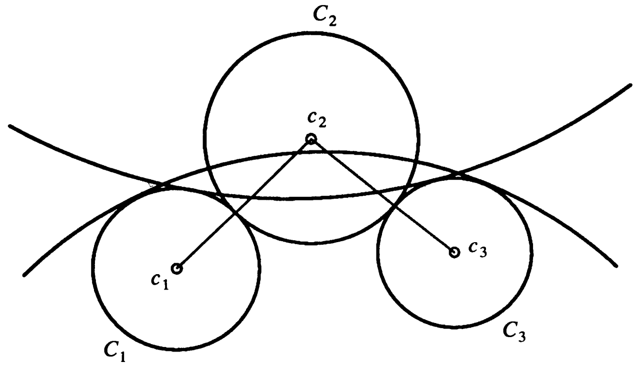
$$l_4 = \pm l_1 \pm l_2 \pm l_3, \tag{1}$$

for some choice of plus or minus signs.

Suppose we start with two circles, C_1 and C_3. What choice do we have for the centers of the remaining two circles? If, for example, the circles C_1 and C_3 are external to each other, as in FIGURE 12a, then a circle C_2 tangent to both has $|l_2 - l_3|$ equal to the sum or difference of the radii of C_1 and C_3. Thus, by a standard property of hyperbolas, the center of C_2 (and likewise of C_4) lies on one of two hyperbolas with foci at c_1 and c_3. For one hyperbola, C_1 and C_3 are both outside or both inside C_2; for the other hyperbola, one is outside and one is inside. If the radii of C_1 and C_3 are equal, then one of the hyperbolas degenerates to a straight line. If C_1 and C_3 are differently placed (for example if one is inside the other), or if one of the circles becomes a straight line (a "circle of infinite radius"), then there may be changes in the locus of possible centers for C_2. However, as the reader may verify, this locus always consists of two *conic sections*, i.e. an ellipse, a parabola, a hyperbola, or a straight line.

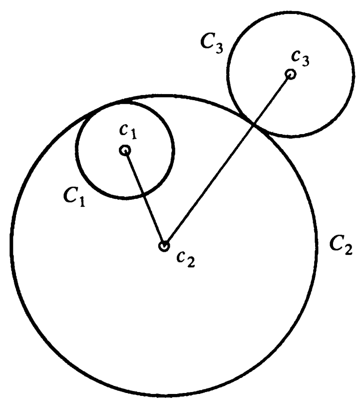
A particularly interesting construction which can be carried out for any PC curve C, is to consider the full locus of centers of *bitangent circles*, i.e. circles tangent to C in at least two places. This locus, which necessarily includes the centers of the arcs making up C, is the *symmetry set* of C. By the above remarks, the symmetry set consists of arcs of conic sections; in fact, two consecutive arcs will always meet with a common tangent line. In the final section of this paper, we give an intriguing example of a four-arc PC curve which has an *isolated point* on its symmetry set; a full discussion of symmetry sets of PC curves appears in **[Ba-G2]**.

Suppose now that we have four circles with each tangent to the next. As pointed out above, exactly two of the sixteen possible PC curves made up from arcs of these four circles are smooth.



$l_2 - l_3 = r_1 - r_3$

(a)



$l_3 - l_2 = r_1 + r_3$

(b)

Figure 12

In the case of four externally tangent circles (as in FIGURE 4), it is clear that the arcs making up one of these smooth PC curves must alternate between clockwise and counter-clockwise orientation. With the usual convention that clockwise curves have negative curvature and counter-clockwise curves have positive curvature, this implies that each of the four arcs is a vertex arc, as defined previously. The reader may like to verify that, with configurations other than externally tangent circles, all four arcs remain vertex arcs so long as the PC curve remains smooth and does not intersect itself. This establishes a version of the Four Vertex Theorem for four-arc PC curves. More generally, one can show that any smooth closed PC curve which does not intersect itself has at least four vertex arcs. This can be proved using an argument analogous to that of Osserman, for the classical Four Vertex Theorem [O].

PC CURVES AND FOUR-BAR LINKAGES. It is instructive to regard the evolute polygon as a linkage in the plane. This amounts to thinking of the edges as rigid rods connected at the endpoints c_i. We usually take c_1 and c_4 as fixed in position in the plane and we allow the other three rods to turn about their endpoints. As the quadrilateral changes in shape, so will any PC curve with this quadrilateral as its evolute. We may note that a collection of four tangential circles centered at the vertices c_i will roll on each other without slipping as the linkage changes shape.

The theory of four-bar linkages has been studied extensively. In that theory, it is shown that a linkage can move continuously from any position into a collapsed position, where all the centers are on a straight line (and hence all nodes lie on the same line, too), provided that some relation of the form (1) holds (where l_4 is the length of the fixed rod). In certain cases, namely

$$l_4 = l_1 + l_2 + l_3 \quad \text{and} \quad l_4 = -l_1 + l_2 - l_3,$$

there is only one possible position for the linkage and that is when it is in a collapsed position, so the result holds automatically in these cases. We state and prove the result below, using an elementary argument; for a more general setting of this result, see for example [G-N].

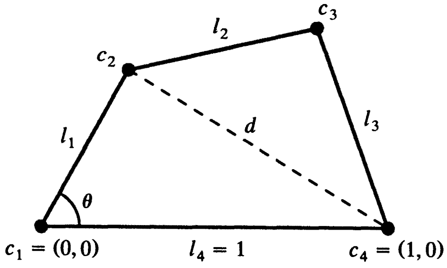
For us, the main significance of the collapsing lemma is a sort of converse construction:

Proposition A. *Every closed four-arc PC curve can be obtained by starting with one based on a collapsed quadrilateral and "uncollapsing" it, keeping the radii of the circles unchanged as the quadrilateral moves away from the collapsed position.*

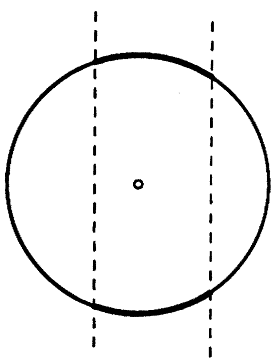
Note that since the radii remain unchanged, so do the edge lengths and the PC curve remains closed as the quadrilateral uncollapses. The proposition is an immediate consequence of the following lemma:

Collapsing Lemma. *Any quadrilateral satisfying* (1) *can be continuously collapsed so that its four vertices are collinear.*

Proof of Lemma: Let us fix $l_4 = 1$ and take $c_1 = (0,0), c_4 = (1,0), c_2 = (l\cos t, l\sin t)$ as in FIGURE 13a. The condition for c_3 to exist for this position of c_2 is $|l_2 - l_3| \leqslant d \leqslant l_2 + l_3$, where d is the distance from c_2 to c_4. This is



(a)



(b)

Figure 13

equivalent to

$$l_1^2 + 1 - (l_2 + l_3)^2 \leqslant 2l_1 \cos t \leqslant l_1^2 + 1 - (l_2 - l_3)^2.$$

The allowable values of t are therefore those points on the unit circle between two vertical lines, one or both of which may actually miss the circle [FIGURE 13b]. We need only check that

(i) if $l_1 \pm l_2 \pm l_3 = 1$, then $t = 0$ is an allowable value, that is, $2l_1 \leqslant l_1^2 + 1 - (l_2 - l_3)^2$;

(ii) if $-l_1 \pm l_2 \pm l_3 = 1$, then $t = \pi$ is an allowable value, that is, $l_1^2 + 1 - (l_2 + l_3)^2 \leqslant -2l_1$.

Since, in (i), $(l_1 - 1)^2 = (l_2 \pm l_3)^2$, and, in (ii), $(l_1 + 1)^2 = (l_2 \pm l_3)^2$ the results are immediate.

FIGURE 14a gives an example of a closed four-arc PC curve in which the four centers have become collinear. Despite its ordinary appearance, however, this construction is very special: in this case, the lengths satisfy $l_1 = l_3$ and $l_2 = l_4$, which implies that the quadrilateral of centers, before collapsing, was a parallelogram. As the quadrilateral unfolds, the PC curve evolves as shown in FIGURE 14b. On the other hand, when no special relation holds among the l_i, besides (1), it turns out that, when the quadrilateral has collapsed, the PC curve has become degenerate. This is the content of the following result:

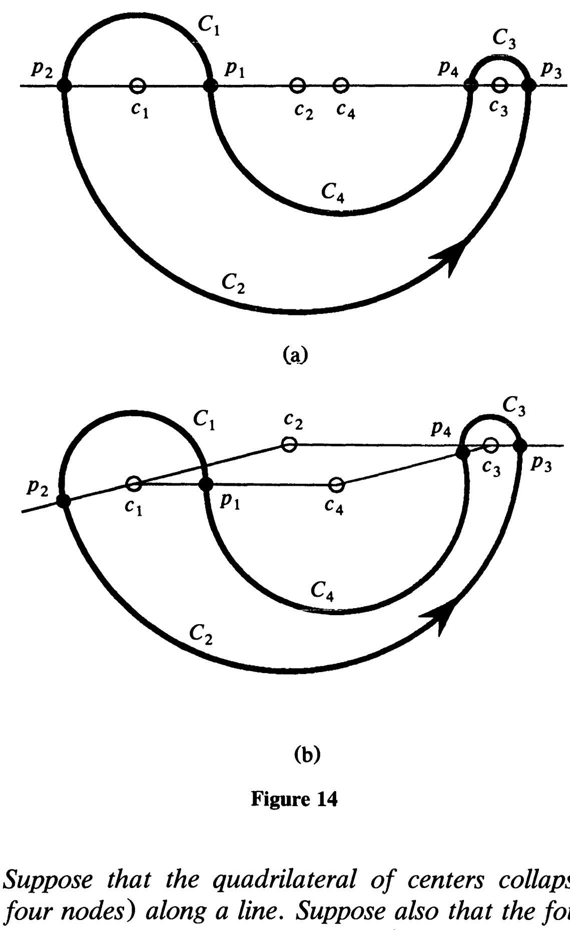


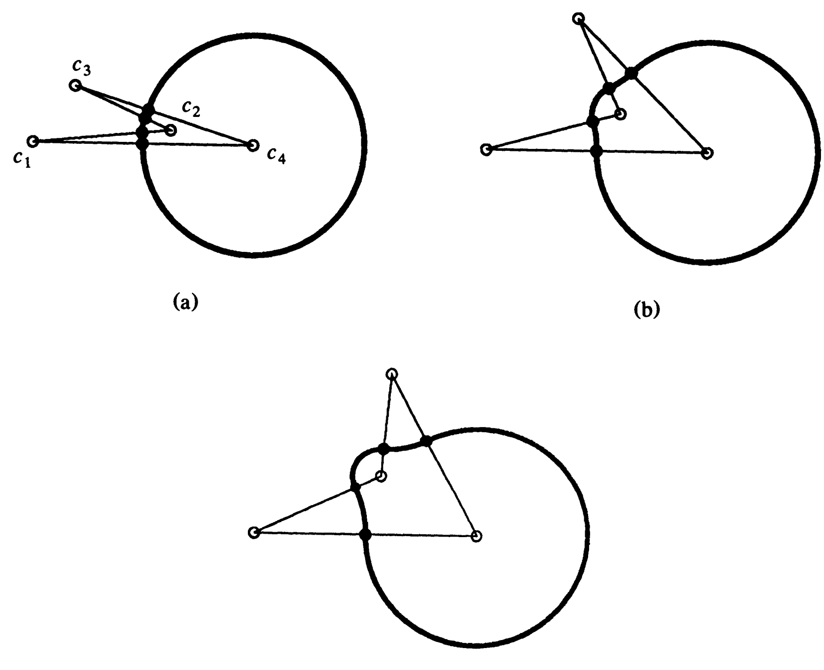
Figure 14

Proposition B. *Suppose that the quadrilateral of centers collapses, with the four centers (and the four nodes) along a line. Suppose also that the four nodes p_i are all distinct. Then the quadrilateral is a parallelogram (i.e. $l_1 = l_3$ and $l_2 = l_4$).*

Remark. It is "usually" true that, in the non-parallelogram case, all four nodes coincide. More precisely, if the centers c_i are all distinct and there is no restriction on the placing of the first node p_1, then either the quadrilateral is a parallelogram or all four nodes p_i coincide.

Proof of Proposition B: Since the centers and nodes are along a line, we can take this line to be the x-axis and describe them by their x-coordinates. Since $p_2 \neq p_1$, we must have $p_2 = 2c_1 - p_1$ since c_1 is the center of the segment from p_1 to p_2. Similarly $p_3 = 2c_2 - 2c_1 + p_1$, and $p_4 = 2c_3 - 2c_2 + 2c_1 - p_1$. Going one more step brings us back to p_1. This gives $c_1 + c_3 = c_2 + c_4$, which implies both $l_1 = l_3$ (i.e., $|c_4 - c_1| = |c_3 - c_2|$) and $l_2 = l_4$ (i.e., $|c_1 - c_2| = |c_4 - c_3|$). The remark is proved by examining all possibilities for p_2, p_3, p_4, given an initial p_1.

When two consecutive nodes of a PC curve coincide, we can take the arc joining them either as a complete circle or as a mere point. FIGURE 15a–c shows a PC curve growing out of a collapsed polygon of centers where all but one of the arcs is taken as a point.



(a) (b)

(c)

Figure 15

CONCLUDING REMARKS. Many of the topics we have introduced in this paper can be taken much further. Here we mention some natural extensions.

Closed PC curves with n arcs and a given evolute polygon fall into two classes, depending on whether the number of cusps is even or odd (smooth PC curves have zero cusps, an even number).

If the number of cusps is even, there is always a relation between the side-lengths of the polygon of the form

$$l_n = \pm l_1 \pm l_2 \pm \ldots \pm l_{n-1} \qquad (2)$$

analogous to (1) above. Furthermore, in this case, the radii of the PC curve can be varied to give a family of parallel PC curves. Examples are given above in FIGURES 1, 2b, 4, and 11.

If the number of cusps is odd, there is no restriction on the sides. There is a *unique* closed curve with a given evolute polygon, and the radii cannot be varied. Examples are the three-arc PC curves in FIGURES 5 and 6. Note that three-arc PC curves always have an odd number of cusps.

The extension of Proposition A and the Collapsing Lemma to PC curves with more than four arcs is straightforward but more complicated. We know of no easy proof that a polygon satisfying (2) for some choice of signs necessarily collapses continuously to a position where all the nodes are collinear. It would appear that this should be easier to achieve as n becomes larger, since the polygon becomes "floppier" as it has more degrees of freedom.

Finally we mention one remarkable example of a symmetry set of a PC curve. The four-arc PC curve C in FIGURE 16a has a *biosculating circle* S, i.e. S is an osculating circle at two points p and q. By definition, the center of S is part of the symmetry set of C, but it is an isolated point of the symmetry set since there are no circles near S that are tangent to C at two points. If we perturb the curve C by moving the node at p slightly counter-clockwise round C_1, and adjusting C_4 and C_3 accordingly, a family of bitangent circles appears to grow out from S. An enlarged picture of the locus of centers of curvature of these bitangent circles is shown in FIGURE 16b. If we move p the other way, all of these bitangent circles come together and disappear. In the study of symmetry sets of one-parameter families of plane curves, this transition is called a *moth*. In **[G-B]** and **[Br-G]** there are extensive discussions of such transition phenomena for general smooth curves. Although many of the transitions for symmetry sets of general smooth curves already appear in the study of plane polygons, as in **[Ba-G1]**, not all of them do, and part of our motivation for studying PC curves was an attempt to find an elementary class of curves for which all general phenomena were already present.

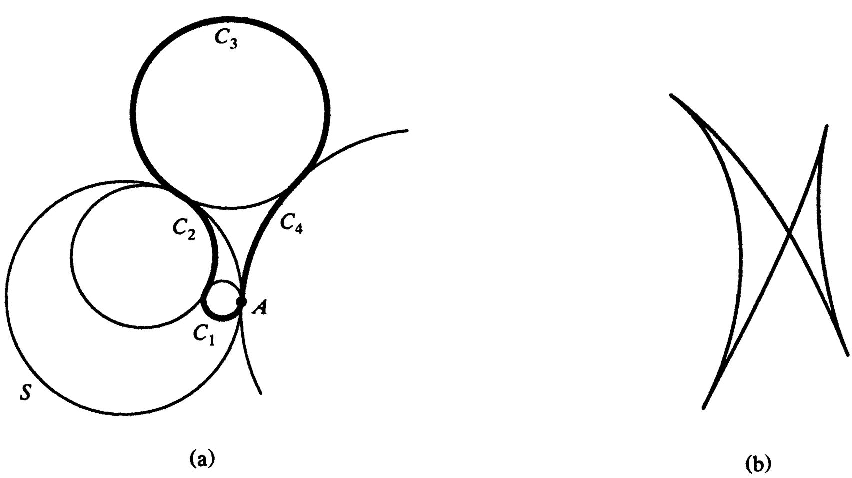


Figure 16

A full discussion of the symmetry sets of PC curves appears in **[Ba-G2]**. Many of the notions in this paper generalize to PC curves in space and in higher dimensions, and we intend to pursue these ideas in a subsequent paper.

We would like to thank Davide Cervone for assistance in producing the computer-generated illustrations for this paper.

REFERENCES

[A] Artmann, Benno "The Cloisters of Hauterive" Math. Intelligencer 13:2 (1991) 44–49.

[Ba-G1] Banchoff, T. and Giblin, P. "Global Theorems for Symmetry Sets of Smooth Curves and Polygons in the Plane" Proc. Royal Soc. of Edinburgh 106A (1987) 221–231.

[Ba-G2] Banchoff, T. and Giblin, P. "Symmetry Sets of Piecewise Circular Curves" Proc. Royal Soc. of Edinburgh 123A (1993) 1135–1149.

[Be] Besicovitch, A. S. "Variants of a Classical Isoperimetric Problem" Quart. J. Math. (2), 3 (1952) 42–9.

[Br-G] Bruce, J. W. and Giblin, P. "Growth, Motion and 1-Parameter Families of Symmetry Sets" Proc. Royal Soc. of Edinburgh 104A (1986) 179–204.

[Ga] Gaba, M. G. "On a Generalization of the Arbelos" Amer. Math. Monthly 47 (1940) 19–24.

[G-Br] Giblin, P. and Brassett, A. "Local Symmetry of Plane Curves" Amer. Math. Monthly 92:10 (1985) 689–707.

[G-N] Gibson, C. G. and Newstead, P. E. "On the Geometry of the Planar 4-Bar Mechanism" Acta Applicandae Mathematicae 7 (1986) 113–135.

[H] Hood, Rodney "A Chain of Circles" The Mathematics Teacher (1961) 134–137.

[M-P] Marciniak, K. and Putz, B. "Approximation of Spirals by Piecewise Circular Curves of Fewest Circular Arc Segments" Computer Aided Design, Vol. 16, No. 2 (1984) 87–90.

[M-N] Martin, R. R. and Nutbourne, A. W. "Differential Geometry Applied to Curve and Surface Design" Vol. 1 (1988) Foundation Ellis Horwood.

[O] Osserman, R. "The Four-or-More Vertex Theorem" Amer. Math. Monthly 92 (1985) 332–337.

[R-R] Rossignac, J. R. and Requicha, A. A. G. Piecewise-Circular Curves for Geometric Modeling, IBM Journal of Research and Development (1987) 296–313.

[S] Sabin, M. "The Use of Piecewise Forms in the Numerical Representation of Shape" Report no. 60, Computer and Automation Institute, Hungarian Academy of Science, Budapest (1977).

Department of Mathematics
Brown University
Providence, RI 02912

Department of Pure Mathematics
University of Liverpool
Liverpool L69 3BX
England

Pascal's Theorem

The proof of Pascal's Theorem mentioned in Professor van Yzeren's article (MONTHLY 100, pp. 930–931) is not my own but the proof I learned in 11th grade Descriptive Geometry and the Mathematisches-Naturwissenschaftliches Gymnasium of Basel, Switzerland. When I wrote the book I therefore assumed that the proof was part of everybody's general mathematical education. I am quite sure that this proof was absorbed by Swiss Type C (science, A is classical languages, B modern languages) students for at least 50 years. It also appears in what was the standard Swiss high school text of Descriptive Geometry (Flükiger). I did check in Italian and German D.G. texts; the Italians do not have Pascal's theorem and a German University text does not prove it. Unfortunately, I do not have Austrian high school D.G. texts but I assume that at least pre-World War I Austrian texts did present a similar proof. To find out whether Dr. van Yzeren's proof was known somewhere one would have to comb through the school literature of the few countries that did require descriptive geometry in their high school curriculum.

Unfortunately, New Math has succeeded in destroying much of European education almost as much as it did American education.

H. Guggenheimer
P.O. Box 401
West Hempstead, *NY* 11552