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On the Geometry of the Tangent Bundle with the Cheeger-Gromoll Metric

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Abstract. Let (M, g) be a Riemannian manifold of constant sectional curvature κ and $(TM, \tilde{g})$ be the tangent bundle of M equipped with the Cheeger-Gromoll metric induced by g. We give necessary and sufficient conditions for TM having positive scalar curvature. This gives counterexamples to a stated theorem of Sekizawa.

1. Introduction.

A Riemannian metric g on a smooth manifold M gives rise to several natural Riemannian metrics on the tangent bundle TM of M. The best known example is the Sasaki metric $\hat{g}$ introduced in [6], see also [2]. In the present paper we study tangent bundles equipped with the so called Cheeger-Gromoll metric. Its construction was suggested in [1] but the first explicit description was given by Musso and Tricerri in [5].

In [7] Sekizawa calculates the Levi-Civita connection $\tilde{\nabla}$, the curvature tensor $\tilde{R}$, the sectional curvatures $\tilde{K}$ and the scalar curvature $\tilde{S}$ of the Cheeger-Gromoll metric. He then states in his Theorem 6.3 that if (M, g) is an m-dimensional manifold of constant sectional curvature $\kappa \geq -3(m-2)/m$, then $(TM, \tilde{g})$ has non-negative scalar curvature.

In this paper we prove the following results giving counterexamples to Sekizawa's statement.

THEOREM 1.1. *Let (M, g) be a surface of constant sectional curvature κ. Then there exists a real number $C_2 \geq 40$ such that the tangent bundle $(TM, \tilde{g})$*
 i. *has positive scalar curvature if and only if $\kappa \in [0, C_2)$,*
 ii. *has non-negative scalar curvature if and only if $\kappa \in [0, C_2]$.*

THEOREM 1.2. *Let (M, g) be a Riemannian manifold of dimension $m > 2$ and of constant sectional curvature κ. Then there exist real numbers $c_m < 0$ and $C_m > 60$ such that the tangent bundle $(TM, \tilde{g})$*
 i. *has positive scalar curvature if and only if $\kappa \in (c_m, C_m)$,*
 ii. *has non-negative scalar curvature if and only if $\kappa \in [c_m, C_m]$.*

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For further details on the geometry of tangent bundles equipped with natural metrics, such as those of Sasaki and Cheeger-Gromoll we refer the reader to the recent survey given in [4].

2. The Cheeger-Gromoll metric.

Let (M, g) be a Riemannian manifold and let (p, u) be a point on the tangent bundle TM of M. Then the Levi-Civita connection ∇ on M induces a natural splitting of the tangent space $T_{(p,u)}TM$ into its vertical and horizontal subspaces

$$T_{(p,u)} = \mathcal{V}_{(p,u)} \oplus \mathcal{H}_{(p,u)} .$$

This gives rise to the vertical and horizontal lifts X^v, X^h of a vector field X on M. The vertical subspace is the kernel of the bundle map $\pi : TTM \to TM$ at the point (p, u). A horizontal curve in TM corresponds to a vector field on M which is parallel with respect to the Levi-Civita connection ∇ on (M, g). The vector field U on TM given by $U_{(p,u)} = (u_p)^v$ is called *the canonical vector field*.

DEFINITION 2.1. Let (M, g) be a Riemannian manifold. Then the *Cheeger-Gromoll metric* $\tilde{g}$ on TM is the Riemannian metric on the tangent bundle TM given by

i. $\tilde{g}_{(p,u)}(X^h, Y^h) = g_p(X, Y)$,
ii. $\tilde{g}_{(p,u)}(X^h, Y^v) = 0$,
iii. $\tilde{g}_{(p,u)}(X^v, Y^v) = \frac{1}{\alpha}(g_p(X, Y) + g_p(X, u)g_p(Y, u))$

for all vectors $X, Y \in T_pM$. Here $\alpha = 1 + g(u, u)$.

In [7] Sekizawa calculates the Levi-Civita connection $\tilde{\nabla}$ and the curvature tensor $\tilde{R}$ of the Cheeger-Gromoll metric on TM. His results are partially contained in Lemmata 2.2 and 2.4.

LEMMA 2.2. *Let (M, g) be a Riemannian manifold and TM be its tangent bundle equipped with the Cheeger-Gromoll metric $\tilde{g}$. Then the Levi-Civita connection $\tilde{\nabla}$ of $(TM, \tilde{g})$ satisfies the following*:

$$\begin{aligned}\tilde{\nabla}_{X^v}Y^v =&\frac{1}{\alpha}((1+\alpha)\tilde{g}(X^v, Y^v)U - \tilde{g}(X^v, U)\tilde{g}(Y^v, U)U \\ &- \tilde{g}(X^v, U)Y^v - \tilde{g}(Y^v, U)X^v) .\end{aligned}$$

COROLLARY 2.3. *Let (M, g) be a Riemannian manifold and TM be its tangent bundle equipped with the Cheeger-Gromoll metric $\tilde{g}$. Then the projection map $\pi : TM \to M$ is a harmonic morphism.*

For further information on harmonic morphisms between Riemannian manifolds see the regularly updated bibliography on [3].

PROOF. It follows from Definition 2.1 and Lemma 2.2 that $\pi : TM \to M$ is a Riemannian submersion with totally geodesic fibres. Hence it is a harmonic morphism. □

LEMMA 2.4. *Let (M, g) be a Riemannian manifold and TM be its tangent bundle equipped with the Cheeger-Gromoll metric $\tilde{g}$. Then the curvature tensor $\tilde{R}$ of $(TM, \tilde{g})$ satisfies the following*:

$$\begin{aligned}\tilde{R}(X^v, Y^v)Z^v =& \frac{\alpha+2}{\alpha^2}(\tilde{g}(X^v, Z^v)g(Y, u)U - \tilde{g}(Y^v, Z^v)g(X, u)U) \\ &+ \frac{1+\alpha+\alpha^2}{\alpha^2}(\tilde{g}(Y^v, Z^v)X^v - \tilde{g}(X^v, Z^v)Y^v) \\ &+ \frac{\alpha+2}{\alpha^2}(g(X, u)g(Z, u)Y^v - g(Y, u)g(Z, u)X^v)\,.\end{aligned}$$

In his paper Sekizawa attempts to calculate the sectional curvatures $\tilde{K}$ and the scalar curvature $\tilde{S}$ for $(TM, \tilde{g})$ but unfortunately his calculations are wrong. In the rest of this section we shall correct the error and obtain valid expressions for $\tilde{K}$ and $\tilde{S}$.

Let $\|\cdot\|$ denote the norm with respect to the Cheeger-Gromoll metric $\tilde{g}$ and let $\tilde{Q}(V, W)$ be the square of the area of the parallelogram with sides $V, W \in T_{(p,u)}TM$ given by

$$\tilde{Q}(V, W) = \|V\|^2\|W\|^2 - \tilde{g}(V, W)^2\,.$$

Furthermore let $\tilde{G}$ be the $(2, 0)$-tensor on the tangent bundle TM given by

$$\tilde{G} : (V, W) \mapsto \tilde{g}(\tilde{R}(V, W)W, V)\,.$$

LEMMA 2.5. [7] *Let $X, Y \in T_pM$ be two orthonormal vectors in the tangent spaces T_pM of M at p. Then*

i. $\tilde{Q}(X^h, Y^h) = 1$,
ii. $\tilde{Q}(X^h, Y^v) = \frac{1}{\alpha}(1 + g(Y, u)^2)$,
iii. $\tilde{Q}(X^v, Y^v) = \frac{1}{\alpha^2}(1 + g(Y, u)^2 + g(X, u)^2)$.

Sekizawa's mistake is contained in the proof of the last part of the following Lemma.

LEMMA 2.6. *Let X, Y be two orthonormal vectors in the tangent space T_pM of M at p. Then*

i. $\tilde{G}(X^h, Y^h) = K(X, Y) - \frac{3}{4\alpha}|R(X, Y)u|^2$,
ii. $\tilde{G}(X^h, Y^v) = \frac{1}{4\alpha^2}|R(u, Y)X|^2$,
iii. $\tilde{G}(X^v, Y^v) = \frac{(1+\alpha+\alpha^2)}{\alpha^2}\tilde{Q}(X^v, Y^v) - \frac{(\alpha+2)}{\alpha^3}(g(X, u)^2 + g(Y, u)^2)$.

PROOF. Here we only prove iii. The rest can be found in [7].

$$\begin{aligned}\tilde{G}(X^v, Y^v) =& \tilde{g}(\tilde{R}(X^v, Y^v)Y^v, X^v) \\ &+ \frac{(\alpha+2)}{\alpha^2}(\tilde{g}(X^v, Y^v)g(Y, u)g(X, u) - \tilde{g}(Y^v, Y^v)g(X, u)^2) \\ &+ \frac{(1+\alpha+\alpha^2)}{\alpha^2}(\tilde{g}(Y^v, Y^v)\tilde{g}(X^v, X^v) - \tilde{g}(X^v, Y^v)) \\ &+ \frac{(\alpha+2)}{\alpha^2}(g(X, u)g(Y, u)\tilde{g}(X^v, Y^v) - g(Y, u)^2\tilde{g}(X^v, X^v))\end{aligned}$$

$$= \frac{(1+\alpha+\alpha^2)}{\alpha^2}\tilde{Q}(X^v, Y^v) - \frac{(\alpha+2)}{\alpha^3}(g(X,u)^2 + g(Y,u)^2)\,.$$

□

PROPOSITION 2.7. *Let (M, g) be a Riemannian manifold and TM be its tangent bundle equipped with the Cheeger-Gromoll metric $\tilde{g}$. Then the sectional curvatures $\tilde{K}$ of $(TM, \tilde{g})$ satisfy the following*:

i. $\tilde{K}(X^h, Y^h) = K(X, Y) - \frac{3}{4\alpha}|R(X, Y)u|^2$,

ii. $\tilde{K}(X^h, Y^v) = \frac{1}{4\alpha}\frac{|R(u,Y)X|^2}{(1+g(Y,u)^2)}$,

iii. $\tilde{K}(X^v, Y^v) = \frac{1-\alpha}{\alpha^2} + \frac{\alpha+2}{\alpha}\frac{1}{(1+g(X,u)^2+g(Y,u)^2)}$.

PROOF. The statements follow directly from the division of $\tilde{G}(X^i, Y^j)$ by $\tilde{Q}(X^i, Y^j)$ for $i, j \in \{h, v\}$. □

For a given point $(p, u) \in TM$ with $u \neq 0$ let $\{e_1, \cdots, e_m\}$ be an orthonormal basis for the tangent space T_pM of M at p such that $e_1 = u/|u|$, where $|u|$ is the norm of u with respect to the metric g on M. Then for $i \in \{1, \cdots, m\}$ and $k \in \{2, \cdots, m\}$ define the horizontal and vertical lifts by $f_i = e_i^h$, $f_{m+1} = e_1^v$ and $f_{m+k} = \sqrt{\alpha}e_k^v$. Then $\{f_1, \cdots, f_{2m}\}$ is an orthonormal basis for the tangent space $T_{(p,u)}M$ with respect to the Cheeger-Gromoll metric.

LEMMA 2.8. *Let (p, u) be a point on TM and $\{f_1, \cdots, f_{2m}\}$ be an orthonormal basis for the tangent space $T_{(p,u)}TM$ as above. Then the sectional curvatures $\tilde{K}$ satisfy the following equations.*

$$\begin{aligned}
\tilde{K}(f_i, f_j) &= K(e_i, e_j) - \frac{3}{4\alpha}|R(e_i, e_j)u|^2\,,\\
\tilde{K}(f_i, f_{m+1}) &= 0\,,\\
\tilde{K}(f_i, f_{m+k}) &= \frac{1}{4}|R(u, e_k)e_i|^2\,,\\
\tilde{K}(f_{m+1}, f_{m+k}) &= \frac{3}{\alpha^2}\,,\\
\tilde{K}(f_{m+k}, f_{m+l}) &= \frac{\alpha^2+\alpha+1}{\alpha^2}\,,
\end{aligned}$$

for $i, j \in \{1, \cdots, m\}$ and $k, l \in \{2, \cdots, m\}$.

PROOF. The result is a direct consequence of Proposition 2.7. □

PROPOSITION 2.9. *Let (M, g) be a Riemannian manifold with scalar curvature S. Let TM be its tangent bundle equipped with the Cheeger-Gromoll metric $\tilde{g}$ and (p, u) be a point on TM. Then the scalar curvature $\tilde{S}$ of $(TM, \tilde{g})$ satisfies the following*:

$$\tilde{S}_{(p,u)} = S_p + \frac{(2\alpha-3)}{4\alpha}\sum_{i,j=1}^{m}|R(e_i, e_j)u|^2$$

$$+\frac{(m-1)}{\alpha^2}(6+(m-2)(\alpha^2+\alpha+1))\,.$$

PROOF. Let $\{f_1, \cdots, f_{2m}\}$ be an orthonormal basis for the tangent space $T_{(p,u)}TM$ as above. By the definition of the scalar curvature we know that

$$\begin{aligned}
\tilde{S} =& \sum_{i\neq j} \tilde{K}(f_i, f_j)\\
=&2\sum_{\substack{i,j=1\\ i<j}}^{m} \tilde{K}(f_i, f_j) + 2\sum_{i,j=1}^{m} \tilde{K}(f_i, f_{m+j}) + 2\sum_{\substack{i,j=1\\ i<j}}^{m} \tilde{K}(f_{m+i}, f_{m+j})\\
=&\sum_{i\neq j}^{m} K(e_i, e_j) - \frac{3}{4\alpha}\sum_{i,j=1}^{m} |R(e_i, e_j)u|^2\\
&+\frac{1}{2}\sum_{i,j=1}^{m} |R(u, e_j)e_i|^2 + 2\sum_{i=2}^{m}\frac{3}{\alpha^2} + \sum_{\substack{i,j=2\\ i\neq j}}^{m} \frac{(1+\alpha+\alpha^2)}{\alpha^2}\\
=&S + \frac{(2\alpha-3)}{4\alpha}\sum_{i,j=1}^{m} |R(e_i, e_j)u|^2 + \frac{(m-1)}{\alpha^2}(6+(m-2)(1+\alpha+\alpha^2))\,.
\end{aligned}$$

For the above calculations we have used Lemma 2.10. □

LEMMA 2.10. *Let (M, g) be a Riemannian manifold with curvature tensor R. If $p \in M$ and $\{e_1, \cdots, e_m\}$ is an orthonormal basis for the tangent space T_pM, then*

$$\sum_{i,j=1}^{m} |R(e_j, u)e_i|^2 = \sum_{i,j=1}^{m} |R(e_j, e_i)u|^2\,.$$

PROOF. With $u = \sum_{i=1}^{m} u_i X_i$ we get

$$\begin{aligned}
\sum_{i,j=1}^{m} |R(e_j, u)e_i|^2 &= \sum_{i,j,k,l=1}^{m} u_k u_l g(R(e_j, e_k)e_i, R(e_j, e_l)e_i)\\
&= \sum_{i,j,k,l,s=1}^{m} u_k u_l g(R(e_s, e_i)e_k, e_j) g(R(e_s, e_i)e_l, e_j)\\
&= \sum_{i,j,k,l=1}^{m} u_k u_l g(R(e_j, e_i)e_k, R(e_j, e_i)e_k)\\
&= \sum_{i,j=1}^{m} |R(e_j, e_i)u|^2\,.
\end{aligned}$$

□

3. The constant curvature case.

In this section we shall assume that (M, g) is an m-dimensional Riemannian manifold of constant sectional curvature κ. This implies that its curvature tensor R takes the special form

$$R(X, Y)Z = \kappa(g(Y, Z)X - g(X, Z)Y)$$

and the scalar curvature S is given by $S = \kappa m(m-1)$.

PROPOSITION 3.1. *Let (M, g) be a Riemannian manifold of constant sectional curvature κ. Let TM be its tangent bundle equipped with the Cheeger-Gromoll metric $\tilde{g}$. Then the sectional curvatures $\tilde{K}$ of $(TM, \tilde{g})$ satisfy the following*:

i. $\tilde{K}(X^h, Y^h) = \kappa - \frac{3\kappa^2}{4\alpha}(g(u, X)^2 + g(u, Y)^2)$,

ii. $\tilde{K}(X^h, Y^v) = \frac{\kappa^2 g(X,u)^2}{4\alpha(1+g(Y,u)^2)}$,

iii. $\tilde{K}(X^v, Y^v) = \frac{1-\alpha}{\alpha^2} + \frac{\alpha+2}{\alpha}\frac{1}{1+g(X,u)^2+g(Y,u)^2}$,

for any orthonormal vectors $X, Y \in T_pM$.

PROOF. This is a simple calculation using the special form of the curvature tensor. □

PROPOSITION 3.2. *Let (M, g) be a Riemannian manifold of constant sectional curvature κ. Let TM be its tangent bundle equipped with the Cheeger-Gromoll metric $\tilde{g}$ and $\tilde{K}$ be the sectional curvatures of $(TM, \tilde{g})$. Then*

i. *$\tilde{K}(X^h, Y^h)$ is non-negative if $0 \leq \kappa \leq \frac{4}{3}$,*

ii. *$\tilde{K}(X^h, Y^v)$ is non-negative,*

iii. *$\tilde{K}(X^v, Y^v)$ is positive.*

PROOF. If $X, Y \in T_pM$ are orthonormal, then obviously

$$g(Y, u)^2 + g(X, u)^2 \leq |u|^2 < \alpha\,.$$

With this in hand the result follows directly by Proposition 3.1. □

PROPOSITION 3.3. *Let (M, g) be a Riemann manifold of constant sectional curvature κ. Let $(TM, \tilde{g})$ be the tangent bundle equipped with the Cheeger-Gromoll metric. Then the scalar curvature $\tilde{S}$ of TM is given by*

$$\begin{aligned}\tilde{S} =& \frac{(m-1)}{2\alpha^2}(\alpha(\alpha-1)(2\alpha-3)\kappa^2 + 2m\alpha^2\kappa \\ &+ 2(6 + (m-2)(1+\alpha+\alpha^2)))\,.\end{aligned}$$

PROOF. The result follows from Proposition 2.9 and the following calculations

$$\begin{aligned}\sum_{i,j=1}^{m} |R(e_i, e_j)u|^2 =& \sum_{i,j=1}^{m} (\alpha-1)\kappa^2|\delta_{1j}e_i - \delta_{i1}e_j|^2 \\ =& 2(m-1)(\alpha-1)\kappa^2\,.\end{aligned}$$

□

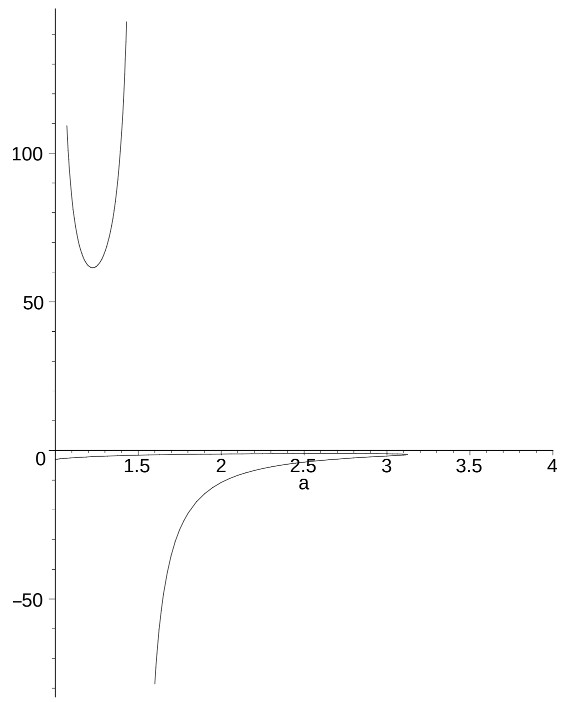

FIGURE 1.

For a given $m \geq 2$, we are now interested in determining the sign of the scalar curvature $\tilde{S}_m(\alpha, \kappa)$ as a function of $(\alpha, \kappa) \in D = [1, \infty) \times \mathbf{R}$. The contour $D_0 = \{(\alpha, \kappa) \in D |\ \tilde{S}_m(\alpha, \kappa) = 0\}$ in the (α, κ)-plane is determined by the second order polynomial equation

$$\alpha(\alpha - 1)(2\alpha - 3)\kappa^2 + 2m\alpha^2\kappa + 2(6 + (m - 2)(1 + \alpha + \alpha^2)) = 0 \tag{1}$$

in κ. If $\alpha \neq 1$, $\alpha \neq 3/2$ and the descriminant of equation (1) is non-negative we get the solutions

$$\kappa_{\pm} = \frac{-m\alpha^2 \pm \sqrt{m^2\alpha^4 - \alpha(\alpha - 1)(2\alpha - 3)2(6 + (m - 2)(1 + \alpha + \alpha^2))}}{\alpha(\alpha - 1)(2\alpha - 3)}.$$

In Figure 1 we have plotted the contour D_0 in the (α, κ)-plane for the case when $m = 3$. When removing D_0 from D the rest falls into three connected components. The scalar curvature is positive in the component D_+ containing the point $(2, 40)$ and negative in the other two.

We are interested in determining those $\kappa \in \mathbf{R}$ such that $S_3(\alpha, \kappa)$ is positive (non-negative) for all $\alpha \in [1, \infty)$ i.e. which are the horizontal half-lines completely contained in the component D_+. The connected component of D_0 contained in the upper halfspace ($\kappa > 0$) is the graph of the solution κ_- for $\alpha \in (1, 3/2)$. It has exactly one minimum $C_3 > 0$.

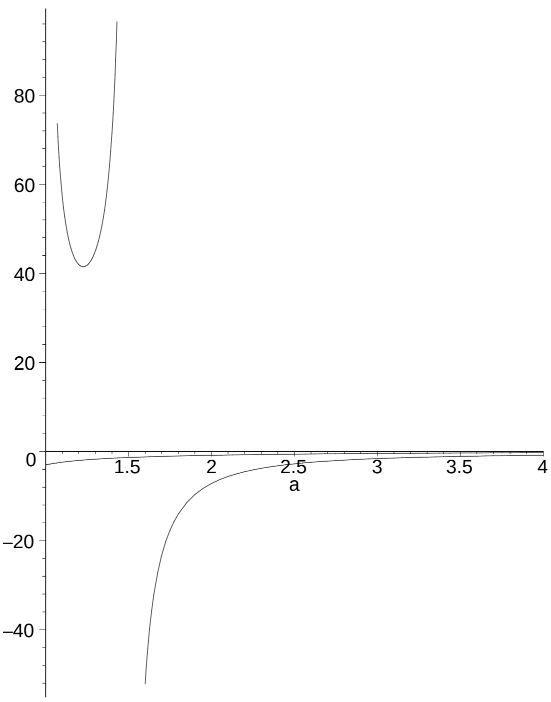

FIGURE 2.

The graph of the other solution κ_+, where defined, has excatly one maximum $c_3 < 0$. The family of horizontal lines that we are looking for are then parametrized by $\kappa \in (c_3, C_3)$.

It is easy to see that for $m > 3$ we get exactly the same qualitative behaviour of the two solutions κ_- and κ_+ as for $m = 3$. This provides us with the following result:

THEOREM 3.4. *Let (M, g) be a Riemannian manifold of dimension $m > 2$ and of constant sectional curvature κ. Then there exist real numbers $c_m < 0$ and $C_m > 60$ such that the tangent bundle $(TM, \tilde{g})$*

i. *has positive scalar curvature if and only if $\kappa \in (c_m, C_m)$,*
ii. *has non-negative scalar curvature if and only if $\kappa \in [c_m, C_m]$.*

Notice that for $\alpha \in (1, 3/2)$ and $m \geq 3$ the function κ is increasing in m, so if $m < \bar{m}$ then $C_m < C_{\bar{m}}$.

When $m = 2$ the descriminant of equation (1) is positive everywhere, and the solution κ_- and κ_+ are given by

$$\kappa_\pm = \frac{-2\alpha^2 \pm 2\sqrt{\alpha^4 - 3\alpha(\alpha - 1)(2\alpha - 3)}}{\alpha(\alpha - 1)(2\alpha - 3)}.$$

In Figure 2 we have plotted the contour D_0 in the (α, κ) for $m = 2$. When this is removed from D the rest falls into four connected components. The scalar curvature is positive in the two components containing the points $(2, \pm 40)$ and negative in the others. When $\alpha > 3/2$ both the solutions $\kappa_{\pm}$ are negative and approaching 0 in the limit $\alpha \to \infty$.

This leads to the following result

THEOREM 3.5. *Let (M, g) be a surface of constant sectional curvature κ. Then there exists a real number $C_2 \geq 40$ such that the tangent bundle $(TM, \tilde{g})$*

i. *has positive scalar curvature if and only if $\kappa \in [0, C_2)$,*

ii. *has non-negative scalar curvature if and only if $\kappa \in [0, C_2]$.*

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