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ON THE GIBBS PHENOMENON III: RECOVERING EXPONENTIAL ACCURACY IN A SUB-INTERVAL FROM A SPECTRAL PARTIAL SUM OF A PIECEWISE ANALYTIC FUNCTION

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ABSTRACT

We continue the investigation of overcoming Gibbs phenomenon, i.e., obtaining exponential accuracy at all points including at the discontinuities themselves, from the knowledge of a spectral partial sum of a discontinuous but piecewise analytic function.

We show that if we are given the first N expansion coefficients of an L_2 function $f(x)$ in terms of either the trigonometrical polynomials or the Chebyshev or Legendre polynomials, we can construct an **exponentially convergent approximation** to the point values of $f(x)$ in any sub-interval in which it is analytic.

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1 Introduction

In this paper we continue our investigation of overcoming Gibbs phenomenon, i.e., recovering pointwise exponential accuracy at all points including at the discontinuities themselves, from the knowledge of a spectral partial sum of a discontinuous but piecewise analytic function, which we started in [5] and [6].

Spectral approximations based upon trigonometric polynomials (Fourier, for periodic problems) or polynomials (Chebyshev or Legendre, for non-periodic problems) are exponentially accurate for analytic functions [4], [3]. However, for discontinuous but piecewise analytic functions, the spectral partial sum approximates the function poorly throughout the domain. Away from the discontinuity only first order accuracy is achieved. Near the discontinuity there are $O(1)$ oscillations which do not decrease with N, the number of terms retained in the spectral sum. This is known as Gibbs phenomenon.

In [5] and [6] we treated a representative problem :

- Let $f(x)$ be an *analytic but nonperiodic function.* Suppose that we are given its first $-N \leq k \leq N$ Fourier coefficients. Can one obtain exponentially convergent pointwise approximations in the maximum norm?

To solve this problem we used the Gegenbauer polynomials $C_n^\lambda(x)$, which are orthogonal in $[-1, 1]$ with the weight function $(1-x^2)^{\lambda-\frac{1}{2}}$.

The procedure of overcoming the Gibbs phenomenon consists of two steps:

1. Given the Fourier partial sum of the first N terms, we first recover the first $m \sim N$ Gegenbauer expansion coefficients with exponential accuracy. This can be achieved for any L_2 function, as long as we choose λ in the weight function of Gegenbauer polynomials to be proportional to N. The error incurred at this stage is called the **truncation error.**

2. The next step is to prove, *for an analytic function*, the exponential convergence of its Gegenbauer expansion, when the parameter λ in the weight function is

proportional to the number of terms retained in the expansion. The error at this stage is labelled the **regularization error.**

Thus we have shown how to overcome the Gibbs phenomenon for an analytic but non-periodic function. By a simple shift this procedure covers the case of any analytic function with one discontinuity.

In this paper we treat a broader class of problems. Here we assume that $f(x)$ is an L_2 function on $[-1,1]$ and analytic in a subinterval $[a,b] \subset [-1,1]$. We assume that the spectral partial sum (based on either the Fourier or Chebyshev or Legendre expansion) of a function over *the full interval* $[-1,1]$ is known, and try to recover exponentially accurate point values over a subinterval $[a,b]$.

We will follow the same path as in [5]. Basically we will show that the first N Fourier (or Chebyshev or Legendre) expansion coefficients contain enough information such that a rapidly converging *Gegenbauer* expansion in the subinterval $[a,b]$ can be constructed. As before, we will separate the analysis of the error into two parts: truncation error and regularization error. Truncation error measures the difference between the exact Gegenbauer coefficients and those obtained by using the spectral partial sum. These will be investigated in Section 3 for the Fourier case and in Section 4 for the Chebyshev and Legendre cases. Regularization error then measures the difference between the Gegenbauer expansion using the first few Gegenbauer coefficients and the function itself in a sub-interval $[a,b]$, in which the function is assumed analytic. The estimation of the regularization error is in Section 5. In Section 2 we will provide some useful properties of Gegenbauer polynomials to be used later. Section 6 contains a summary theorem and some remarks.

Throughout this paper, we will use A to denote a generic constant or at most a polynomial in the growing parameters. It may not be the same at different locations.

The message of this paper is that the knowledge of the partial spectral sum of an L_2 function in $[-1,1]$ furnishes enough information so that an *exponentially convergent approximation* can be constructed in any subinterval in which $f(x)$ is analytic.

In particular, if $f(x)$ is *piecewise analytic*, then an exponentially convergent approximation *in the maximum norm* can be recovered from its partial spectral expansion.

2 Preliminaries

This section is devoted to a collection of results about the Gegenbauer polynomials. Though the results are classical, they are not widely known and certainly not in our context. We rely heavily on the standardization in Bateman [2].

We start by defining the Gegenbauer polynomials $C_n^\lambda(x)$ in the following

Definition 2.1

The Gegenbauer polynomial $C_n^\lambda(x)$ is defined by

$$(1-x^2)^{\lambda-\frac{1}{2}} C_n^\lambda(x) = \frac{(-1)^n}{2^n n!} G(\lambda,n) \frac{d^n}{dx^n}\left[(1-x^2)^{n+\lambda-\frac{1}{2}}\right] \tag{2.1}$$

where $G(\lambda,n)$ is given by

$$G(\lambda,n) = \frac{\Gamma(\lambda+\frac{1}{2})\Gamma(n+2\lambda)}{\Gamma(2\lambda)\Gamma(n+\lambda+\frac{1}{2})} \tag{2.2}$$

□

Under this definition we have

$$C_n^\lambda(1) = \frac{\Gamma(n+2\lambda)}{n!\Gamma(2\lambda)} \tag{2.3}$$

and

$$|C_n^\lambda(x)| \leq C_n^\lambda(1), \qquad -1 \leq x \leq 1 \tag{2.4}$$

The Gegenbauer polynomials are orthogonal under their weight function, in fact we have

Lemma 2.2

The Gegenbauer polynomials satisfy the following orthogonality condition:

$$\int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} C_k^\lambda(x) C_n^\lambda(x) dx = \delta_{k,n} h_n^\lambda \tag{2.5}$$

where

$$h_n^\lambda = \pi^{\frac{1}{2}} C_n^\lambda(1) \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)(n+\lambda)} \tag{2.6}$$

$\square$

We will need to use heavily the asymptotics of the Gegenbauer polynomials for large n and λ. For this we need

Lemma 2.3 (Stirling)

For any number x such that $x \geq 1$ we have

$$(2\pi)^{\frac{1}{2}} x^{x+\frac{1}{2}} e^{-x} \leq \Gamma(x+1) \leq (2\pi)^{\frac{1}{2}} x^{x+\frac{1}{2}} e^{-x} e^{\frac{1}{12x}} \tag{2.7}$$

$\square$

Lemma 2.4

There exists a constant A independent of λ and n such that

$$A^{-1} \frac{\lambda^{\frac{1}{2}}}{(n+\lambda)} C_n^\lambda(1) \leq h_n^\lambda \leq A \frac{\lambda^{\frac{1}{2}}}{(n+\lambda)} C_n^\lambda(1) \tag{2.8}$$

The proof follows from (2.6) and Stirling's formula (2.7).

$\square$

In the analysis we will extensively use the relationship between the Fourier functions $e^{ik\pi x}$ and the Gegenbauer polynomials $C_n^\lambda(x)$. In particular, we will focus our attention on the *Legendre polynomials*

$$P_n(x) = C_n^{\frac{1}{2}}(x) \tag{2.9}$$

and the *Chebyshev polynomials*

$$T_n(x) = \frac{n}{2} C_n^0(x) \tag{2.10}$$

Our basic formula is taken from [2], page 213:

Lemma 2.5

$$\frac{1}{h_l^\lambda} \int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} e^{in\pi x} C_l^\lambda(x) dx = \Gamma(\lambda) \left(\frac{2}{\pi n}\right)^\lambda i^l (l+\lambda) J_{l+\lambda}(\pi n) \tag{2.11}$$

where $J_\nu(x)$ is the Bessel function.

$\square$

Lemma 2.5 gives us the Gegenbauer expansion coefficients of the *Fourier functions.* However, what we really want are the *Fourier coefficients* of the Gegenbauer polynomials. We have those coefficients, luckily, for the Legendre and Chebyshev polynomials.

Lemma 2.6

Let a_k^N be the Fourier coefficients of the Legendre polynomial $P_N(x)$ defined in (2.9) and b_k^N be the Fourier coefficients of the Chebyshev polynomial $T_N(x)$ defined in (2.10), i.e.

$$P_N(x) = \sum_{k=-\infty}^{\infty} a_k^N e^{ik\pi x} \tag{2.12}$$

and

$$T_N(x) = \sum_{k=-\infty}^{\infty} b_k^N e^{ik\pi x} \tag{2.13}$$

Then

$$a_k^N = \frac{i^N}{\sqrt{2k}} J_{N+\frac{1}{2}}(\pi k) \tag{2.14}$$

and

$$b_k^N = \frac{(-i)^N \pi}{2} J_N(\pi k) \tag{2.15}$$

$\square$

Equation (2.14) follows from (2.11) upon substituting $\lambda = \frac{1}{2}$ and using (2.6) and (2.3). Equation (2.15) can be found in [7], page 836.

$\square$

Lemma 2.6 provides us with a better understanding of the terms "small scales" and "large scales". The question is the following: it is clear that $e^{iN\pi x}$ is small scale for large N, intuitively $T_N(x)$ and $P_N(x)$ are also small scale functions, however the expansions (2.12) and (2.13) contain *all* the scales. So, in what sense are $T_N(x)$ and $P_N(x)$ small scales?

The answer is given in the following Lemma:

Lemma 2.7

Let a_k^N and b_k^N be the Fourier coefficients of the Legendre polynomial $P_N(x)$ and the Chebyshev polynomial $T_N(x)$ respectively, given in (2.12)-(2.15). Then

$$|a_k^N| \leq A \min\left(1, (\frac{e\pi k}{2N})^N\right) \tag{2.16}$$

$$|b_k^N| \leq A \min\left(1, (\frac{e\pi k}{2N})^N\right) \tag{2.17}$$

where A is a constant independent of k or N.

Proof:

We start with the following estimate for the Bessel function $J_n(nz)$ (see [1], page 362):

$$|J_n(nz)| \leq \left(\frac{ze^{\sqrt{1-z^2}}}{1+\sqrt{1-z^2}}\right)^n, \qquad 0 \leq z \leq 1 \tag{2.18}$$

From (2.18) we can deduce trivially that

$$|J_n(nz)| \leq (\frac{ez}{2})^n \qquad 0 \leq z \leq 1 \tag{2.19}$$

We also have the uniform bound on Bessel functions

$$|J_n(z)| \leq 1 \tag{2.20}$$

Using (2.19) and (2.20) in (2.14) and (2.15), with $z = \frac{\pi k}{N}$ and $n = N$, we obtain (2.16) and (2.17).

□

Lemma 2.7 provides the answer why $P_N(x)$ and $T_N(x)$ (for large N) can be considered as small scale functions. Basically only the terms that are greater than N appear in their Fourier expansions (2.12) and (2.13). The lower terms decay *exponentially* with N.

3 Truncation Error in a Sub-interval—Fourier Method

Consider an arbitrary L_2 function $f(x)$ defined in $[-1,1]$. Suppose that the first $2N+1$ Fourier coefficients of this function are given

$$\hat{f}(k) = \frac{1}{2}\int_{-1}^{1} f(x)e^{-ik\pi x}dx \qquad |k| \leq N \tag{3.1}$$

In [5] we showed that we can recover (with exponential accuracy) the first $m \sim N$ terms in the *Gegenbauer* expansion of $f(x)$ based on the interval $[-1,1]$.

Here we are interested to find the Gegenbauer expansion of $f(x)$ based on a sub-interval $[a,b] \subset [-1,1]$. This will allow us to handle multiple discontinuities and discontinuities of unknown locations. We will show that in this case too, we can get exponential accuracy. We start by introducing the local variable ξ:

Definition 3.1

The local variable ξ is defined by

$$x = x(\xi) = \epsilon\xi + \delta \tag{3.2}$$

where

$$\epsilon = \frac{b-a}{2}, \qquad \delta = \frac{b+a}{2} \tag{3.3}$$

Thus when $a \leq x \leq b$, $-1 \leq \xi \leq 1$.

□

Denote now the Fourier partial sum by $f_N(x)$, namely

$$f_N(x) = \sum_{k=-N}^{N} \hat{f}(k)e^{ik\pi x} \tag{3.4}$$

As mentioned before, we are trying to recover the Gegenbauer expansion coefficients based on the sub-interval $[a,b]$, i.e. we would like to find the first $m \sim N$ coefficients $\hat{f}_\epsilon^\lambda(l)$ in the expansion

$$f(x(\xi)) = \sum_{l=0}^{\infty} \hat{f}_\epsilon^\lambda(l) C_l^\lambda(\xi) \tag{3.5}$$

where

$$\hat{f}_\epsilon^\lambda(l) = \frac{1}{h_l^\lambda}\int_{-1}^{1}(1-\xi^2)^{\lambda-\frac{1}{2}} f(x(\xi))C_l^\lambda(\xi)d\xi \tag{3.6}$$

Of course we do not have $\hat{f}_\epsilon^\lambda(l)$ at our disposal but only an approximation based on the Fourier partial sum $f_N(x)$, thus we have

$$\hat{g}_\epsilon^\lambda(l) = \frac{1}{h_l^\lambda}\int_{-1}^{1}(1-\xi^2)^{\lambda-\frac{1}{2}} f_N(x(\xi))C_l^\lambda(\xi)d\xi \tag{3.7}$$

The *Truncation Error* is the measure of deviation of the approximate Gegenbauer expansion from the true expansion:

$$TE(\lambda, m, N, \epsilon) = \max_{-1\le\xi\le 1} \left|\sum_{l=0}^{m}(\hat{f}_\epsilon^\lambda(l) - \hat{g}_\epsilon^\lambda(l))C_l^\lambda(\xi)\right| \tag{3.8}$$

We can show that the truncation error is exponentially small:

Theorem 3.2:

Let $f(x)$ be an L_2 function whose first $2N+1$ Fourier coefficients $\hat{f}(k)$ defined in (3.1) are known. Let $\hat{g}_\epsilon^\lambda(l)$ be the Gegenbauer coefficients, based on the sub-interval $[a, b]$ expansion of the Fourier partial sum $f_N(x)$ defined in (3.7), and $\hat{f}_\epsilon(l)$ be those of $f(x)$ defined in (3.6). Assume that

$$\lambda = \alpha\epsilon N \qquad m = \beta\epsilon N \tag{3.9}$$

Where ϵ is the half length of the sub-interval $[a, b]$ defined in (3.3). Then the truncation error is bounded by

$$TE(\alpha\epsilon N, \beta\epsilon N, N, \epsilon) \le Aq^{\epsilon N}, \tag{3.10}$$

where

$$q = \frac{(\beta+2\alpha)^{\beta+2\alpha}}{(2\pi e)^\alpha \alpha^\alpha \beta^\beta} \tag{3.11}$$

and A grows at most quadratically with N. In particular, if $\alpha = \beta \approx \frac{1}{4}$, then

$$q \approx 0.8$$

Proof:

The proof follows exactly the one in [5], Theorem 3.1 and 3.3, with proper accounting for the interval length ϵ.

$\square$

4 Truncation Error in a Sub-interval—Chebyshev and Legendre Methods

In this section we assume that we have the first $N+1$ Chebyshev or Legendre coefficients in the expansion of an L_2 function $f(x)$ based on the interval $-1 \le x \le 1$. We show how to get its Gegenbauer expansion based on a sub-interval $[a,b] \subset [-1,1]$ with exponential accuracy in N. The proof makes use of Lemma 2.7, which states that the *low modes* of the Fourier expansion of the Chebyshev or Legendre polynomial of degree N decay exponentially with N. Thus only the high modes count, and those are shown in Section 2 to have small Gegenbauer coefficients.

For simplicity of notation we will deal with $C_k^{\frac{1}{2}}(x)$, which is the Legendre polynomial $P_k(x)$ (see (2.9)), and $C_k^0(x)$, which is $\frac{2}{k}$ times the Chebyshev polynomial $T_k(x)$ (see (2.10)). The Chebyshev or Legendre expansion coefficients of $f(x)$ are given by

$$\hat{f}^\mu(k) = \frac{1}{h_k^\mu} \int_{-1}^{1} (1-x^2)^{\mu-\frac{1}{2}} f(x) C_k^\mu(x) dx \tag{4.1}$$

for $\mu = 0$ or $\mu = \frac{1}{2}$.

In this section we consider functions $f(x)$ satisfying

Assumption 4.1:

$|\hat{f}^\mu(k)| \le A$ for all k.

$\square$

We remark that if $f(x)$ is an L_2 function the assumption is fulfilled.

We assume that we know the first $N+1$ Chebyshev or Legendre coefficients, $\hat{f}^\mu(k)$ for $0 \le k \le N$, and define

$$f_N^\mu(x) = \sum_{k=0}^{N} \hat{f}^\mu(k) C_k^\mu(x) \tag{4.2}$$

Note that $f_N^\mu(x)$ *does not* converge fast to $f(x)$ if there exist discontinuities inside the domain.

The function $f(x)$ has also a Gegenbauer expansion in a sub-interval $[a,b]$. With ξ, ϵ and δ defined in (3.2)-(3.3), we have

$$f(\epsilon\xi+\delta)=\sum_{l=0}^{\infty}\hat{f}_\epsilon^\lambda(l)C_l^\lambda(\xi) \tag{4.3}$$

where the Gegenbauer coefficients $\hat{f}_\epsilon^\lambda(l)$ are defined by

$$\hat{f}_\epsilon^\lambda(l)=\frac{1}{h_l^\lambda}\int_{-1}^{1}(1-\xi^2)^{\lambda-\frac{1}{2}}f(\epsilon\xi+\delta)C_l^\lambda(\xi)d\xi \tag{4.4}$$

As before, we do not have $\hat{f}_\epsilon^\lambda(l)$ at our disposal, but only an approximation based on the partial Chebyshev or Legendre sum $f_N^\mu(x)$, thus we have

$$\hat{g}_\epsilon^\lambda(l)=\frac{1}{h_l^\lambda}\int_{-1}^{1}(1-\xi^2)^{\lambda-\frac{1}{2}}f_N^\mu(\epsilon\xi+\delta)C_l^\lambda(\xi)d\xi \tag{4.5}$$

How well do $\hat{g}_\epsilon^\lambda(l)$ approximate $\hat{f}_\epsilon^\lambda(l)$? To answer this question we define

Definition 4.2:

The truncation error is defined by

$$TE(\lambda,m,N,\epsilon)=\max_{-1\le\xi\le1}|\sum_{l=0}^{m}(\hat{f}_\epsilon^\lambda(l)-\hat{g}_\epsilon^\lambda(l))C_l^\lambda(\xi)| \tag{4.6}$$

where $\hat{f}_\epsilon^\lambda(l)$ are defined by (4.4) and $\hat{g}_\epsilon^\lambda(l)$ are defined by (4.5).

□

The truncation error is the measure of the distance between the true Gegenbauer expansion in the interval $[a,b]$ and its approximation based on the Chebyshev or Legendre partial sum in $[-1,1]$.

We first have the following lemma:

Lemma 4.3:

$$TE(\lambda,m,N,\epsilon)\le\sum_{q=N+1}^{\infty}|\hat{f}^\mu(q)|\sum_{l=0}^{m}\left|\frac{C_l^\lambda(1)}{h_l^\lambda}\int_{-1}^{1}(1-\xi^2)^{\lambda-\frac{1}{2}}C_q^\mu(\epsilon\xi+\delta)C_l^\lambda(\xi)d\xi\right| \tag{4.7}$$

Proof:

From (4.4) and (4.5) we have

$$\left|\hat{f}_\epsilon^\lambda(l) - \hat{g}_\epsilon^\lambda(l)\right| = \frac{1}{h_l^\lambda}\left|\int_{-1}^{1}(1-\xi^2)^{\lambda-\frac{1}{2}}(f(\epsilon\xi+\delta) - f_N^\mu(\epsilon\xi+\delta))C_l^\lambda(\xi)d\xi\right| \tag{4.8}$$

Substituting (4.8) into (4.6), recalling (2.4) and

$$f(\epsilon\xi+\delta) - f_N^\mu(\epsilon\xi+\delta) = \sum_{q=N+1}^{\infty} \hat{f}^\mu(q)C_q^\mu(\epsilon\xi+\delta) \tag{4.9}$$

we obtain (4.7).

□

For simplicity of notations we denote:

$$F_{q,l,\lambda} = \left|\frac{C_l^\lambda(1)}{h_l^\lambda}\int_{-1}^{1}(1-\xi^2)^{\lambda-\frac{1}{2}}C_q^\mu(\epsilon\xi+\delta)C_l^\lambda(\xi)d\xi\right|, \qquad q > N \tag{4.10}$$

In order to estimate this term, we consider the Fourier series approximation of $C_q^\mu(\epsilon\xi+\delta)$ from (2.12) and (2.13) to get

Lemma 4.4:

$$F_{q,l,\lambda} \leq AC_l^\lambda(1)\Gamma(\lambda)\sum_{k=1}^{\infty}\left(\frac{2}{k\pi\epsilon}\right)^\lambda |J_{l+\lambda}(k\pi\epsilon)|\,|a_k^q| \tag{4.11}$$

where a_k^q is given either by (2.14) or by (2.15), and A grows at most as fast as a linear polynomial in λ and l.

Proof:

We expand $C_q^\mu(x)$ using (2.12) or (2.13):

$$C_q^\mu(x) = \sum_{k=-\infty}^{\infty} a_k^{q,\mu}e^{ik\pi x} = \sum_{k=-\infty}^{\infty} a_k^{q,\mu}e^{ik\pi\epsilon\xi}e^{ik\pi\delta} \tag{4.12}$$

where $a_k^{q,\frac{1}{2}} = a_k^q$ is given by (2.14) and $a_k^{q,0} = \frac{2}{q}b_k^q$ is given by (2.15).

We now use (2.11) to get

$$\left|\frac{1}{h_l^\lambda}\int_{-1}^{1}(1-\xi^2)^{\lambda-\frac{1}{2}}e^{ik\pi\epsilon\xi}e^{ik\pi\delta}C_l^\lambda(\xi)d\xi\right| = \Gamma(\lambda)\left(\frac{2}{|k|\pi\epsilon}\right)^\lambda (l+\lambda)\,|J_{l+\lambda}(k\pi\epsilon)| \tag{4.13}$$

Substituting (4.12) and (4.13) into (4.10) gives us (4.11).

$\square$

We are ready now to estimate directly $F_{q,l,\lambda}$ defined in (4.10).

Lemma 4.5:

Consider $F_{q,l,\lambda}$ defined in (4.10). Let $\lambda = \frac{2}{27e}\epsilon N$ and $l = \gamma\lambda$ where $\gamma < 1$. Then for every $q > N$ we have

$$F_{q,l,\lambda} \leq Ae^{-\lambda}\left(\frac{N}{q}\right)^{\lambda}. \tag{4.14}$$

where A grows at most quadratically with N.

Proof:

Let $k_q = \left[\frac{q}{\pi e}\right]$ be the integer part of $\frac{q}{\pi e}$. We split the sum on the right hand side of (4.11) into

$$F_{q,l,\lambda} \leq A\left(\sum_{k=1}^{k_q} + \sum_{k=k_q+1}^{\infty}\right)\left[C_l^{\lambda}(1)\Gamma(\lambda)\left(\frac{2}{k\pi\epsilon}\right)^{\lambda}|J_{l+\lambda}(k\pi\epsilon)|\,|a_k^q|\right] = A\left(F_{q,l,\lambda}^{(1)} + F_{q,l,\lambda}^{(2)}\right) \tag{4.15}$$

We first use (2.3) and (2.7) to obtain:

$$C_l^{\lambda}(1) \leq A\left(\frac{(\gamma+2)^{\gamma+2}}{4\gamma^{\gamma}}\right)^{\lambda} \leq A\left(\frac{27}{4}\right)^{\lambda} \tag{4.16}$$

and

$$\Gamma(\lambda) \leq A\lambda^{\lambda}e^{-\lambda} \tag{4.17}$$

We start with an estimation of the second sum

$$F_{q,l,\lambda}^{(2)} = \sum_{k=k_q+1}^{\infty} C_l^{\lambda}(1)\Gamma(\lambda)\left(\frac{2}{k\pi\epsilon}\right)^{\lambda}|J_{l+\lambda}(k\pi\epsilon)|\,|a_k^q|$$

We use (4.16), (4.17), the fact that $|J_{l+\lambda}(x)| \leq 1$ and $|a_k^q| \leq 1$, and $k > k_q = \left[\frac{q}{\pi e}\right]$ and $\lambda = \frac{2}{27e}\epsilon N$ to get

$$F_{q,l,\lambda}^{(2)} \leq A\left(\frac{27\lambda}{2\epsilon q}\right)^{\lambda} \leq Ae^{-\lambda}\left(\frac{N}{q}\right)^{\lambda}. \tag{4.18}$$

where A grows at most linearly with N, the growth coming from the summation. For the first sum

$$F_{q,l,\lambda}^{(1)} = \sum_{k=1}^{k_q} C_l^{\lambda}(1)\Gamma(\lambda)\left(\frac{2}{k\pi\epsilon}\right)^{\lambda}|J_{l+\lambda}(k\pi\epsilon)|\,|a_k^q|,$$

we still use (4.16), (4.17) and the bound $|J_{l+\lambda}(x)| \le 1$. However, for the estimate on $|a_k^q|$, we use (2.16), (2.17):

$$|a_k^q| \le A\left(\frac{e\pi k}{2q}\right)^q \tag{4.19}$$

and the fact $q > N > \lambda$ to get

$$F_{q,l,\lambda}^{(1)} \le A\sum_{k=1}^{k_q}\left(\frac{e\pi k}{2q}\right)^q\left(\frac{27\lambda}{2e\pi k\epsilon}\right)^\lambda \le Ak_q\left(\frac{e\pi k_q}{2q}\right)^q\left(\frac{27\lambda}{2e\pi k_q\epsilon}\right)^\lambda \tag{4.20}$$

and, upon substituting for $k_q = \left[\frac{q}{\pi e}\right]$ and $\lambda = \frac{2}{27e}\epsilon N$:

$$F_{q,l,\lambda}^{(1)} \le Ak_q\left(\frac{1}{2}\right)^q e^{-\lambda}\left(\frac{N}{q}\right)^\lambda \le Ae^{-\lambda}\left(\frac{N}{q}\right)^\lambda \tag{4.21}$$

Combining (4.18) and (4.21) we obtain (4.14).

□

We are now ready for the main theorem of this section:

Theorem 4.6:

Let the truncation error be defined in (4.6). Let $\lambda = \frac{2\epsilon}{27e}N$ and $m \le \lambda$, then

$$TE(\lambda, m, N, \epsilon) \le Ae^{-\lambda} \tag{4.22}$$

where A grows at most as a fourth order polynomial in N.

Proof:

The theorem follows from (4.7), the Assumption 4.1, and (4.14).

□

5 Regularization Error for a Sub-interval

In this section we study the second part of the error, which is caused by using a finite Gegenbauer expansion based on a sub-interval $[a, b] \subset [-1, 1]$, to approximate a function $f(x)$ which is assumed analytic in this sub-interval $[a, b]$. Since in previous sections we have established exponentially small truncation errors if λ and m are both growing linearly with N, in this section we will consider the case of $\lambda \sim m$.

We will assume that $f(x)$ is an analytic function on $[a,b]$ satisfying

Assumption 5.1

There exists constants $\rho \geq 1$ and $C(\rho)$ such that, for every $k \geq 0$,

$$\max_{a\leq x\leq b} \left| \frac{d^k f}{dx^k}(x) \right| \leq C(\rho) \frac{k!}{\rho^k} \tag{5.1}$$

□

This is a standard assumption for analytic functions. ρ is the distance from $[a,b]$ to the nearest singularity of $f(x)$ in the complex plane (see, for example, [8]). This assumption and the following proof can be modified to allow for an arbitrary ellipse region around $[a,b]$ for the analyticity of $f(x)$ (H. Vandeven, private communication).

Let us consider the Gegenbauer partial sum of the first m terms for the function $f(\epsilon\xi + \delta)$:

$$f_m^{\lambda,\epsilon}(\xi) = \sum_{l=0}^{m} \hat{f}_\epsilon^\lambda(l) C_l^\lambda(\xi) \tag{5.2}$$

with ξ, ϵ and δ defined by (3.2) and (3.3), and the Gegenbauer coefficients based on [a,b] defined by

$$\hat{f}_\epsilon^\lambda(l) = \frac{1}{h_l^\lambda} \int_{-1}^{1} (1-\xi^2)^{\lambda-\frac{1}{2}} f(\epsilon\xi + \delta) C_l^\lambda(\xi) d\xi \tag{5.3}$$

We want to estimate the regularization error in the maximum norm:

$$RE(\lambda, m, \epsilon) = \max_{-1\leq\xi\leq 1} \left| f(\epsilon\xi + \delta) - \sum_{l=0}^{m} \hat{f}_\epsilon^\lambda(l) C_l^\lambda(\xi) \right| \tag{5.4}$$

We have the following result for the estimation of the regularization error, when $\lambda \sim m$:

Theorem 5.2:

Assume $\lambda = \gamma m$ where γ is a positive constant. If $f(x)$ is analytic in $[a,b] \subset [-1,1]$ satisfying the Assumption 5.1, then the regularization error defined in (5.4) can be bounded by

$$RE(\gamma m, m, \epsilon) \leq A q^m \tag{5.5}$$

where q is given by

$$q = \frac{\epsilon(1+2\gamma)^{1+2\gamma}}{\rho 2^{1+2\gamma}\gamma^{\gamma}(1+\gamma)^{1+\gamma}} \tag{5.6}$$

which is always less than 1. In particular, if $\gamma = 1$ and $m = \beta N$ where β is a positive constant, then

$$RE(\beta N, \beta N, \epsilon) \leq A q^N \tag{5.7}$$

with

$$q = \left(\frac{27\epsilon}{32\rho}\right)^{\beta} \tag{5.8}$$

Proof:

The proof follows in exactly in the same fashion as in [5], Lemma 4.2, Theorems 4.3 and 4.4, with proper accounting for the transformation $x(\xi) = \epsilon\xi + \delta$.

□

6 Concluding Remarks

In this Section, we combine the estimates for truncation errors and regularization errors in previous sections, to obtain the main theorem of this paper:

Theorem 6.1: Removal of the Gibbs Phenomenon for the sub-interval case

Consider an L_2 function $f(x)$ on $[-1,1]$, which is analytic in a sub-interval $[a,b] \subset [-1,1]$ and satisfies

$$\max_{a \leq x \leq b} \left| \frac{d^k f}{dx^k}(x) \right| \leq C(\rho)\frac{k!}{\rho^k}, \qquad \rho \geq 1 \tag{6.1}$$

Assume that either the first $2N+1$ Fourier coefficients

$$\hat{f}(k) = \frac{1}{2}\int_{-1}^{1} f(x) e^{-ik\pi x} dx, \qquad -N \leq k \leq N$$

or the first $N+1$ Chebyshev or Legendre coefficients

$$\hat{f}^{\mu}(k) = \frac{1}{h_k^{\mu}}\int_{-1}^{1} (1-x^2)^{\mu-\frac{1}{2}} f(x) C_k^{\mu}(x) dx, \qquad \mu = 0 \ or\ \mu = \frac{1}{2}$$

are known. Let $\hat{g}_\epsilon^\lambda(l)$, $0 \le l \le m$ be the Gegenbauer expansion coefficients, based on the subinterval $[a,b]$, of the spectral partial sum $f_N(x)$ in (3.4) or $f_N^\mu(x)$ in (4.2). These Gegenbauer coefficients are defined in (3.7) for the Fourier case and in (4.5) for the Chebyshev or Legendre case. Then for $\lambda = m = \beta\epsilon N$ where $\beta < \frac{2\pi e}{27}$ (Fourier case) or $\beta \le \frac{2}{27e}$ (Chebyshev or Legendre case), we have

$$\max_{-1\le\xi\le 1}\left|f(\epsilon\xi+\delta) - \sum_{l=0}^{m}\hat{g}_\epsilon^\lambda(l)C_l^\lambda(\xi)\right| \le A\left(q_T^{\epsilon N} + q_R^{\epsilon N}\right) \tag{6.2}$$

where

$$q_T = \left(\frac{27\beta}{2\pi e}\right)^\beta < 1, \qquad q_R = \left(\frac{27\epsilon}{32\rho}\right)^\beta < 1$$

for the Fourier case and

$$q_T = e^{-\beta} < 1, \qquad q_R = \left(\frac{27\epsilon}{32\rho}\right)^\beta < 1$$

for the Chebyshev or Legendre case, and A is at most a fourth order polynomial in N.

Proof:

Just combine the results of Theorems 3.2, 4.6 and 5.2.

□

We finally give two remarks:

Remark 6.2

It can be seen from (6.2) that the truncation error gets bigger when the interval half-length ϵ decreases, because the parameters m and λ must decrease with ϵ. This in turn also affects the regularization error, although there is a factor ϵ in q_R which offsets this effect.

□

Remark 6.3

No attempt has been made in this paper to optimize the parameters.

□

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13. ABSTRACT *(Maximum 200 words)*
We continue the investigation of overcoming Gibbs phenomenon, i.e., obtaining exponential accuracy at all points including at the discontinuities themselves, from the knowledge of a spectral partial sum of a discontinuous but piecewise analytic function.

We show that if we are given the first N expansion coefficients of an L_2 function $f(x)$ in terms of either the trigonometrical polynomials or the Chebyshev or Legendre polynomials, we can construct an **exponentially convergent approximation** to the point values of $f(x)$ in any sub-interval in which it is analytic.

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