# On the global existence and asymptotic behavior of solutions of reaction-diffusion equations 

By Kyûya Masuda

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## 1. Introduction

Let $\Omega$ be a bounded domain with smooth boundary $\Gamma$ in $R^{n}$. Let $\beta \geq 1$, and $\mu_{j}>0(j=1,2)$. R. Martin posed a problem on the existence and uniform bounds of solutions $u=\left\{u_{1}, u_{2}\right\}$ of the reaction-diffusion equation of the form :

$$
\left\{\begin{array}{l}
\frac{\partial u_{1}}{\partial t}=\mu_{1} \Delta u_{1}-u_{1} u_{2}^{\beta}  \tag{1}\\
\frac{\partial u_{2}}{\partial t}=\mu_{2} \Delta u_{2}+u_{1} u_{2}^{\beta}
\end{array} \quad x \in \Omega, t>0\right.
$$

under various boundary conditions and non-negative initial data; this equation is related to the Rosenzweig-MacArthur equation in ecology (see J. MaynardSmith [5] ; D. Conway and A. Smoller [2]). N. Alikakos [1] obtained $L^{\infty}$ bounds of solutions of (1) subject to the homogeneous Neumann boundary condition under the assumption $1 \leq \beta<(n+2) / n$, and gave a positive partial answer to it. The purpose of the present paper is to give a complete answer to the problem of Martin.

We consider a solution $u=\left\{u_{1}, u_{2}\right\}$ of the more general type of reactiondiffusion equations

$$
\begin{equation*}
\frac{\partial u_{j}}{\partial t}=\mu_{j} \Delta u_{j}+f_{j}(u), x \in \Omega, t>0(j=1,2) \tag{2}
\end{equation*}
$$

subject to the boundary condition :

$$
\begin{equation*}
\alpha_{j}(x) \frac{\partial u_{j}}{\partial n}+\left(1-\alpha_{j}(x)\right) u_{j}=0, x \in \Gamma,(j=1,2) \tag{3}
\end{equation*}
$$

and with the initial condition :

$$
\begin{equation*}
\left.u_{j}\right|_{t=0}=a_{j}(x), \quad x \in \Omega(j=1,2), \tag{4}
\end{equation*}
$$

$(\partial / \partial n$ denotes differentiation in the direction of the exterior normal to $\Gamma$ ). Here we make the following assumptions:

Assumption 1. $\alpha_{j}(x)(j=1,2)$ is a non-negative $C^{2}$-function on $\Gamma$ such
that $0 \leq \alpha_{j}(x) \leq 1 \quad(x \in \Gamma)$.
Assumption 2. $a_{j}(x)(j=1,2)$ is a non-negative $C^{2}$-function on $\bar{\Omega}$, satisfying (3);

Assumption 3. $f_{j}(y)(j=1,2)$ is a $C^{1}$-function on $\bar{R}_{+}^{2}=\left\{y=\left(y_{1}, y_{2}\right)\right.$; $\left.y_{1} \geq 0, y_{2} \geq 0\right\}$ such that
i) $\quad-f_{1}(y), f_{2}(y)$ are non-negative and $f_{1}((0, s))=f_{2}((s, 0))=0$ for $s \geq 0$;
ii) there is a monotonically increasing function $\omega(s)(s \geq 0)$ and a positive constant $r$ with

$$
\begin{align*}
& f_{2}(y) \leq \omega\left(y_{1}\right)\left(y_{2}+y_{2}^{r}\right), \\
& f_{2}(y) \leq \omega\left(y_{1}\right)\left|f_{1}(y)\right|
\end{align*} \quad y=\left(y_{1}, y_{2}\right) \in \bar{R}_{+}^{2}
$$

We note that $f_{1}(y)=-y_{1} y_{2}^{\beta}, f_{2}(y)=y_{1} y_{2}^{\beta}$ (see (1)) satisfies the assumption 3. Our result is now given by the following

Theorem. Let the assumptions 1, 2 and 3 hold. Then the initialboundary value problem (2), (3), (4) has a unique global solution $u=\left\{u_{1}, u_{2}\right\}$. Moreover, $u$ tends to some constant vector of the form $c=\left\{c_{1}, c_{2}\right\}$, as $t \rightarrow \infty$, uniformly on $\bar{\Omega}$, where $c_{j} \geq 0$ and $f_{j}\left(c_{j}\right)=0(j=1,2)$.

The following corollary follows immediately from the theorem above, and gives a complete answer to the problem of Martin.

Corollary. Let the assumptions 1 and 2 hold. Then the initialboundary value problem (1), (3), (4) has a unique global solution $u=\left\{u_{1}, u_{2}\right\}$. Moreover, $u$ tends to a constant vector $c=\left\{c_{1}, c_{2}\right\}$, as $t \rightarrow \infty$, uniformly on $\bar{\Omega}$, where $c_{j} \geq 0, c_{1} c_{2}=0$, and

$$
\begin{equation*}
c_{1}+c_{2}=\int_{0}^{\infty} \int_{\Gamma}\left(\frac{\partial u_{1}}{\partial n}+\frac{\partial u_{2}}{\partial n}\right) d S_{x} d t+\int_{\Omega}\left(a_{1}(x)+a_{2}(x)\right) d x \tag{6}
\end{equation*}
$$

In the next two sections we shall give some a priori estimates, which are the core of the proof of our theorem. In the final section we shall give the proof of the theorem.

## 2. $L^{p}$ bounds of solutions

Here we shall give some a priori estimates for a solution $u$ of (2), (3), (4) which is supposed to exist in the interval $[0, T)$. We first see that $u_{j}$ $(j=1,2)$ is non-negative;

$$
\begin{equation*}
u_{j}(x, t) \geq 0, \quad x \in \Omega, \quad 0 \leq t<T,(j=1,2) \tag{7}
\end{equation*}
$$

since the initial data $a_{j}(x)$ is non-negative, and since $f_{j}(u)$ has, by the assump-
tion 3 (i), the form : $f_{j}(u)=c_{j}(u) u_{j}$ where $c_{j}(u)=c_{j}(u(x, t))$ is some cotinuous function of $x, t$. By the maximum principle,

$$
\begin{equation*}
0 \leq u_{1}(x, t) \leq\left\|a_{1}\right\|_{L^{\infty}} \tag{8}
\end{equation*}
$$

since $f_{1}(u) \leq 0$ by the assumption 3 . To get the $L^{\infty}$-bounds for $u$ is the essential part of the proof of our theorem. This follows from the following key a priori estimate : for $p>1$,

$$
\begin{equation*}
\left\|u_{2}(t)\right\|_{L^{p}} \leq M_{p}, \quad 0 \leq t<T \tag{9}
\end{equation*}
$$

$M_{p}$ being a constant independent of $T$ (we shall simply write $u_{j}(t)$ for $\left.u_{j}(x, t)\right)$, which we shall prove in a series of lemmas; $\|\cdot\|_{L_{p}}$ denotes the usual $L^{p}$-norm over $\Omega$. In doing so, we set

$$
g_{p}\left(u_{2}\right)=\left(1+u_{2}\right)^{p}, \quad u_{2} \geq 0
$$

for $p>0$. We note

$$
\begin{equation*}
g_{p}^{\prime}\left(u_{2}\right) \leq(p /|p-1|)^{1 / 2} g_{p}\left(u_{2}\right)^{1 / 2}\left|g_{p}^{\prime \prime}\left(u_{2}\right)\right|^{1 / 2} \tag{10}
\end{equation*}
$$

for $p \neq 1$. We denote the inner product of $L^{2}(\Omega)$ by $(\cdot, \cdot)$, and define $(\nabla b$, $c \nabla d)$ by :

$$
(\nabla b, c \nabla d)=\sum_{i=1}^{n} \int_{\Omega} \frac{\partial b(x)}{\partial x_{i}} c(x) \frac{\partial d(x)}{\partial x_{i}} d x
$$

for scalar functions $b, c$ and $d$ on $\Omega$.
Lemma 1. Let $u=\left\{u_{1}, u_{2}\right\}$ be a solution of (2), (3), (4) in [0,T). Then for $w$ in $W^{1,2}(\Omega)$ and $0 \leq t<T$,

$$
\begin{equation*}
\left(\Delta u_{j}(t), w\right)=\int_{\Gamma} \frac{\partial u_{j}(x, t)}{\partial n} w(x) d S_{x}-\left(\nabla u_{j}(t), \nabla w\right) \tag{11}
\end{equation*}
$$

Moreover, if wis non-negative, then

$$
\begin{equation*}
\left(\Delta u_{j}(t), w\right) \leq-\left(\nabla u_{j}(t), \nabla w\right), \quad 0 \leq t<T \tag{12}
\end{equation*}
$$

( $W^{1,2}(\Omega)$ is the usual Sobolev space).
Proof. By integration by parts, we have (11). By (3) and (7), it is easy to see that

$$
\begin{equation*}
\partial u_{j}(x, t) / \partial n \leq 0, \quad x \in \Omega, 0 \leq t<T, j=1,2 \tag{13}
\end{equation*}
$$

from which (12) follows by (11).
Lemma 2. Let $u$ be as in Lemma 1. Then

$$
\begin{gather*}
\int_{\Omega} u_{j}(x, t) d x-\int_{0}^{t} \int_{F} \frac{\partial u_{j}(x, s)}{\partial n} d S_{x} d s-\int_{0}^{t} \int_{\Omega} f_{j}(u(x, s)) d x d s  \tag{14}\\
=\int_{\Omega} a_{j}(x) d x, \quad 0 \leq t<T, j=1,2
\end{gather*}
$$

Proof. Integrating (2) with respect to $x$, $t$ over $\Omega \times[0, t]$, we have, by (11) with $w=1$, (14).

Lemma 3. Let $u$ be as in Lemma 1. Then

$$
\begin{gathered}
\left(g_{p}\left(u_{2}(t)\right), 1\right)-\mu_{2} \int_{0}^{t} \int_{T} \frac{\partial u_{2}(x, s)}{\partial n} g_{p}^{\prime}\left(u_{2}(x, s)\right) d S_{x} d s \\
=\left(g_{p}\left(a_{2}\right), 1\right)-\mu_{2} \int_{0}^{t}\left(\nabla u_{2}, g_{p}^{\prime \prime}\left(u_{2}\right) \nabla u_{2}\right) d s+\int_{0}^{t}\left(f_{2}(u), g_{p}^{\prime}\left(u_{2}\right)\right) d s \\
0 \leq t<T
\end{gathered}
$$

Proof. By (2) with $j=2$, and (12),

$$
\begin{aligned}
\frac{d}{d t}\left(g_{p}\left(u_{2}(t)\right), 1\right) & =\left(g_{p}^{\prime}\left(u_{2}(t)\right), \frac{\partial u_{2}(t)}{\partial t}\right) \\
& =\mu_{2}\left(g_{p}^{\prime}\left(u_{2}(t)\right), \Delta u_{2}(t)\right)+\left(g_{p}^{\prime}\left(u_{2}(t)\right), f_{2}(u(t))\right)
\end{aligned}
$$

Integrating the above inequality in $t$, we have, by (11), (15).
Lemma 4. Let $u$ be as in Lemma 1. Let $p \neq 1$. Then

$$
\begin{gathered}
\int_{0}^{t}\left(f_{2}(u), g_{p}\left(u_{2}\right)\right) d s \leq M+M \int_{0}^{t}\left(\nabla u_{2},\left|g_{p}^{\prime \prime}\left(u_{2}\right)\right| \nabla u_{2}\right) d s \\
\quad+M \int_{0}^{t}\left(g_{p}^{\prime}\left(u_{2}\right), f_{2}(u)\right) d s \\
0 \leq t<T
\end{gathered}
$$

$M$ being a constant independent of $T$ (which may depend on $\left\|a_{j}\right\|_{L^{\infty}, j}, j, 2$ )
Proof. We shall denote by the same $M$ various constant independent of $T$ (which may depend on $\left.p,\left\|a_{1}\right\|_{L^{\infty},},\left\|a_{2}\right\|_{L^{\infty}}\right)$. By (2) and (12), $\left(u_{j}=u_{j}(t)\right)$

$$
\begin{align*}
\frac{d}{d t}\left(u_{1}+\right. & \left.u_{1}^{2}, g_{p}\left(u_{2}\right)\right) \\
=( & \left.\left(1+2 u_{1}\right)\left(\mu_{1} \Delta u_{1}+f_{1}(u)\right), g_{p}\left(u_{2}\right)\right) \\
& +\left(u_{1}+u_{1}^{2}, g_{p}^{\prime}\left(u_{2}\right)\left(\mu_{2} \Delta u_{2}+f_{2}(u)\right)\right) \\
\leq- & \left(\mu_{1}+\mu_{2}\right)\left(\nabla u_{1},\left(1+2 u_{1}\right) g_{p}^{\prime}\left(u_{2}\right) \nabla u_{2}\right)-2 \mu_{1}\left(\nabla u_{1}, g_{p}\left(u_{2}\right) \nabla u_{1}\right)  \tag{17}\\
& +\left(f_{1}(u)+2 u_{1} f_{1}(u), g_{p}\left(u_{2}\right)\right)-\mu_{2}\left(\nabla u_{2},\left(u_{1}+u_{1}^{2}\right) g_{p}^{\prime \prime}\left(u_{2}\right) \nabla u_{2}\right)
\end{align*}
$$

$$
\begin{aligned}
& \quad+\left(u_{1}+u_{1}^{2}, g_{p}^{\prime}\left(u_{2}\right) f_{2}(u)\right) \\
& \left(\equiv I_{1}+I_{2}+I_{3}+I_{4}+I_{5}\right)
\end{aligned}
$$

We shall estimate each term on the right-side of (17). By the Schwarz inequality, (10), and (8),

$$
\begin{aligned}
I_{1}+I_{2} & \leq M\left(\nabla u_{2},\left(1+2 u_{1}\right)^{2}\left|g_{p}^{\prime \prime}\left(u_{2}\right)\right| \nabla u_{2}\right) \\
& \leq M\left(\nabla u_{2},\left|g_{p}^{\prime \prime}\left(u_{2}\right)\right| \nabla u_{2}\right) .
\end{aligned}
$$

By the assumption 3 (i) and (ii), $\left(M_{0}=\omega\left(\left\|a_{1}\right\|_{L^{\infty}}\right) /\left(1+2\left\|a_{1}\right\|_{L^{\infty}}\right)\right)$

$$
I_{3} \leq\left(1+2\left\|a_{1}\right\|_{L^{\infty}}\right)\left(f_{1}(u), g_{p}\left(u_{2}\right)\right) \leq-M_{0}^{-1}\left(f_{2}(u), g_{p}\left(u_{2}\right)\right)
$$

By (8),

$$
I_{4} \leq M\left(\nabla u_{2},\left|g_{p}^{\prime \prime}\left(u_{2}\right)\right| \nabla u_{2}\right) ; \quad I_{5} \leq M\left(g_{p}^{\prime}\left(u_{2}\right), f_{2}(u)\right)
$$

Collecting all the estimates above, we see :

$$
\text { the left hand side of } \begin{aligned}
(17) \leq & -M_{0}^{-1}\left(f_{2}(u), g_{p}\left(u_{2}\right)\right)+M\left(g_{p}^{\prime}\left(u_{2}\right), f_{2}(u)\right) \\
& +M\left(\nabla u_{2},\left|g_{p}^{\prime \prime}\left(u_{2}\right)\right| \nabla u_{2}\right)
\end{aligned}
$$

Integrating the above inequality in $t$ over $[0, t]$, we get (16).
Lemma 5. Let $u$ be as in Lemma 1. Let $0<\theta<1$. Then

$$
\begin{equation*}
\int_{0}^{t}\left(f_{2}(u(s)), g_{k-\theta}(s)\right) d s \leq M_{k} . \quad 0 \leq t<T,(k=1,2, \cdots), \tag{18}
\end{equation*}
$$

$M_{k}$ being a constant independent of $T$.
Proof. We first show (18) holds for $k=1$. By (14) with $j=1$, the assumption 3 (ii), and (13), $\left(M_{0}^{\prime}=\omega\left(\left\|a_{1}\right\|_{L^{\infty}}\right)\right)$

$$
\begin{equation*}
0 \leq \int_{0}^{t}\left(f_{2}(u), 1\right) d s \leq-M_{0}^{\prime} \int_{0}^{t}\left(f_{1}(u), 1\right) d s \leq M_{0}^{\prime}\left(a_{1}, 1\right) \tag{19}
\end{equation*}
$$

Hence the first and second terms on the left hand side of (14) with $j=2$ are bounded by some constant (independent of $T$ ). Hence the left hand side of (15) with $p=1-\theta$ is bounded by some constant (independent of $T$ ). Since the left hand side of (18) with $k=1$ is, by (16) with $p=1-\theta$, bounded by the left hand side of (15) with $p=1-\theta$, we see that (18) holds for $k=1$; note $g_{p}^{\prime \prime}\left(u_{2}\right)$ is negative for $p=1-\theta$. We suppose that (18) holds for $k=m$. Then since $g_{p}^{\prime \prime}\left(u_{2}\right)>0$ for $p>1$, it follows from (15) and (16) that

$$
\int_{0}^{t}\left(f_{2}(u), g_{p}\left(u_{2}\right)\right) d s \leq M_{2}+M_{2} \int_{0}^{t}\left(f_{2}(u), g_{p}^{\prime}\left(u_{2}\right)\right) d s
$$

$M_{2}$ being a constant independent of $T$, which shows (18) holds for $k=$ $m+1-\theta$. By induction on $k$, the proof of the lemma is completed.

Lemma 6. For any $p>1$, we have

$$
\begin{equation*}
\left\|u_{2}(t)\right\|_{L^{p}} \leq M, \quad 0 \leq t<T, \tag{9}
\end{equation*}
$$

$M_{p}$ being a constant independent of $T$.
Proof. From (13), (15) and (18), the estimate (9) follows.

## 3. $L^{\infty}$-bounds of solutions

A) Let $p>\max \{n, 2\}$. We then define the operator $A_{j, p}$ in $L^{p}(\Omega)$ by :

$$
\begin{aligned}
D\left(A_{j, p}\right) & =\left\{u \in W^{2, p}(\Omega) ; \alpha_{j}(x) \frac{\partial u}{\partial n}+\left(1-\alpha_{j}(x)\right) u=0 \text { on } \Gamma\right\} \\
A_{j, p} u & =-\mu_{j} \Delta u
\end{aligned}
$$

where $W^{2, p}(\Omega)$ is the usual Sobolev space; We shall simply write $A_{j}$ for $A_{j, p}$ unless otherwise stated. Here we recall the basic properties of the $A_{j}$ (For the detail see, e.g., H. Tanabe [7]). We know that the estimate

$$
\|u\|_{2, p} \leq M\left\{\|u\|_{L^{p}}+\left\|A_{j} u\right\|_{L^{p}}\right\}, \quad u \in D\left(A_{j}\right)
$$

holds ( $M$ : constant) where $\|\cdot\|_{k . p}$ is the norm of the Sobolev space $W^{k, p}(\Omega)$. Furthermore, $-A_{j}$ generates the holomorphic semi-groups $\left\{e^{-t A_{j}}\right\}_{t>0}$ in $L^{p}(\Omega)$. We also know that the spectrum of $A_{j}$ consists of non-negative eigenvalues $\left\{\lambda_{i}\right\}_{i=1}^{\infty}$ with finite multiplicity: $0 \leq \lambda_{1} \leq \lambda_{2} \leq \lambda_{3} \leq \cdots$. Since any eigenvalue $\lambda_{i}$ of $A_{j, p}$ is also eigenvalue of the self-adjoint operator $A_{j, 2}$ in $L^{2}(\Omega)$, it is easy to see that the first eigenvalue $\lambda_{1}$ of $A_{j}$ is positive if $\alpha_{j}(x) \not \equiv 1$; and zero if $\alpha_{j}(z) \equiv 1$. For $f \in L^{p}(\Omega)$, we set

$$
P_{j} f= \begin{cases}\int_{\Omega} f(x) d x| | \Omega \mid & \left(\text { if } \alpha_{j}(x) \equiv 1\right) \\ 0 & \text { (if } \left.\alpha_{j}(x) \not \equiv 1\right)\end{cases}
$$

Then $P_{j}$ is the projection operator onto the eigenspace corresponding to the first eigenvalue $\lambda_{1}=0$ if $\alpha_{j}(x) \equiv 1$. Define the operator $Q_{j}$ in $L^{p}(\Omega)$ by

$$
Q_{j}=I-P_{j}
$$

( $I$ denotes the identity operator). Then

Lemma 7. Let $0<\theta \leq 1$. For $0 \leq s \leq t$, we have

$$
\begin{align*}
& \left\|e^{-t A_{j}}\right\| \leq M  \tag{20}\\
& \left\|Q_{j} e^{-t A_{j}}\right\| \leq M e^{-\beta t}  \tag{21}\\
& \left\|A_{j} e^{-t A_{j}}\right\| \leq M t^{-1} e^{-\beta t} ;  \tag{22}\\
& \left\|e^{-t A_{j}}-e^{-s A_{j}}\right\| \leq M(t-s)^{\theta} s^{-\theta} e^{-\beta s} ;  \tag{23}\\
& \left\|\nabla e^{-t A_{j}}-\nabla e^{-s A_{j}}\right\| \leq M(t-s)^{\theta / 2} s^{-(\theta+1) / 2} e^{-\beta s} ;  \tag{24}\\
& \left\|e^{-t A_{j}} w-e^{-s A_{j}} w\right\|_{L^{\rho}} \leq M(t-s) e^{-\beta s}\left\|A_{j} w\right\|_{L^{p}}, \quad w \in D\left(A_{j}\right) ;  \tag{25}\\
& \left\|\nabla e^{-t A_{j}} w-\nabla e^{-s A_{j}} w\right\|_{L^{p}} \leq M(t-s)^{1 / 2} e^{-\beta s}\left\|A_{j} w\right\|_{L}^{p}, w \in D\left(A_{j}\right), \tag{27}
\end{align*}
$$

$(j=1,2)$ where $M, \beta$ are positive constants independent of $t ;\|\cdot\|$ denotes the operator norm in $L^{p}(\Omega)$.

Proof. The proofs of the inequalities above are standard in the theory of semi-groups. We omitt the subscript $j$. Since the spectrum of $A Q$ in $Q L^{p}(\Omega)$ consists of positive eigenvalues, and since $A Q=O,(21)$ and (22) follow from the representation of the holomorphic semi-groups by the DufordTaylor integral. Since $P e^{-t t}=P$, using (21) just proved, we have (20). (23) and (25) follow from (21), (22) and the following inequalities :

$$
\begin{aligned}
& \left\|e^{-t A} w-e^{-s A} w\right\|_{L^{p}}^{\theta} \leq(t-s)^{\theta}\left\|\frac{I-e^{-(t-s) A}}{(t-s) A} Q\right\|^{\theta}\left\|A e^{-s A} w\right\|_{L^{\theta}}^{\theta} ; \\
& \left\|e^{-t A} w-e^{-s A} w\right\|_{L^{p}}=\left\|e^{-t A} w-e^{-s A} w\right\|_{L^{p}}^{\theta}\left\|e^{-t A} w-e^{-s A} w\right\|_{L^{p}}^{1-\theta} .
\end{aligned}
$$

Since $A$ has a bounded inverse in $Q L^{p}(\Omega)$, we have

$$
\|Q w\|_{L^{p}} \leq M\|A w\|_{L^{p}}, \quad w \in D(A) .
$$

Hence, using the interpolation theorem :

$$
\|\nabla w\|_{L^{p}}^{2} \leq M\|Q w\|_{L^{p}}\|w\|_{2, p}
$$

(see, e. g., S. Mizohata [6: Theorem 3. 26]) we find

$$
\begin{align*}
& \|\nabla w\|_{L^{p}}^{2} \leq M\|Q w\|_{L^{p}}\|A w\|_{L^{p}} ;  \tag{28}\\
& \|\nabla w\|_{L^{p}} \leq M\|A w\|_{L^{p}} . \tag{29}
\end{align*}
$$

Hence, by (28),

$$
\left\|\nabla e^{-t A} w-\nabla e^{-s A} w\right\|_{L^{p}} \leq M\left\|e^{-t A} w-e^{-s A} w\right\|_{L^{p}}^{1 / 2}\left\|\left(I-e^{-(t-s) A}\right) A e^{-s A} w\right\|_{L^{p}}^{1 / 2}
$$

from which (24) follows by (22) and (23). Similarly, (27) can be proved by (29), (22), and (25).

Let $q>2$. Let $f(t), 0 \leq t<T$, be an $L^{q}$-function of $t$ with values in $L^{p}(\Omega)$. We then define the function $\dot{v}_{j}(t)$ by:

$$
\begin{equation*}
v(t)=e^{-t A_{j}} a_{j}+\int_{0}^{t} e^{-(t-s) A_{j}} f(s) d s, \quad j=1,2 \tag{30}
\end{equation*}
$$

Then :
Lemma 8. Let $v_{j}, f$ be as above. Let $0<\theta<1-2 / q$. We have; $(0 \leq$ $s \leq t<T$ )

$$
\begin{align*}
& \left\|v_{j}(t)\right\|_{L^{p}} \leq M+M \int_{0}^{t}\|f(\sigma)\|_{L^{p}} d \sigma  \tag{31}\\
& \left\|Q_{j} v_{j}(t)\right\|_{L^{p}} \leq M e^{-\beta t}+M \int_{0}^{t} e^{-\beta(t-\sigma)}\|f(\sigma)\|_{L^{p}} d \sigma \tag{32}
\end{align*}
$$

$$
\begin{aligned}
& \left\|Q_{j} v_{j}(t)\right\|_{L^{p}} \leq M e^{-\beta t}+M \int_{0}^{t} e^{-\beta(t-\sigma)}\|f(\sigma)\|_{L^{p}} d \sigma \\
& \left\|\nabla v_{j}(t)\right\|_{L^{p}} \leq M e^{-\beta t}+M \int_{0}^{t}(t-\sigma)^{-1 / 2} e^{-\beta(t-o)}\|f(\sigma)\|_{L^{p}} d \sigma \\
& \left\|v_{j}(t)-v_{j}(s)\right\|_{L^{p}} \leq M(t-s)+M\left\{(t-s)^{\theta_{1}}+(t-s)^{\theta}\right\}\left(\int_{0}^{t}\|f(\sigma)\|_{L^{p}}^{q} d \sigma\right)^{1 / q} \\
& \quad\left(\theta_{1}=1-1 / q\right)
\end{aligned}
$$

$$
\left\|\nabla v_{j}(t)-\nabla v_{j}(s)\right\|_{L^{p}} \leq M(t-s)+M\left\{(t-s)^{\theta_{2}}+(t-s)^{\theta_{3}}\right\}\left(\int_{0}^{t}\|f(\sigma)\|_{L^{p}}^{q} d \sigma\right)^{1 / q}
$$

$$
\left(\theta_{2}=\theta / 2 ; \theta_{3}=1 / 2-1 / q\right) .
$$

Proof. (31) and (32) follow easily from (20), (21). We shall show (35). We omitt the subscript $j$, and write $v, A$, etc. for $v_{j}, A_{j}$, etc. We have

$$
\begin{align*}
v(t)-v(s)= & \int_{s}^{t} P f(s) d s+Q\left(e^{-t A}-e^{-s A}\right) a+\int_{s}^{t} Q e^{-(t-\sigma) A} f(\sigma) d \sigma \\
& +\int_{0}^{s} Q\left(e^{-(t-\sigma) A}-e^{-(s-\sigma) A}\right) f(\sigma) d \sigma, \quad\left(\equiv I_{1}+I_{2}+I_{3}+I_{4}\right) \tag{36}
\end{align*}
$$

By (28) and (22), we see

$$
\begin{equation*}
\left\|\nabla e^{-t A}\right\| \leq M t^{-1 / 2} e^{-\beta t} \tag{37}
\end{equation*}
$$

We have : $\nabla I_{1}=0$. By (25), $\left\|\nabla I_{2}\right\|_{L^{p}} \leq M(t-s)$. By (37) and the Hölder inequalty,

$$
\begin{aligned}
\left\|\nabla I_{3}\right\|_{L^{p}} & \leq M \int_{s}^{t}(t-\sigma)^{-1 / 2} e^{-\beta(t-\sigma)}\|f(\sigma)\|_{L^{p}} d \sigma \\
& \leq M(t-s)^{\theta_{3}}\left(\int_{0}^{t}\|f(\sigma)\|_{L^{p}}^{q} d \sigma\right)^{1 / q} .
\end{aligned}
$$

By (24) and the Hölder inequality,

$$
\left\|\nabla I_{4}\right\|_{L^{p}} \leq M(t-s)^{\theta_{2}}\left(\int_{0}^{t}\|f(\sigma)\|_{L^{p}}^{q} d \sigma\right)^{1 / q}
$$

Collecting all the estimates above, we get (35). Similarly, using (23), (25), and (37), we can prove (33) and (34).

Lemma 9. Let $v_{j}, f$ be as in Lemma 8. Suppose that $f(t)$ is Hölder continuous for $t(0 \leq t<T)$ :

$$
\begin{equation*}
\|f(t)-f(s)\|_{L^{p}} \leq L\left(|t-s|^{\theta_{4}}+|t-s|^{\theta_{5}}\right) \tag{38}
\end{equation*}
$$

where $L, \theta_{4}, \theta_{5}$ are constants, $0<\theta_{4}, \theta_{5} \leq 1$. Then $v_{j}(t) \in D\left(A_{j}\right)$, and

$$
\begin{equation*}
\left\|A_{j} v_{j}(t)\right\|_{L^{p}} \leq M+M L+M\|f(t)\|_{L^{p}} \tag{39}
\end{equation*}
$$

$(0 \leq t<T)$, $M$ being a constant independent of $T, f$. Moreover, for any $\varepsilon>0$ we have $(\varepsilon \leq s<t<T)$

$$
\begin{align*}
& \left\|A_{j} v_{j}(t)-A_{j} v(s)\right\|_{L^{p}} \leq M(t-s) \varepsilon^{-1}+M L\left((t-s)^{\theta_{4}}+(t-s)^{\theta_{j}}\right)  \tag{40}\\
& \quad+M(t-s) \varepsilon^{-1}\|f(t)\|_{L^{p}} .
\end{align*}
$$

(For the proof, see T. Kato [4: Lemma IX 1.28]).
B) We now proceed to the derivation of $L^{\infty}$-bounds of solutions. Let $p>\max \{n, 2\} . \quad M$ denotes, as before, various constants independent of $T$ (which may depend on $p,\left\|a_{j}\right\|_{2, p}$ ). We have :

Lemma 10. Let $u=\left\{u_{1}, u_{2}\right\}$ be as in Lemma 1. Then

$$
\begin{align*}
& \left\|u_{j}(t)\right\|_{2, p} \leq M ;  \tag{41}\\
& \left\|u_{j}(t)\right\|_{L^{\infty}} \leq M ; \tag{42}
\end{align*}
$$

$$
\begin{equation*}
\int_{0}^{t}\left\|f_{j}(u(s))\right\|_{L^{p}}^{3 p} d s \leq M \tag{43}
\end{equation*}
$$

for $0 \leq t<T ; j=1,2$.
Proof. We claim : $(0 \leq t<T ; j=1,2)$

$$
\begin{equation*}
\left\|f_{j}(u(t))\right\|_{L^{q}} \leq M \quad(q \geq 1) \tag{44}
\end{equation*}
$$

$$
\begin{equation*}
\int_{0}^{t}\left\|f_{j}(u(s))\right\|_{L^{p}}^{3 p} d s \leq M \int_{0}^{t}\left|\left(f_{j}(u(s)), 1\right)\right|\|u(s)\|_{L^{(3 p-1) / 2}}^{3 p-1} d s \tag{45}
\end{equation*}
$$

$$
\begin{equation*}
\left\|\nabla u_{j}(t)\right\|_{L^{p}} \leq M \tag{46}
\end{equation*}
$$

Indeed, by (5), (8), and (9), (44) holds for $j=2$. By the Hölder inequality,
(45) with $j=2$ follows from (44), and hence we have, by (19), (43) with $j=2$. Since $u_{j}(t)$ satisfies (30) with $f(t)$ replaced by $f_{j}(u(t))$ :

$$
\begin{equation*}
u_{j}(t)=e^{-t A_{j}} a_{j}+\int_{0}^{t} e^{-(t-s) A_{j}} f_{j}(u(s)) d s \tag{47}
\end{equation*}
$$

it follows from (33), (43) with $j=2$, and the Hölder inequality that (46) holds for $j=2$. (42) with $j=2$ now follows from (46), (9) and the Sobolev lemma. On the other hand, (42) with $j=2$, and (8) give (44) with $j=1$. Hence by the Hölder inequality, we have (45) with $j=1$, which shows (43) with $j=1$. From (45), (33), we can see, as before, that (46), (42) hold for $j=1$. By (34) and (43), $(q=3 p ; \theta=1 / 4)$

$$
\left\|u_{j}(t)-u_{j}(s)\right\|_{L^{p}} \leq M(t-s)+M(t-s)^{2 / 3}+M(t-s)^{1 / 4}
$$

and so

$$
\left\|f_{j}(u(t))-f_{j}(u(s))\right\|_{L^{p}} \leq M(t-s)+M(t-s)^{2 / 3}+M(t-s)^{1 / 4}
$$

By (39) and (44), we have (41).

## 4. The proof of the theorem

Let $p>n+1$. Let $T>0$. Set $X_{1}=C\left([0, T] ; W^{1, p}(\Omega)\right)$. Then $X_{1}$ is the Banach space with an appropriate norm. We define a sequence $\left\{u^{(k)}\right\}_{k=1}^{\infty}$ in $X_{1} \times X_{1}$ inductively by: $\left(u^{(k)}=\left\{u_{1}^{(k)}, u_{2}^{(k)}\right\}\right)$

$$
\begin{aligned}
& u_{j}^{(1)}(t)=e^{-t A_{j}} a_{j} \\
& u_{j}^{(k+1)}(t)=u_{j}^{(1)}(t)+\int_{0}^{t} e^{-(t-s) A_{j}} f_{j}\left(u^{(k)}(s)\right) d s
\end{aligned}
$$

By the standard argument, we can see from (31)-(35) that $\left\{u^{(k)}(t)\right\}$ is a Cauchy sequence in $X_{1} \times X_{1}$ if $T$ is sufficiently small. Let $u(t)$ be the limit of the Cauchy sequence $\left\{u^{(k)}\right\}$ in $X_{1} \times X_{1}$. Then the $u(t)$ is cleary a solution of (47). Hence, in the same way as in the proof of Lemma 9, we can show that $u_{j}(t) \in D\left(A_{j}\right), A_{j} u_{j}(t)$ is Hölder continuous for $0<t<T, u_{j}(t)$ is continuously differentiable for $0<t<T$, its derivative $d u(t) / d t$ is Hölder continuous for $0<t<T$, and $u_{j}(t)$ satisfies the (abstract) equation in $L^{p}(\Omega)$ : $d u_{j}(t) / d t+A_{j} u_{j}(t)=f_{j}(u(t))$ with $u_{j}(0)=a_{j}$. On the other hand this implies that $u_{j} \in W^{1, p}(\bar{\Omega} \times[0, T))$. Hence, by the Sobolev imbedding theorem, $u$ is Hölder continuous on $\bar{\Omega} \times(0, T)$, and so is $f_{j}(u(t))$. Hence, by the existence and regularity theorem for solutions of paraboic equation (see S. Itô [3]) a classical solution $v_{j}$ of $\partial v_{j} / \partial t=\mu_{j} \Delta v_{j}+f_{j}(u(x, t))$ with conditions (3) and (4) exists and coincides with $u_{j}$; compute $d\left\|v_{j}(t)-u_{j}(t)\right\|_{L^{2}}^{2} / d t$. This implies $u_{j}$
is a classical solution of (2), (3) and (4). The uniqueness of solution can be proved by a routin argument. Since we have a priori bounds (41), we have actually a unique global solution; note it suffices to assume that $a_{j} \in$ $W^{2, p}(\Omega)$.

We finally show the asymptotic behavior in $t$ of the solution of (2), (3), (4). By (43), the right-hand sides of (32), (33) with $f(t)$ replaced by $f_{j}(u(t))$
 the Sobolev imbedding theorem, $Q_{j} u_{j}(t)$ converges to zero as $t \rightarrow \infty$, uniformly on $\bar{\Omega}$. By (13) and the assumption 3 (i), the second and third terms on the right-hand side of (14) with $j=1$ are monotone and bounded, and hence are convergent for $t \rightarrow \infty$. Hence $\left(u_{1}(t), 1\right)$ converges to some non-negative constant $\alpha_{1}$ as $t \rightarrow \infty$. By (19), $\left(f_{2}(u(t)), 1\right)$ is integrable on $[0, \infty)$, since $-\left(f_{1}(u(t)), 1\right)$ is integrable on $[0, \infty)$. Hence by (14) with $j=2, \quad\left(u_{2}(t), 1\right)$ converges to some non-negative constant $\alpha_{2}$ as $t \rightarrow \infty$. Hence, $P_{j} u_{j}(t)$ converges to $c_{j}$ as $t \rightarrow \infty$, where $c_{j}=\alpha_{j} /|\Omega|$. Hence, $u_{j}(t)$ converges to $c_{j}$ as $t \rightarrow \infty$, uniformly on $\bar{\Omega} ; c_{j} \geq 0$. Note $c_{j}=0$ if $\alpha_{j}\left(x_{0}\right)<1$ for some $x_{0} \in \Gamma$. Since $\left\|f_{j}(u(t))\right\|_{R^{p}}^{3 p}$ is, by (43), integrable on $[0, \infty)$; there is a sequence $\left\{t_{k}\right\}$, tending to the infinity, such that $\left\|f_{j}\left(u\left(t_{k}\right)\right)\right\|_{L^{p} \rightarrow 0}$. Since $u_{j}(t) \rightarrow c_{j}$, uniformly on $\bar{\Omega}($ as $t \rightarrow \infty)$, we have: $\left\|f_{j}(c)\right\|_{L^{p}}=0\left(c=\left\{c_{1}, c_{2}\right\}\right)$, which implies $f_{j}(c)=0$. Thus the proof of the theorem is completed.

Proof of the corollary. It suffices to show (6). Adding (14) with $j=1$, and (14) with $j=2$, and letting $t$ tend to the infinity, we have (6); note $f_{1}(u)+f_{2}(u)=0$.

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