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ON THE GLOBAL MONODROMY OF A LEFSCHETZ FIBRATION ARISING FROM THE FERMAT SURFACE OF DEGREE 4

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Abstract

A complete description of the global monodromy of a Lefschetz fibration arising from the Fermat surface of degree 4 is given. As a by-product we get a positive relation among right hand Dehn twists in the mapping class group of a closed orientable surface of genus 3.

1. Introduction

The motivation of this work is an interest in the topological monodromy of surface bundles obtained by the following way. Let $X \subset \mathbf{P}_N$ be a complex surface embedded in the complex projective space of dimension N. We denote by $\mathbf{P}^N$ the dual projective space of $\mathbf{P}_N$, i.e., the space of all hyperplanes of $\mathbf{P}_N$. The dual variety $X^\vee$ of X is, by definition, the set of all hyperplanes of $\mathbf{P}_N$ tangent to X at some point. Then we have a complex analytic family of compact Riemann surfaces over $\mathbf{P}^N \backslash X^\vee$; the fiber over $H \in \mathbf{P}^N \backslash X^\vee$ is the hyperplane section $H \cap X$.

If we regard such a family as an oriented surface bundle, its bundle structure is totally encoded (at least when the genus of $H \cap X$ is ≥ 2) in the associated topological monodromy ρ from the fundamental group $\pi_1(\mathbf{P}^N \backslash X^\vee)$, which is nontrivial when $X^\vee$ is a hypersurface, to the mapping class group Γ_g of a closed orientable surface of genus g, where g is the genus of $H \cap X$. If a finite presentation of $\pi_1(\mathbf{P}^N \backslash X^\vee)$ and a description of ρ in terms of this presentation are obtained, we might say that the topological monodromy ρ is understood. However such a nice situation may not be expected in general. One reason for this is the difficulty of the computations of $\pi_1(\mathbf{P}^N \backslash X^\vee)$, see [4].

Instead we consider to cut $\mathbf{P}^N \backslash X^\vee$ by a generic line. Let L be a line (1-dimensional projective subspace) of $\mathbf{P}^N$ and consider the restriction of the family over $\mathbf{P}^N \backslash X^\vee$ to $L \backslash (L \cap X^\vee)$. We focus on the associated topological monodromy ρ' from $\pi_1(L \backslash (L \cap X^\vee))$ to Γ_g. If L meets $X^\vee$ transversely, $L \cap X^\vee$ consists of finitely many points and the inclusion $L \backslash (L \cap X^\vee) \hookrightarrow \mathbf{P}^N \backslash X^\vee$ induces the surjection on the fundamental group level (the Zariski theorem of Lefschetz

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type, see [6]). Thus for instance to know the group $\mathrm{Im}(\rho)$, called *the universal monodromy group* in [4], it suffices to consider ρ' instead of ρ. Moreover, theory of Lefschetz pencils can be applied to the study of ρ', as follows. There is a natural family of algebraic curves over L; the fiber over $H \in L$ is the (possibly singular) hyperplane section $H \cap X$. As in [8] or [9], this family turns out to be a Lefschetz fibration in the sense of [5], Definition 8.1.4. In particular all the singular fibers, which are over $L \cap X^\vee$, have one nodal singularity and the local monodromy around each point of $L \cap X^\vee$ is the right hand Dehn twist along a simple closed curve, called *the vanishing cycle*. Thus the determination of the positions of all the vanishing cycles on a fixed reference fiber will lead to a complete description of *the global monodromy* ρ'. Also, as a by-product we will get a positive relation among right hand Dehn twists in Γ_g, since $\pi_1(L \backslash L \cap X^\vee)$ admits a presentation by a standard generating system subject to one relation, see the paragraph before Theorem 1.1.

In this paper we investigate a particular example. Hereafter X is the Fermat surface of degree 4, namely the smooth hypersurface in $\mathbf{P}_3$ defined by the equation

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0,$$

where $[x_0 : x_1 : x_2 : x_3]$ is a homogeneous coordinate system of $\mathbf{P}_3$. In this case $X^\vee$ is an irreducible hypersurface of $\mathbf{P}^3$, whose defining equation will be given in section 2. Let

$$\mathscr{F} := \{(x, H) \in \mathbf{P}_3 \times (\mathbf{P}^3 \backslash X^\vee); x \in H \cap X\}$$

and let

$$\rho : \pi_1(\mathbf{P}^3 \backslash X^\vee, b_0) \to \Gamma_3 \tag{1.1}$$

be the associated topological monodromy of the second projection $\pi : \mathscr{F} \to \mathbf{P}^3 \backslash X^\vee$, where b_0 is a base point. Note that for each $H \in \mathbf{P}^3 \backslash X^\vee$, the hyperplane section $\pi^{-1}(H) = H \cap X \subset H \cong \mathbf{P}_2$ is a non-singular plane curve of degree 4.

To state the result we prepare a terminology. Let $\{v_i\}_i$ be a set of n points in $\mathbf{P}_1$ and choose a base point b_0 of $\mathbf{P}_1 \backslash \{v_i\}_i$. We say a set of n based loops $\{\lambda_i\}_i$ is a *standard generating system for* $\pi_1(\mathbf{P}_1 \backslash \{v_i\}_i, b_0)$ if each λ_i is free homotopic to a loop nearby v_i going once around v_i by counter-clockwise manner, and their product $\lambda_1 \lambda_2 \cdots \lambda_n$ is trivial as an element of $\pi_1(\mathbf{P}_1 \backslash \{v_i\}_i, b_0)$.

Theorem 1.1. *Let X be the Fermat surface of degree* 4 *and L a line of $\mathbf{P}^3$ meeting $X^\vee$ transversely. Choose a base point b_0 of $L \backslash (L \cap X^\vee)$. Then there is a standard generating system $\lambda_1, \ldots, \lambda_{36}$ for $\pi_1(L \backslash (L \cap X^\vee), b_0)$ such that the monodromy $\rho'(\lambda_i)$ is given by the right hand Dehn twist along the simple closed curve C_i on a genus* 3 *surface as shown in Figure* 1.1. *Here, ρ' is the composition of $\pi_1(L \backslash (L \cap X^\vee), b_0) \to \pi_1(\mathbf{P}^3 \backslash X^\vee, b_0)$ induced by the inclusion and* (1.1).

Since $\lambda_1 \lambda_2 \cdots \lambda_{36} = 1$ and ρ' is an anti-homomorphism (see the conventions below), we immediately have the following

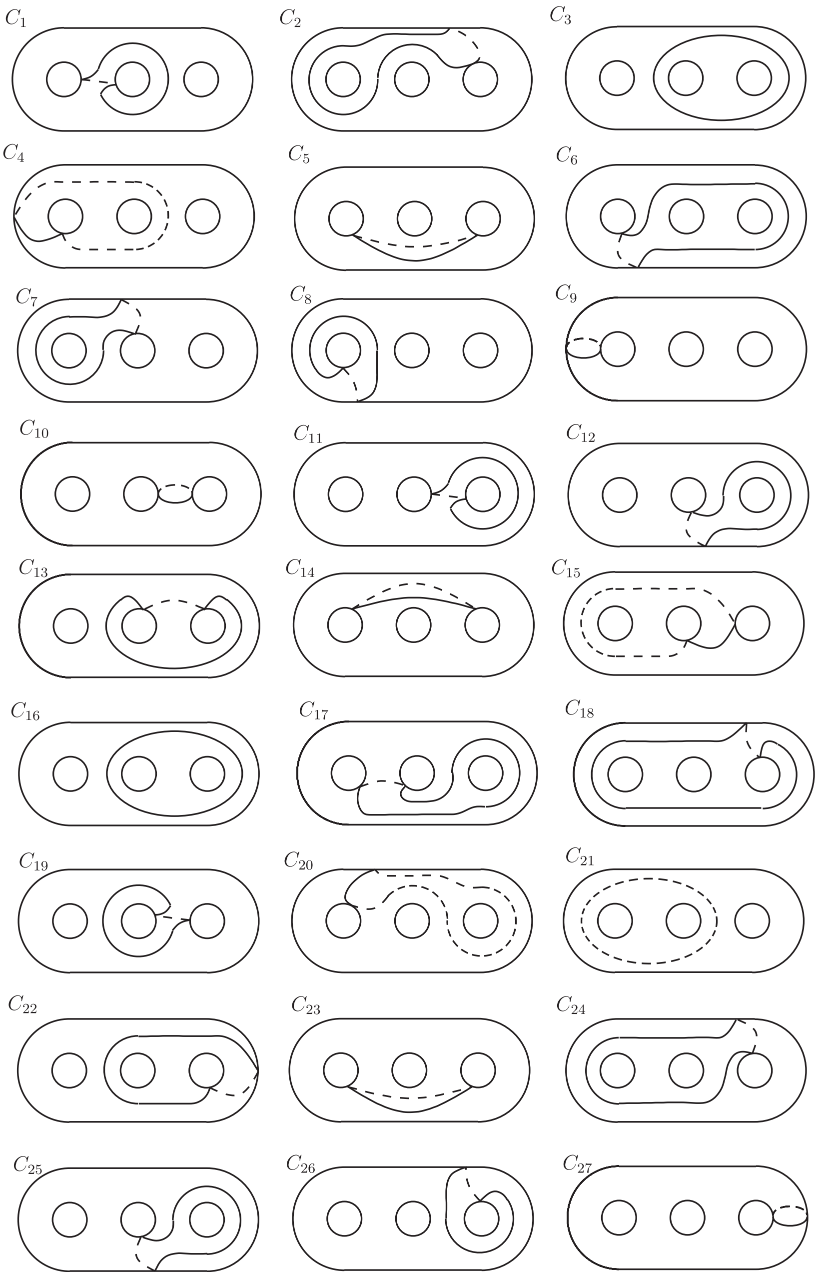
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FIGURE 1.1

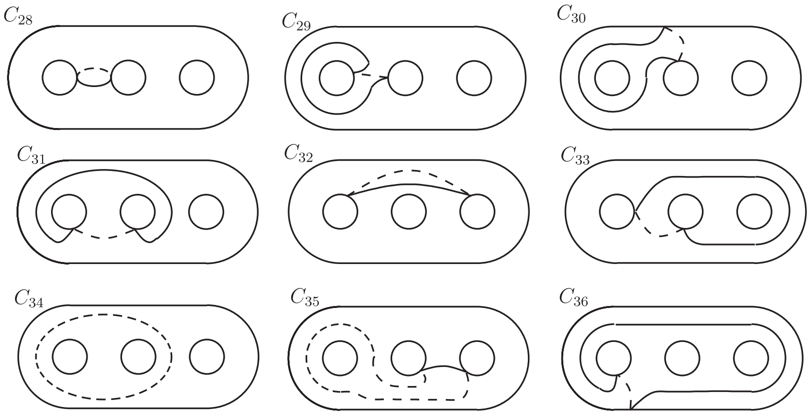

FIGURE 1.1 (continued)

COROLLARY 1.2. *Let us denote by t_i the right hand Dehn twist along C_i. Then the relation $t_{36}t_{35}\cdots t_2t_1 = 1$ holds in the mapping class group Γ_3.*

Also we can show the following

COROLLARY 1.3. *The topological monodromy* (1.1) *is surjective. In other words, the universal monodromy group* $\mathrm{Im}(\rho)$ *coincides with* Γ_3.

Proof. The set of the right hand Dehn twists along the seven simple closed curves C_9, $t_9(C_8)$, C_{28}, $t_{28}^{-1}(C_1)$, C_{10}, $t_{10}^{-1}(C_{11})$, and $t_3^{-1}t_{28}^{-1}(C_6)$ constitutes a Dehn-Lickorish-Humphries generating system of Γ_3 (see [7], Corollary 4.2.F). Thus ρ' is surjective, so is ρ. $\square$

The study of the global monodromy of a holomorphic fibration of Riemann surfaces over a Riemann surface via numerical analysis is initiated by Ahara [2] and Matsumoto [10]. They introduced a holomorphic fibration $f : V_n \to \mathbf{P}_1$, where V_n is the Fermat surface of degree n. Their fibration is not a Lefschetz fibration and has more degenerated singular fibers. Their method was to express the general fibers as branched coverings of $\mathbf{P}_1$ and analyze the motions of the critical points of these branched coverings. The analysis is based on Newton approximation, see [2], section 3. Based on the result of [2], the global monodromy was described in terms of Dehn twists for the case $n = 5$ in [10] (in this case the genus of the general fibers is 3). Recently, Ahara and Awata [1] determined how general fibers of f degenerate to the singular fibers for all n, without numerical analysis.

The rest of the paper is devoted to the proof of Theorem 1.1. Note that the total space of our Lefschetz fibration $\bar{\pi} : \bar{\bar{\mathscr{F}}} \to L$ (see section 5) is the blow up of X at 4 points. We adopt the same method as [2], [10]. In section 2 we give the defining equation of $X^\vee$. In section 3 we cut $\mathbf{P}^3 \setminus X^\vee$ by a line $L = L(c_1, c_2)$ whose defining equation has two parameters c_1 and c_2. For a suitable choice of c_1 and c_2, L will meet $X^\vee$ transversely. We will introduce a homogeneous coordinate system $[u : v]$ to L and will denote by $\mathbf{C}_L$ the set $L \setminus \{[0:1]\}$. Then we proceed to express general fibers as 4-branched coverings of $\mathbf{P}_1$. In section 4 we introduce a projection p_v from $X_v = v \cap X$ to $\mathbf{P}_1$ for $v = [1 : v] \in \mathbf{C}_L \subset \mathbf{P}^3$ and prove its "tameness" over $[1:0] \in \mathbf{P}_1$ (see Lemma 4.1). Section 5 is a preparation for sections 6 and 7. We choose explicit values for c_1 and c_2. Most of the results in sections 6 and 7 depend on numerical analysis using a computer. In section 6 we describe the projection $p_0 : X_0 \to \mathbf{P}_1$ and in section 7 we analyze motions of the critical values of p_v caused by movements of v along suitable chosen paths in L and give a complete description of the topological monodromy $\rho' : \pi_1(L \setminus (L \cap X^\vee)) \to \Gamma_3$. Theorem 1.1 will easily follow from Proposition 7.1.

Conventions about topological monodromy

It is sometimes confusing that there are different kinds of conventions about product of paths or product of maps, so let us fix the conventions in this paper: 1) for any two mapping classes f_1 and f_2, the multiplication $f_1 \circ f_2$ means that f_2 is applied first, 2) for any two homotopy classes of based loops ℓ_1 and ℓ_2, their product $\ell_1 \cdot \ell_2$ means that ℓ_1 is traversed first.

Let Σ be a closed oriented surface and $\pi : E \to B$ an oriented Σ-bundle. Choose a base point $b_0 \in B$ and fix an identification $\phi : \Sigma \xrightarrow{\cong} \pi^{-1}(b_0)$. For each based loop $\ell : [0,1] \to B$, consider the pull back $\ell^*(E) \to [0,1]$. Since $[0,1]$ is contractible there exists a trivialization $\Phi : \Sigma \times [0,1] \to \ell^*(E)$ such that $\Phi(x,0) = \phi(x)$. By assigning the isotopy class of $\phi^{-1} \circ \Phi(x,1)$ to the homotopy class of ℓ, we obtain a map ρ, called *the topological monodromy of* $\pi : E \to B$, from $\pi_1(B, b_0)$ to the mapping class group of Σ. Under the conventions above, ρ is an *anti-homomorphism*, i.e., for $\ell_1, \ell_2 \in \pi_1(B, b_0)$ we have

$$\rho(\ell_1 \ell_2) = \rho(\ell_2)\rho(\ell_1).$$

2. The defining equation of $X^\vee$

Our first task is to describe the defining equation of $X^\vee$. The result might be known, but we give it here since our numerical analysis by a computer performed in sections 6 and 7 will heavily use it. To begin with, we compute the degree of $X^\vee$. By using the formula of Katz [8] (5.5.1), it is computed as

$$\deg(X^\vee) = \int_X \frac{(1+h)^2}{c([X])} = \int_X \frac{(1+h)^2}{1+4h} = 36.$$

Here, $h \in H^2(\mathbf{P}_3; \mathbf{Z})$ denotes the hyperplane class and $c([X]) \in H^*(\mathbf{P}_3; \mathbf{Z})$ denotes the total Chern class of the divisor X. Let $[\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3]$ be the homogeneous coordinate system of $\mathbf{P}^3$ dual to $[x_0 : x_1 : x_2 : x_3]$. Namely, $[\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3] \in \mathbf{P}^3$ is the hyperplane of $\mathbf{P}_3$ defined by

$$\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0.$$

Proposition 2.1. *Let* $\omega = \exp(2\pi\sqrt{-1}/3)$ *and let* β_i *be a formal indeterminate such that* $\beta_i^3 = \alpha_i$ *for* $i = 0, 1, 2,$ *and* 3. *Then the defining equation of* $X^\vee$ *is given by*

$$\prod_{0 \le i_1, i_2, i_3 \le 2} (\beta_0^4 + \omega^{i_1}\beta_1^4 + \omega^{i_2}\beta_2^4 + \omega^{i_3}\beta_3^4) = 0. \tag{2.1}$$

Remark that the left hand side of (2.1) is invariant under the transformations $\{\phi_i\}_{i=0}^3$ where ϕ_i is defined by $\phi_i(\beta_j) = \omega\beta_j$ for $j \neq i$ and $\phi_i(\beta_i) = \beta_i$. Thus it is in fact a homogeneous polynomial in α_i's and the degree is 36.

Proof of Proposition 2.1. Since we know the degree of $X^\vee$ is also equal to 36, it suffices to show that $\alpha \in X^\vee$ if and only if α satisfies the equation (2.1). Let $\alpha = [\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3] \in \mathbf{P}^3$ and assume that $\alpha_0 = 1$. Let $P(x_0, x_1, x_2, x_3) = x_0^4 + x_1^4 + x_2^4 + x_3^4$. By definition, $\alpha \in X^\vee$ if and only if there exists a point $y = [y_0 : y_1 : y_2 : y_3] \in \mathbf{P}_3$ such that

$$\begin{cases} P(y_0, y_1, y_2, y_3) = 0 \\ y_0 + \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 = 0 \\ [P_{x_0}(y) : P_{x_1}(y) : P_{x_2}(y) : P_{x_3}(y)] = [1 : \alpha_1 : \alpha_2 : \alpha_3], \end{cases} \tag{2.2}$$

where P_{x_0} is the partial derivative of P with respect to x_0, etc. Since $P_{x_i} = 4x_i^3$ we see that $y_0 \neq 0$, by the third equation of (2.2). Thus we may assume $y_0 = 1$ and we have

$$y_1^3 = \alpha_1, \quad y_2^3 = \alpha_2, \quad y_3^3 = \alpha_3. \tag{2.3}$$

Under (2.3), the first and the second equations of (2.2) are equivalent. Therefore $\alpha \in X^\vee$ if and only if there exists $(y_1, y_2, y_3) \in \mathbf{C}^3$ such that

$$\begin{cases} y_1^3 = \alpha_1,\ y_2^3 = \alpha_2,\ y_3^3 = \alpha_3 \\ 1 + \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 = 0. \end{cases}$$

Let β_i be a complex number such that $\beta_i^3 = \alpha_i$ for $i = 1, 2,$ and 3. Then, $\alpha \in X^\vee$ if and only if there exist $i_1, i_2, i_3 \in \{0, 1, 2\}$ such that

$$1 + \omega^{i_1}\beta_1^4 + \omega^{i_2}\beta_2^4 + \omega^{i_3}\beta_3^4 = 0,$$

namely $\beta = [1 : \beta_1 : \beta_2 : \beta_3]$ satisfies the equation (2.1). This completes the proof. □

3. Cutting $X^\vee$ by a line of special type

Let c_1 and c_2 be complex numbers and $L = L(c_1, c_2)$ the line of $\mathbf{P}^3$ defined by

$$c_1^3\alpha_0 - \alpha_1 = c_2^3\alpha_0 - \alpha_2 = 0.$$

We introduce a homogeneous coordinate system $[u : v]$ of L by assigning $[\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3] = [u : c_1^3 u : c_2^3 u : v]$ to $[u : v]$.

PROPOSITION 3.1. *The defining equation of* $L \cap X^\vee \subset L$ *is given by*

$$\prod_{0 \le i_1, i_2 \le 2} ((1 + \omega^{i_1} c_1^4 + \omega^{i_2} c_2^4)^3 u^4 + v^4) = 0.$$

Proof. Let β_i, $0 \le i \le 3$ be the formal elements as in Proposition 2.1 and suppose $c_1\beta_0 - \beta_1 = c_2\beta_0 - \beta_2 = 0$. Then $\beta_0^4 + \omega^{i_1}\beta_1^4 + \omega^{i_2}\beta_2^4 + \omega^{i_3}\beta_3^4$ is equal to

$$\beta_0^4 + \omega^{i_1}(c_1\beta_0)^4 + \omega^{i_2}(c_2\beta_0)^4 + \omega^{i_3}\beta_3^4 = (1 + \omega^{i_1} c_1^4 + \omega^{i_2} c_2^4)\beta_0^4 + \omega^{i_3}\beta_3^4,$$

and

$$\begin{aligned}\prod_{0 \le i_3 \le 2} (1 + \omega^{i_1} c_1^4 + \omega^{i_2} c_2^4)\beta_0^4 + \omega^{i_3}\beta_3^4 &= ((1 + \omega^{i_1} c_1^4 + \omega^{i_2} c_2^4)\beta_0^4)^3 + (\beta_3^4)^3 \\ &= (1 + \omega^{i_1} c_1^4 + \omega^{i_2} c_2^4)^3 \alpha_0^4 + \alpha_3^4.\end{aligned}$$

Note that ω is a primitive third root of unity. Combining this computation with Proposition 2.1, we have the result. □

Suppose c_1 and c_2 are chosen so that

1. for any pair (i_1, i_2), we have $1 + \omega^{i_1} c_1^4 + \omega^{i_2} c_2^4 \neq 0$,
2. for any two distinct pairs $i = (i_1, i_2)$ and $j = (j_1, j_2)$, the roots of $f_i(v) = v^4 + (1 + \omega^{i_1} c_1^4 + \omega^{i_2} c_2^4)^3$ and those of $f_j(v) = v^4 + (1 + \omega^{j_1} c_1^4 + \omega^{j_2} c_2^4)^3$ are all different.

Then by Proposition 3.1, $L \cap X^\vee$ consists of $\deg(X^\vee) = 36$ points therefore L meets $X^\vee$ transversely. Moreover, $L \cap X^\vee$ is contained in $L \backslash \{[1 : 0], [0 : 1]\}$. For simplicity we write $\mathbf{C}_L$ instead of $L \backslash \{[0 : 1]\}$, and we identify $\mathbf{C}_L$ with $\mathbf{C}$ by $v \mapsto [1 : v]$, $v \in \mathbf{C}$. Choose $0 \in \mathbf{C}_L$ as a base point of $L \backslash (L \cap X^\vee)$. By the Zariski theorem of Lefschetz type [6], (for our purpose, a weaker statement in [9], (7.4.1) is sufficient) the natural homomorphism

$$\pi_1(L \backslash (L \cap X^\vee), 0) \to \pi_1(\mathbf{P}^3 \backslash X^\vee, b_0) \tag{3.1}$$

induced by the inclusion is surjective (we denote by b_0 the image of $0 \in \mathbf{C}_L$ by the inclusion). From now on, we assume that c_1 and c_2 satisfy the two conditions above and will focus on the surface bundle

$$\pi' : \mathscr{F}' \to L \backslash (L \cap X^\vee)$$

where $\mathscr{F}' = \pi^{-1}(L \backslash (L \cap X^\vee))$ and $\pi' = \pi|_{\mathscr{F}'}$. The associated topological monodromy

$$\rho' : \pi_1(L \backslash (L \cap X^\vee), 0) \to \Gamma_3$$

is the composition of (3.1) and (1.1).

4. A lemma on the hyperplane section by $v \in \mathbf{C}_L$

Let $v \in \mathbf{C}_L$. We denote by X_v the hyperplane section $v \cap X$, whose defining equation is

$$\begin{cases} x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0 \\ x_0 + c_1^3 x_1 + c_2^3 x_2 + v x_3 = 0. \end{cases}$$

Eliminating the indeterminate x_0, we obtain

$$(c_1^3 x_1 + c_2^3 x_2 + v x_3)^4 + x_1^4 + x_2^4 + x_3^4 = 0.$$

Let $E_v = E_v(x_1, x_2, x_3)$ be the left hand side of this equation. Then by regarding $[x_1 : x_2 : x_3]$ as a homogeneous coordinate system of $\mathbf{P}_2$, X_v is identified with the plane curve determined by E_v. Under this identification, consider the projection

$$p_v : X_v \to \mathbf{P}_1, \quad [x_1 : x_2 : x_3] \mapsto [x_1 : x_3].$$

LEMMA 4.1. *If* $|c_1|^4 + |c_2|^4 < 1$, *the following holds: for any* $v \in \mathbf{C}_L$,
1. *the plane curve* X_v *has no singularities on the line* $x_3 = 0$, *and*
2. *the projection* p_v *does not branch over* $[1 : 0] \in \mathbf{P}^1$.

Proof. For simplicity, we write E instead of E_v. Suppose $E = E_{x_1} = E_{x_2} = E_{x_3} = 0$ has a solution $[x_1 : x_2 : 0]$ for some $v \in \mathbf{C}_L$. If $v \neq 0$, we have $c_1^3 x_1 + c_2^3 x_2 = 0$ since $E_{x_3} = 0$. Substituting this into $E_{x_1} = E_{x_2} = 0$ we have $x_1 = x_2 = 0$, a contradiction. Thus it suffices to consider the case when $v = 0$. Suppose $x_2 = 1$. Then we have

$$\begin{cases} E_{x_1} = 4c_1^3 (c_1^3 x_1 + c_2^3)^3 + 4x_1^3 = 0 \\ E_{x_2} = 4c_2^3 (c_1^3 x_1 + c_2^3)^3 + 4 = 0. \end{cases} \tag{4.1}$$

By the second equation of (4.1), we have

$$(c_1^3 x_1 + c_2^3)^3 = -c_2^{-3}. \tag{4.2}$$

Substituting this into the first equation of (4.1), we have $x_1^3 = (c_1/c_2)^3$ therefore we can write $x_1 = \omega^j c_1/c_2$ for some j, $0 \le j \le 2$. Substituting this into (4.2) we have a necessary condition $(c_1^4 \omega^j + c_2^4)^3 = -1$. But this is impossible by our assumption $|c_1|^4 + |c_2|^4 < 1$. If we assume $x_1 = 1$ a similar argument leads to a contradiction. This establishes the first part.

To show the second part, it suffices to show the following: for $(x_1, x_3) = (1, 0)$, the equation $E = E_{x_2} = 0$ does not have any solution in x_2. The argument is similar to the first part. Suppose $x_2 \in \mathbf{C}$ satisfies

$$\begin{cases} E = (c_1^3 + c_2^3 x_2)^4 + 1 + x_2^4 = 0 \\ E_{x_2} = 4c_2^3 (c_1^3 + c_2^3 x_2)^3 + 4x_2^3 = 0. \end{cases} \tag{4.3}$$

By the second equation of (4.3), we have

$$(c_1^3 + c_2^3 x_2)^3 = -\frac{x_2^3}{c_2^3}. \tag{4.4}$$

Substituting this into the first equation of (4.3), we see that $x_2 = \omega^j c_2/c_1$ for some j, $0 \le j \le 2$. Substituting this into (4.4) we have $(c_1^4 + c_2^4\omega^j)^3 = -1$, a contradiction. □

5. A special choice of c_1 and c_2

Henceforth, let $c_1 = 7/8$ and $c_2 = 3/4$. For this choice, the conditions for c_1 and c_2 given in section 3 and the assumption of Lemma 4.1 are satisfied.

To study ρ' (see section 3) we also consider $\overline{\overline{\mathscr{F}}} := \{(x, H) \in \mathbf{P}_3 \times L; x \in H \cap X\}$ and the second projection $\bar{\pi} : \overline{\overline{\mathscr{F}}} \to L$. By the transversality of L and $X^\vee$, it follows that $\overline{\overline{\mathscr{F}}}$ is non-singular and $\bar{\pi} : \overline{\overline{\mathscr{F}}} \to L$ is a Lefschetz fibration (see section 1). The set of critical values of $\bar{\pi}$ is $L \cap X^\vee = \{v_1, \dots, v_{36}\}$. For each v_i, there is a unique critical point $\tilde{v}_i$ in $\bar{\pi}^{-1}(v_i)$ and for a suitable choice of local holomorphic coordinates, the projection $\bar{\pi}$ looks like $(z_1, z_2) \mapsto z_1^2 + z_2^2$ near $\tilde{v}_i$. In this local model, the singular fiber $\bar{\pi}^{-1}(v_i)$ looks like $\Sigma_0 = \{z_1^2 + z_2^2 = 0\}$, which is obtained from the smooth fibers $\Sigma_\varepsilon = \{z_1^2 + z_2^2 = \varepsilon\}$, $\varepsilon > 0$ by collapsing the simple closed curves $C_\varepsilon = \{(x_1, x_2) \in \mathbf{R}^2; x_1^2 + x_2^2 = \varepsilon\}$. The curve C_ε is called *the vanishing cycle*. By the Picard-Lefschetz formula ([5], p. 295), the local monodromy around each v_i is the right hand Dehn twist along the corresponding vanishing cycle.

Recall that the defining equation of $X_v = v \cap X$ is

$$(c_1^3 x_1 + c_2^3 x_2 + v x_3)^4 + x_1^4 + x_2^4 + x_3^4 = 0.$$

By Lemma 4.1, p_v is unramified over $[1:0] \in \mathbf{P}_1$. Thus we focus on p_v restricted to $\mathbf{P}_1 \backslash \{[1:0]\}$, which is identified with $\mathbf{C}$ by $x_1 \mapsto [x_1 : 1]$, $x_1 \in \mathbf{C}$. Let

$$F_{x_1}^v(x_2) := (c_1^3 x_1 + c_2^3 x_2 + v)^4 + x_1^4 + x_2^4 + 1$$

and $G^v(x_1)$ the discriminant of $F_{x_1}^v$ regarded as a polynomial in x_2 and $Q(v)$ the discriminant of $G^v(x_1)$ regarded as a polynomial in x_1. $G^v(x_1)$ is a polynomial of degree 12 in x_1. By definition $v \in \mathbf{C}_L$ is a root of Q if and only if there is a root of G^v with multiplicity ≥ 2. As we will see in section 7, G^{v_i} has this property hence $Q(v_i) = 0$ for $i = 1, \dots, 36$. Therefore if v is not a root of Q the curve X_v is non-singular and all the roots of G^v, which correspond to the critical values of p_v, are simple. By the Riemann-Hurwitz formula we see that the total branching order of each critical value of p_v is 1. This means that over each critical value there is an exactly one critical point of p_v, near which p_v looks like $z \mapsto z^2$ for a suitable choice of local coordinates.

6. Description of the reference fiber

In this section, we describe the reference fiber $X_0 = \pi'^{-1}(0)$ as a 4-fold branched covering $p_0 : X_0 \to \mathbf{P}_1$. As in the last section we focus on p_0 restricted to $\mathbf{P}_1 \backslash \{[1:0]\} \cong \mathbf{C}$.

The roots of $G^0(x_1)$ are numerically computed and we denote them by $a_1, \dots, a_{12}$ as shown in the following schematic figure:

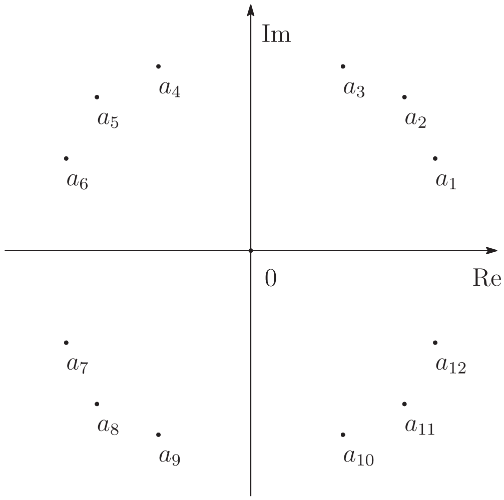

FIGURE 6.1

Here, $a_1 \approx 0.709187 + 0.642143\sqrt{-1}$, $a_2 \approx 0.692307 + 0.692307\sqrt{-1}$, $a_3 \approx 0.642143 + 0.709187\sqrt{-1}$ and $a_{i+3} = \sqrt{-1}a_i$ for $1 \leq i \leq 9$.

For $x_1 \in \mathbf{C}$, the points in the fiber $p_0^{-1}(x_1)$ correspond to the roots of $F_{x_1}^0$ by $[x_1 : x_2 : 1] \mapsto x_2$. Now we choose 0 as a base point of $\mathbf{C}\backslash\{a_i\}_i$. The fiber $p_0^{-1}(0)$ corresponds to the roots of

$$F_0^0(x_2) = (c_2^{12} + 1)x_2^4 + 1,$$

i.e., $\{s_k\}_{k=1}^4$ where $s_k = (1 + c_2^{12})^{-1/4} \exp((2k-1)\pi\sqrt{-1}/4)$.

We will investigate the monodromy

$$\chi : \pi_1(\mathbf{C}\backslash\{a_i\}_i, 0) \to \mathfrak{S}_4$$

of the unramified 4-covering $p_0^{-1}(\mathbf{C}\backslash\{a_i\}_i) \to \mathbf{C}\backslash\{a_i\}_i$. Here, $\mathfrak{S}_4$ is the symmetric group on the four letters s_1, s_2, s_3, and s_4.

For each $j = 1, \ldots, 12$, let m_j be the straight line segment from 0 to a_j and ℓ_j be a based loop in $\mathbf{C}\backslash\{a_i\}_i$ going from 0 to a point nearby a_j along m_j, then going once around a_j by counter-clockwise manner and then coming back to 0 along m_j, as shown in the following figure.

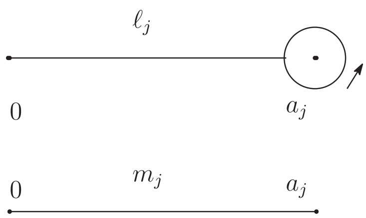

FIGURE 6.2

By numerical analysis using a computer, we see that $\chi(\ell_j)$ is given by the following table:

j	$\chi(\ell_j)$	j	$\chi(\ell_j)$
1	(12)	7	(34)
2	(13)	8	(13)
3	(14)	9	(23)
4	(23)	10	(14)
5	(24)	11	(24)
6	(12)	12	(34)

For example, $\chi(\ell_1) = (12)$ means $\chi(\ell_1)$ is the transposition of s_1 and s_2, etc. Let $S_k = [0 : s_k : 1]$ and $\tilde{a}_j$ the unique critical point of p_0 over a_j, and $\tilde{m}_j$ the connected component of $p_0^{-1}(m_j)$ containing $\tilde{a}_j$ as an interior point. Then $p_0^{-1}(0) = \{S_k\}_{k=1}^4$ and we can draw the picture of S_k, $\tilde{a}_j$, and $\tilde{m}_j$ on X_0 by using the table above, which determines the topological type of the branched covering $p_0 : X_0 \to \mathbf{P}_1$. See the figure below.

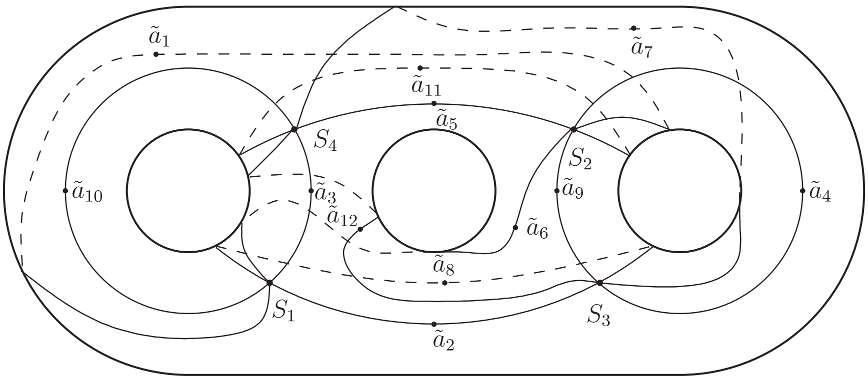

FIGURE 6.3

For example, $\tilde{m}_1$ is the unique path from S_1 through $\tilde{a}_1$ to S_2, corresponding to the data $\chi(\ell_1) = (12)$. In section 7 this figure will be a key to find the vanishing cycles.

7. Finding the vanishing cycles

In this section we give a complete description of

$$\rho' : \pi_1(L \backslash (L \cap X^\vee), 0) \to \Gamma_3$$

and finish the proof of Theorem 1.1. Our task is to determine the position of all the vanishing cycles in X_0. We will achieve this by investigating the motions of the critical values of p_v along a suitably chosen path from 0 to each point of $L \cap X^\vee = \{v_1, \dots, v_{36}\}$.

Now we arrange indices of v_i's and let $\mu_i : [0,1] \to \mathbf{C}_L$ be a simple path from 0 to v_i, satisfying $Q(\mu_i(t)) \neq 0$ for $t \in [0,1)$, as shown in Figure 7.1.

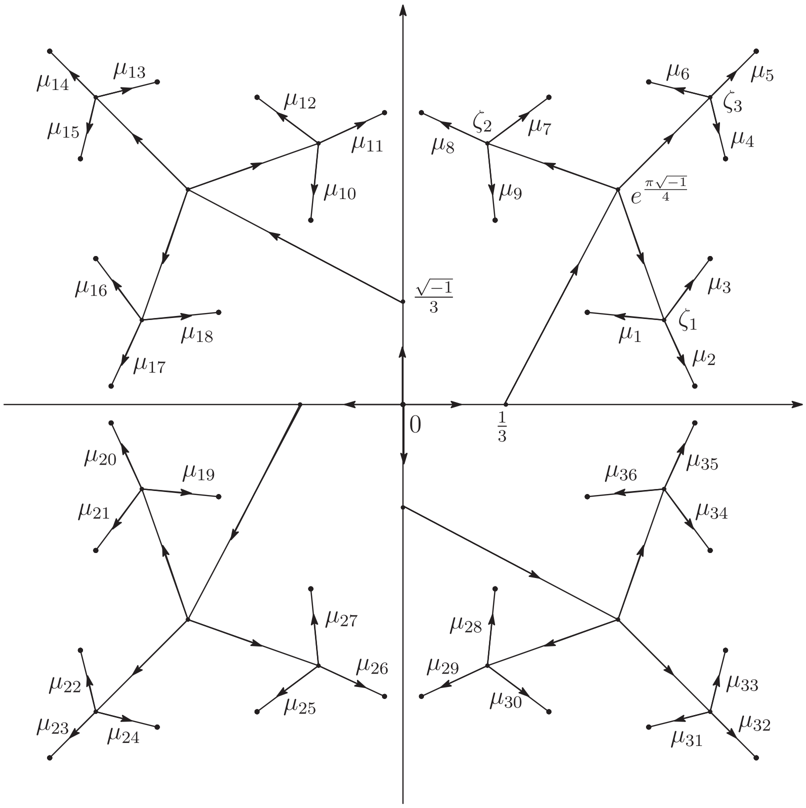

FIGURE 7.1

Approximate values of v_i's are: $v_1 \approx 0.600851 + 0.315483\sqrt{-1}$, $v_2 \approx 0.963952 + 0.064039\sqrt{-1}$, $v_3 \approx 0.999689 + 0.470655\sqrt{-1}$, $v_4 \approx 1.059535 + 0.794167\sqrt{-1}$, $v_5 \approx 1.145495 + 1.145495\sqrt{-1}$, $v_i = \mathrm{Im}(v_{10-i}) + \mathrm{Re}(v_{10-i})$ for $6 \leq i \leq 9$, and $v_{i+9} = \sqrt{-1}v_i$ for $1 \leq i \leq 27$. Each μ_i consists of 4 straight line segments, as shown in the figure. Here, ζ_i is a root of $(1 + \omega^i c_1^4)^3 + \zeta_i^4$ such that $\mathrm{Re}(\zeta_i) > 0$, $\mathrm{Im}(\zeta_i) > 0$ for $i = 1, 2, 3$ and $\mu_{i+9} = \sqrt{-1}\mu_i$ for $1 \leq i \leq 27$.

Let λ_i be a based loop in $\mathbf{C}_L \backslash (L \cap X^\vee)$ going from 0 to a point nearby v_i along μ_i, then going once around v_i by counter-clockwise manner and then coming back to 0 along μ_i. Then $\{\lambda_1, \dots, \lambda_{36}\}$ is a standard generating system for $\pi_1(L \backslash (L \cap X^\vee), 0)$ in the sense of section 1.

For a while we fix i, $1 \le i \le 36$. For each $t \in [0,1)$ the roots of $G^{\mu_i(t)}(x_1)$ are all simple, therefore we can choose complex valued continuous functions $a_1(t), \dots, a_{12}(t)$ such that $G^{\mu_i(t)}(a_j(t)) = 0$ and $a_j(0) = a_j$ for $j = 1, \dots, 12$. We have $a_j(t) \neq a_k(t)$ for $t \in [0,1)$ and (j,k) with $j \neq k$.

By continuity, $a_j(t)$ is uniquely extended to a continuous function on the unit interval $[0,1]$. We would like to study what happens when t approaches 1. By numerical analysis using a computer, we can investigate the motions of $a_j(t)$, $1 \le j \le 12$.

Observation 1. There exist two indices $\delta = \delta(i)$ and $\varepsilon = \varepsilon(i)$, $1 \le \delta < \varepsilon \le 12$ such that $a_\delta(1) = a_\varepsilon(1)$ and $a_j(1) \neq a_k(1)$ for any pair (j,k) with $j < k$ other than (δ, ε), see the table below. In particular, the number of roots of G^{v_i} is 11.

i	$(\delta(i), \varepsilon(i))$	i	$(\delta(i), \varepsilon(i))$	i	$(\delta(i), \varepsilon(i))$	i	$(\delta(i), \varepsilon(i))$
1	$(3,6)$	10	$(6,9)$	19	$(9,12)$	28	$(3,12)$
2	$(1,4)$	11	$(4,7)$	20	$(7,10)$	29	$(1,10)$
3	$(2,5)$	12	$(5,8)$	21	$(8,11)$	30	$(2,11)$
4	$(1,7)$	13	$(4,10)$	22	$(1,7)$	31	$(4,10)$
5	$(2,8)$	14	$(5,11)$	23	$(2,8)$	32	$(5,11)$
6	$(3,9)$	15	$(6,12)$	24	$(3,9)$	33	$(6,12)$
7	$(2,11)$	16	$(2,5)$	25	$(5,8)$	34	$(8,11)$
8	$(3,12)$	17	$(3,6)$	26	$(6,9)$	35	$(9,12)$
9	$(1,10)$	18	$(1,4)$	27	$(4,7)$	36	$(7,10)$

Observation 2. For any root $a_j(1)$ of G^{v_i}, the number of roots of $F^{v_i}_{a_j(1)}$ is 3.

Let $\{\gamma_i^t : [0,1] \to \mathbf{C}\}_{0 \le t \le 1}$ be a continuous family of paths constructed by the following way. First choose a real number $t_0 < 1$ sufficiently near 1, and for $t \in [t_0, 1]$, let γ_i^t be the straight path joining $a_\delta(t)$ and $a_\varepsilon(t)$. Next extending the motions of $a_i(t)$'s for $t \in [0, t_0]$, we have an ambient isotopy $\tau : \mathbf{C} \times [0, t_0] \to \mathbf{C}$ of $\mathbf{C}$ such that $\tau(x_1, t_0) = x_1$ and $\tau(a_i(t_0), t) = a_i(t)$, $1 \le i \le 12$. Finally we set $\gamma_i^t(s) = \tau(\gamma_i^{t_0}(s), t)$ for $t \in [0, t_0]$. Note that we may assume that $\gamma_{i+9}^t = \sqrt{-1}\gamma_i^t$. This follows from the fact that $x_1 \in \mathbf{C}$ is a root of G^v if and only if $\sqrt{-1}x_1$ is a root of $G^{\sqrt{-1}v}$. Then we have

Observation 3. γ_i^0, $1 \le i \le 9$ look like Figure 7.2.

By construction the family $\{\gamma_i^t\}_{0 \le t \le 1}$ satisfies the following three conditions:

1. for each $t \in [0,1]$, $\gamma_i^t(0) = a_\delta(t)$ and $\gamma_i^t(1) = a_\varepsilon(t)$,
2. for each $t \neq 1$, γ_i^t is a simple path not meeting $\{a_j(t)\}_{j \neq \delta, \varepsilon}$,
3. $\gamma_i^1(s) = a_\delta(1) = a_\varepsilon(1)$, for $s \in [0,1]$.

Let $C_i(t)$ be the connected component of $p_{\mu_i(t)}^{-1}(\gamma_i^t([0,1]))$ containing the critical points of $p_{\mu_i(t)}$ over $a_\delta(t)$ and $a_\varepsilon(t)$. We can draw the picture of $C_i(0)$ on X_0 in Figure 6.3, then we see that it is a simple closed curve in X_0, and isotopic to C_i if we identify X_0 with the genus 3 surfaces in Figure 1.1 by an obvious manner. The simplicity of the roots of $G^{\mu_i(t)}(x_1)$ for $t \in [0,1)$ implies that the topological type of $p_{\mu_i(t)}$ is the same as p_0, therefore $C_i(t)$ is also a simple closed curve in $X_{\mu_i(t)}$ for $t \in [0,1)$. On the other hand $C_i(1) = p_{v_i}^{-1}(a_\delta(1))$ consists of a single point, which is a unique singular point of X_{v_i}.

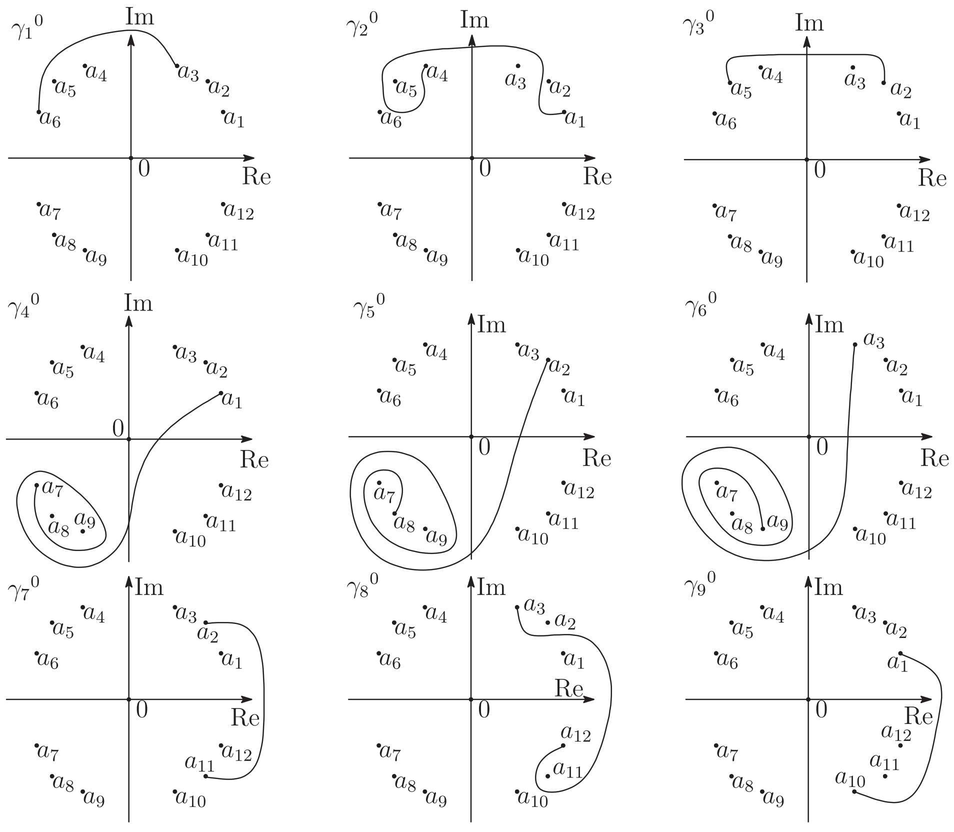

FIGURE 7.2

Let D be the unit closed disk and choose a continuous family $\{\iota_i^t : D \to \mathbf{C}\}_{0 \le t \le 1}$ of embeddings of D such that $\iota_i^t(D)$ contains $\gamma_i^t([0,1])$ and does not meet $\{a_j(t)\}_{j \ne \delta, \varepsilon}$. Let $A_i(t)$ be the connected component of $p_{\mu_i(t)}^{-1}(\iota_i^t(D))$ containing $C_i(t)$. For $t \in [0,1)$, $A_i(t)$ is homeomorphic to an annulus, and $A_i(1)$ is homeomorphic to the space obtained from an annulus by collapsing a non null-homologous simple closed curve in it.

Let M_i be the quotient space of $X_0 \times [0,1]$ obtained by identifying all of $C_i(0) \times \{1\}$ to a single point. Using $\{\iota_i^t\}_{0 \le t \le 1}$, we have a diffeomorphism

$$\bigcup_{0 \le t \le 1} \partial A_i(t) \cong \partial A_i(0) \times [0,1]$$

($\partial A_i(t)$ is the boundary of $A_i(t)$) compatible with the natural projections onto $[0,1]$. By the observations, we can extended it to a diffeomorphism

$$\bigcup_{0\le t\le 1} X_{\mu_i(t)} \backslash \mathrm{Int}\, A_i(t) \cong (X_0 \backslash \mathrm{Int}\, A_i(0)) \times [0,1] \tag{7.1}$$

($\mathrm{Int}\, A_i(t)$ is the interior of $A_i(t)$). Moreover, using $\{\iota_i^t\}_{0\le t\le 1}$ again we can extend (7.1) to a homeomorphism from $\bar{\pi}^{-1}(\mu_i) = \bigcup_{0\le t\le 1} X_{\mu_i(t)}$ to M_i also compatible with the projections onto $[0,1]$. Here $(X_0\backslash \mathrm{Int}\, A_i(0)) \times [0,1]$ is understood to be a subspace of M_i by an obvious manner.

The exsistence of the homeomorphism $\bar{\pi}^{-1}(\mu_i) \cong M_i$ implies that $C_i(0)$ is the vanishing cycle along μ_i. In summary, we have proved the following.

PROPOSITION 7.1. *The monodromy $\rho'(\lambda_i) \in \Gamma_3$ is the right hand Dehn twist along C_i.*

Now we can complete the proof of Theorem 1.1.

Proof of Theorem 1.1. We write L_0 instead $L = L(7/8, 3/4)$ and let L_1 be a line of $\mathbf{P}^3$ meeting $X^\vee$ transversely. Choose a base point $b_1 \in L_1\backslash(L_1 \cap X^\vee)$. Since the set of all lines of $\mathbf{P}^3$ meeting $X^\vee$ transversely is Zariski open hence path connected, there exist a continuous family $\{L(t)\}_{t\in[0,1]}$ of lines of $\mathbf{P}^3$ such that $L(t)$ meets $X^\vee$ transversely and $L(0) = L_0$, $L(1) = L_1$. Let $\mathscr{F}_t' := \{(x,H) \in \mathbf{P}_3 \times (L(t)\backslash(L(t)\cap X^\vee)); x \in H \cap X\}$. Then there exist continuous families of homeomorphisms $\{\psi_t : L_0\backslash(L_0\cap X^\vee) \to L(t)\backslash(L(t)\cap X^\vee)\}_{0\le t\le 1}$ and $\{\Psi_t : \mathscr{F}_0' \to \mathscr{F}_t'\}_{0\le t\le 1}$ such that $\pi_t' \circ \Psi_t = \psi_t \circ \pi_0'$ where π_t' is the second projection. Now $\{\psi_1(\lambda_i)\}_i$ is a standard generating system for $\pi_1(L_1\backslash(L_1\cap X^\vee), \psi_1(0))$ such that the image of $\psi_1(\lambda_i)$ under the associated topological monodromy is the right hand Dehn twist along C_i. The result follows by considering an isomorphism $\pi_1(L_1\backslash(L_1\cap X^\vee),\ \psi_1(0)) \cong \pi_1(L_1\backslash(L_1\cap X^\vee), b_1)$ induced by a path from $\psi_1(0)$ to b_1. □

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