

ON THE GLOBAL STRUCTURE OF HOPF HYPERSURFACES IN A COMPLEX SPACE FORM

A. A. BORISENKO

ABSTRACT. It is known that a tube over a Kähler submanifold in a complex space form is a Hopf hypersurface. In some sense the reverse statement is true: a connected compact generic immersed C^{2n-1} regular Hopf hypersurface in the complex projective space is a tube over an irreducible algebraic variety. In the complex hyperbolic space a connected compact generic immersed C^{2n-1} regular Hopf hypersurface is a geodesic hypersphere.

Introduction

A natural class of real hypersurfaces in a complex space form $\overline{M}(c)$ of constant holomorphic curvature $4c$ is the class of Hopf hypersurfaces. For a unit normal vector ξ of a hypersurface M the vector $J\xi$ is a tangent vector to M , where J is the complex structure of the complex space form $\overline{M}(c)$.

DEFINITION. A hypersurface $M \subset \overline{M}(c)$ is called a Hopf hypersurface if the vector $J\xi$ is a principal direction at every point of M .

Y. Maeda [11] proved that for Hopf hypersurfaces in the n -dimensional complex projective space $\mathbf{C}P^n$ the corresponding principal curvature in the direction $J\xi$ is constant. It is known that a tube over a Kähler submanifold in a complex projective space is a Hopf hypersurface. T.E. Cecil and P.J. Ryan studied the local and global structure of Hopf hypersurfaces with constant rank of the focal map Φ_r .

Let M be an embedded hypersurface of $\overline{M}(c)$ of the regularity class C^2 . Let NM be the normal bundle of M with projection $p : NM \rightarrow M$ and let BM be the unit normal bundle. For $\xi \in NM$ let $F(\xi)$ be the point in $\overline{M}(c)$ reached by traversing a distance $|\xi|$ along the geodesic in $\overline{M}(c)$ originating at $x = p(\xi)$ with the initial tangent vector ξ .

A point $P \in \overline{M}(c)$ is called a focal point of multiplicity $\nu > 0$ of (M, x) if $P = F(\xi)$ and the Jacobian of the map F has nullity ν at ξ .

Received October 18, 1999; received in final form July 25, 2000.

2000 *Mathematics Subject Classification*. Primary 53C40; Secondary 53C15.

DEFINITION. The tube of radius r over M is the image of the map $\Phi_r : BM \rightarrow \overline{M}(c)$ given by $\Phi_r(\xi) = F(r\xi)$, $\xi \in BM$.

T.E. Cecil and P.J. Ryan proved the following result.

LEMMA 1 [1]. *Let M be a connected, orientable Hopf hypersurface of $\mathbf{C}P^n$ with corresponding constant principal curvature $\mu = 2 \cot 2r$. Suppose the map Φ_r has constant rank q on M . Then q is even and every point $x_0 \in M$ has a neighborhood U such that $\Phi_r(U)$ is an embedded complex $q/2$ -dimensional submanifold of $\mathbf{C}P^n$.*

We remark that, in Lemma 1 and in Lemma 13 below, C^3 regularity is enough. From Lemmas 1 and 13 we obtain that a Hopf hypersurface with Φ_r of constant rank is an analytical hypersurface. It follows from this fact that $\Phi_r(U)$ is a complex submanifold and parametrization functions of $\Phi_r(U)$ satisfy an elliptic system of the PDE's with analytical coefficients. From C^2 regularity of $\Phi_r(U)$ we obtain that $\Phi_r(U)$ is analytic.

The global version of Lemma 1 has the following form [1] :

Let M be a connected compact embedded real Hopf hypersurface in $\mathbf{C}P^n$ with corresponding constant principal curvature $\mu = 2 \cot 2r$. Suppose the map Φ_r has constant rank q on M . Then Φ_r factors through a holomorphic immersion of the complex $q/2$ -dimensional manifold M/T_0 into $\mathbf{C}P^n$, where T_0 are $(2n - q - 1)$ -dimensional spheres, the leaves of the distribution

$$T_0(x) = \{y \in T_x M, (\Phi_r)_*(y) = 0\}.$$

Acknowledgements. The author began to study Hopf hypersurfaces when he was a visiting professor at the University of Valencia (Spain) from January 1997 to June 1997. The author would like to thank Vicente Miquel for his hospitality and for useful conversations. The author wishes to thank Vicente Miquel and Yury Nikolaevsky for closely reading a preliminary version of this paper and for their remarks. The author would also like to thank the referee for valuable remarks.

1. The main results

The following theorem gives a complete description of the global structure of Hopf hypersurfaces in complex space forms.

Let M be an immersed regular hypersurface in a regular manifold N . Suppose that for a point $P \in N$ of self-intersection the linear span of the tangent hyperplanes to the branches of M coincides with the tangent space $T_P N$ of the ambient manifold. This point is called a generic point of self-intersection. If every point of self-intersection of the hypersurface M is a generic point of self-intersection then the hypersurface M is called a generic immersed hypersurface.

THEOREM 1. *Let M be a C^{2n-1} regular compact generic immersed orientable Hopf hypersurface in the complex projective space $\mathbf{C}P^n$ ($n \geq 2$). Then M is a tube over an irreducible algebraic variety.*

COROLLARY. *Let M be a C^{2n-1} regular connected compact embedded Hopf hypersurface in the complex projective space $\mathbf{C}P^n$ ($n \geq 2$). Then M is a tube over an irreducible algebraic variety.*

The following are some standard examples of Hopf hypersurfaces in $\mathbf{C}P^n$ of constant holomorphic curvature 4.

1. A geodesic hypersphere M is the set of points at a fixed distance $r < \frac{\pi}{2}$ from a point $P \in \mathbf{C}P^n$. It is obvious that M is also the tube of radius $\frac{\pi}{2} - r$ over the hyperplane $\mathbf{C}P^{n-1} \subset \mathbf{C}P^n$ dual to the point P .

2. A tube over a totally geodesic space $\mathbf{C}P^k$ ($1 \leq k \leq n - 1$).

3. A tube over a totally geodesic real projective space RP^n and over a complex quadric $Q^{n-1} = \{(z_0, \dots, z_n) \in \mathbf{C}P^n : z_0^2 + z_1^2 + \dots + z_n^2 = 0\}$.

A tube of small radius r over a closed irreducible algebraic manifold in $\mathbf{C}P^n$ is an analytic Hopf hypersurface. But let $f = x_0^6 x_3^2 + x_1^3 x_2^5 = 0$ be the algebraic variety M in $\mathbf{C}P^3$. Then the point $P(1, 0, 0, 0)$ is a singular point (since $\text{grad } f/P = 0$). In any neighborhood of the point P the normal curvatures at smooth points vary from $-\infty$ to $+\infty$. From Lemma 12 below it follows that normal curvatures of the tube of any radius r tend to $+\infty$. It follows that the tube of any radius r has regularity less than $C^{1,1}$.

V. Miquel proved the following theorem:

THEOREM (V. MIQUEL, [13]). *Let M be a connected compact embedded Hopf hypersurface in $\mathbf{C}P^n$ contained in a geodesic ball of radius $R < \frac{\pi}{2}$. Suppose that*

- (1) M has constant mean curvature H .
- (2) The principal curvature μ in the direction $J\xi$ satisfies the inequality

$$\mu \geq 2 \cot \left(2 \arccot \cot \left[\frac{(2n-1)H - \mu}{2n-2} \right] \right).$$

Then M is a geodesic hypersphere.

We prove the following theorem.

THEOREM 2. *Let M be a C^{2n-1} regular connected compact generic immersed orientable Hopf hypersurface in the complex projective space $\mathbf{C}P^n$ ($n \geq 2$) contained in a geodesic ball of radius $R < \frac{\pi}{2}$. Then M is a geodesic hypersphere.*

Let $\mathbf{C}H^n$ be the complex hyperbolic space of constant holomorphic curvature -4 . We prove the following theorem.

THEOREM 3. *Let M be a connected compact generic immersed orientable C^{2n-1} regular Hopf hypersurface in the complex hyperbolic space \mathbf{CH}^n ($n \geq 2$). Then the Hopf hypersurface M is a geodesic hypersphere.*

2. Lemmas

LEMMA 2 (Y. MAEDA, [11]). *Let M be a connected Hopf hypersurface in the complex projective space \mathbf{CP}^n . Then the principal curvature μ of M in the direction $J\xi$ is constant.*

Let A_ξ be the shape operator of M .

LEMMA 3 (T.E. CECIL, P.J. RYAN [1]). *Suppose $J\xi$ is an eigenvector of A_ξ with an eigenvalue μ . Then we have:*

- (a) $(F_*)_{r\xi}(X, 0) = 0$ if $\lambda = \cot r$ is an eigenvalue of A_ξ and X is a vector in the eigenspace T_λ corresponding to the eigenvalue λ .
- (b) $(F_*)_{r\xi}(J\xi, 0) = 0$ if $\mu = 2 \cot 2r$.
- (c) $(F_*)_{r\xi}(X, V) \neq 0$ except as determined by (a) and (b).

Now, let M be a real hypersurface of a complex space form $\overline{M}^n(c)$ of constant holomorphic curvature $4c$ and let ξ be a unit normal field on M . If $X \in T_P M$, $P \in M$, then one has a decomposition

$$JX = \phi X + f(X)\xi$$

into the tangent and normal components, respectively. So, ϕ is a $(1, 1)$ -tensor field and f is a 1-form. These satisfy

$$\phi^2 X = -X + f(X)U, \quad \phi U = 0, \quad f(\phi X) = 0$$

for any vector field X tangent to M , where $U = -J\xi$. Moreover, we have

$$\begin{aligned} g(\phi X, Y) + g(X, \phi Y) &= 0, & f(X) &= g(X, U), \\ g(\phi X, \phi Y) &= g(X, Y) - f(X)f(Y) \end{aligned}$$

with g the metric tensor in $\overline{M}^n(c)$. We denote by A the shape operator on $T_P M$ associated with ξ .

LEMMA 4.

1. ([9]) *Let M be a Hopf hypersurface in $\overline{M}^n(c)$. Then we have:*

- (a) $-2c\phi = \mu(\phi A + A\phi) - 2A\phi A$,
- (b) $X\mu = (U\mu)f(X)$,

and

$$(U\mu)g((\phi A + A\phi)X, Y) = 0,$$

where μ is the principal curvature in the direction $U = -J\xi$, X, Y are vectors tangent to M , and $U\mu$ is the derivative of the function μ in the direction U .

Moreover, if $\phi A + A\phi = 0$ then

$$\begin{aligned} cg(X, \phi Y) &= -g(\phi AX, AY) = g(A\phi X, AY), \\ cg(\phi X, \phi X) &= -g(A\phi X, A\phi X) \end{aligned}$$

and so $c \leq 0$.

2. ([11]) Let M be a Hopf hypersurface in $\mathbf{C}P^n$. If $X \in T_\alpha \subset T_P M$, then

$$JX \in T_{\mu\alpha+2/2\alpha-\mu} \subset T_P M,$$

where T_α is an eigenspace corresponding to a principal curvature α .

It follows from equation (a) of the first part of the lemma that α cannot be equal to μ or $\mu/2$.

DEFINITION. Let A be a subset of a metric space X . Let $\delta(A)$ denote the diameter of A , and let

$$\begin{aligned} \delta^p(A) &= [\delta(A)]^p \text{ for } p > 0, \\ \delta^0(A) &= \begin{cases} 1 & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset. \end{cases} \end{aligned}$$

For $p \geq 0$ and $\varepsilon > 0$ define

$$\begin{aligned} H_\varepsilon^p(A) &= \inf \left\{ \sum_{i=1}^\infty \delta^p(A_n) : A \subset \cup A_n \text{ and } \delta(A_n) < \varepsilon \right\}, \\ H^p(A) &= \lim_{\varepsilon \rightarrow 0^+} H_\varepsilon^p(A) = \sup H_\varepsilon^p(A). \end{aligned}$$

We call H^p the Hausdorff p -measure.

LEMMA 5 (H. FEDERER, [4]). If $m > \nu \geq 0$ and $k \geq 1$ are integers, A is an open subset of R^m , $B \subset A$, Y is a normed vector space and $f : A \rightarrow Y$ is a map of class C^k such that

$$\text{Dim im } f_*(x) \leq \nu \text{ for } x \in B,$$

then

$$H^{\nu+(m-\nu)/k}[f(B)] = 0.$$

DEFINITION. Let Ω be a complex manifold. A set $A \subset \Omega$ is called an analytic set in Ω if for each point $a \in \Omega$ there exists a neighborhood U of a and functions f_1, \dots, f_N holomorphic in U such that $A \cap U = Z_{f_1} \cap \dots \cap Z_{f_k} \cap U$, where Z_f is the set of zeros of a holomorphic function f .

A point a of an analytic set A is called a regular point if there exists a neighborhood U of a in Ω such that $A \cap U$ is a complex submanifold of U . The complex dimension of $A \cap U$ is then called the dimension of A at the point a and is denoted by $\text{dim}_a A$. The set of all regular points of an analytic set is denoted by $\text{reg } A$. Its complement $A \setminus \text{reg } A$ is denoted by $\text{sng } A$. The

set $\text{sng } A$ is called the set of singular points of the set A . It can be shown by induction on the dimension of the manifold Ω that $\text{sng } A$ is nowhere dense and closed. This allows us to define the dimension of A at any point a of A as

$$\dim_a A = \lim_{z \rightarrow a} \dim_z A \quad (z \in \text{reg } A).$$

The set A is called purely p -dimensional if $\dim_z A = p$ for all $z \in A$ (see [2], [3]).

LEMMA 6 (B. SHIFFMAN, [16]). *Let E be a closed subset of a complex manifold Ω and let A be a purely q -dimensional analytic subset of $\Omega \setminus E$. If $H^{2q-1}(E) = 0$ then the closure \overline{A} of the set A in Ω is a purely q -dimensional analytic subset of Ω .*

DEFINITION (D. MUMFORD, [14]). Let $U \subset \mathbf{C}^n$ be an open set. A closed subset $X \subset U$ is a $*$ -analytic subset of U if X can be decomposed as

$$X = X^{(r)} \cup X^{(r-1)} \cup \dots \cup X^{(0)},$$

where for all i , $X^{(i)}$ is an i -dimensional complex submanifold of U and $\overline{X^{(i)}} \subset X^{(i)} \cup X^{(i-1)} \dots \cup X^{(0)}$. If $X^{(r)} \neq \emptyset$, then r is called the dimension of X .

An analytic set is always $*$ -analytic [14].

LEMMA 7 (CHOW'S THEOREM, [14]). *If $X \subset \mathbf{C}P^n$ is a closed $*$ -analytic subset, then X is a finite union of algebraic varieties.*

LEMMA 8 [3]. *An analytic set A in a complex manifold Σ is irreducible if and only if the set $\text{reg } A$ is connected.*

Let $X \subset \mathbf{C}P^n$ denote a closed irreducible algebraic variety of dimension l (which may have singularities), and let $X_e \subset X$ denote the (non-empty) open subset of its smooth points. (For the definitions of irreducible, singular and smooth points see [14].) Define

$$V'_X = \left\{ (x, y) \in \mathbf{C}P^n \times \mathbf{C}\check{P}^n \mid x \in X_e \text{ and } y \text{ is tangent hyperplane at } x \right\},$$

where $\mathbf{C}\check{P}^n$ is the dual complex projective space.

The closure V_X of V'_X in the Zariski topology on $\mathbf{C}P^n \times \mathbf{C}\check{P}^n$ is called the tangent hyperplane bundle of X . It is a closed irreducible algebraic variety of dimension $(n-1)$. The first projection maps V_X onto X :

$$\pi_1 : V_X \rightarrow X, \quad (x, y) \rightarrow x.$$

Consider now the second projection

$$\pi_2 : V_X \rightarrow \mathbf{C}\check{P}^n, \quad (x, y) \rightarrow y.$$

Its image $\check{X} = \pi_2(V_X)$ is a closed irreducible variety of $\mathbf{C}\check{P}^n$ of dimension at most $(n-1)$, the dual variety of X [9].

LEMMA 9 (DUALITY THEOREM) [6, 10]. *The tangent hyperplane bundles of a closed irreducible algebraic variety X and its dual variety \check{X} coincide: We have $V_{\check{X}} = V_X$ and hence $\check{\check{X}} = X$.*

Let $\mathbf{C}P^n$ be the complex projective space with a standard Fubini-Study metric. To a hyperplane $L \subset \mathbf{C}P^n$ passing through a point $x \in \mathbf{C}P^n$ we associate the point $y \in \mathbf{C}P^n$ representing the complex line in \mathbf{C}^{n+1} orthogonal to L . Then the distance $\rho(x, y)$ is equal to $\pi/2$. One can identify $\mathbf{C}\check{P}^n$ with $\mathbf{C}P^n$ in this way and consider \check{X} as a subset in $\mathbf{C}P^n$.

It is possible to define a tube over a closed irreducible algebraic variety $X \subset \mathbf{C}P^n$ which may have singularities. Let $(x, y) \in V_X \subset \mathbf{C}P^n \times \mathbf{C}\check{P}^n = \mathbf{C}P^n \times \mathbf{C}P^n$, $x \in X$, $y \in \check{X}$, and let $L(x, y)$ be a complex projective line through x , $y \in \mathbf{C}P^n$. Then $L(x, y)$ is a totally geodesic two-dimensional sphere in $\mathbf{C}P^n$ of curvature 4, the distance $\rho(x, y)$ is equal to $\pi/2$, and x and y are poles of the sphere $L(x, y)$. The set of points of $L(x, y)$ at a distance r from the point x is a circle $S_r(x, y)$ with the center x . The union

$$S_r = \bigcup_{(x,y) \in V_X} S_r(x, y)$$

is called the tube of radius r over X . The set S_r is also the tube of radius $\frac{\pi}{2} - r$ over the dual variety \check{X} .

If all the points of X are regular, this definition coincides with one given above.

The set of points $\text{sng } V_X \subset V_X$ such that $(x, y) \in \text{sng } V_X$ if $x \in \text{sng } X$ or $y \in \text{sng } \check{X}$ is a closed algebraic subvariety of V_X , $\text{reg } V_X = V_X \setminus \text{sng } V_X$ is an open set of V_X in the Zariski topology.

Let $X \subset \mathbf{C}P^n$ be a closed irreducible algebraic variety and let x_0 be a Zariski open set in X . Then the closure of x_0 in the classical topology is X [14].

Consider the Segre map

$$\sigma : \mathbf{C}P^n \times \mathbf{C}\check{P}^n \rightarrow \mathbf{C}P^{(n+1)^2-1}.$$

Then $\sigma(V_X)$ is a closed irreducible algebraic variety in $\mathbf{C}P^{(n+1)^2-1}$, and the set $\text{reg } V_X$ is an open set of V_X in the Zariski topology.

As corollary we obtain the following result.

LEMMA 10. *The closure of the set $\text{reg } V_X \subset \mathbf{C}P^n \times \mathbf{C}P^n$ in the standard topology coincides with the tangent bundle V_X .*

Therefore the tube over X is the closure of the set

$$\bigcup_{(x,y) \in \text{Reg } V_X} S_r(x, y).$$

LEMMA 11 [5]. *Let X be a compact topological space. Suppose A is a closed subset such that $X \setminus A$ is a smooth n -dimensional orientable manifold without boundary. Then*

$$H_q(X, A) \simeq H^{n-q}(X \setminus A),$$

where H_i and H^i are homology and cohomology groups.

LEMMA 12 [1]. *Suppose $J\xi$ is an eigenvector of the shape operator A_ξ of a Hopf hypersurface M in the complex projective space, with the corresponding eigenvalue $2 \cot 2\Theta$, $0 < \Theta < \frac{\pi}{2}$. Suppose $J\xi, X_2, \dots, X_n$ is a basis of principal vectors of A_ξ with $A_\xi X_j = \cot \Theta_j X_j$, $2 \leq j \leq n$, $0 < \Theta_j < \pi$; $\frac{\partial}{\partial t_j}$ ($2 \leq j \leq k$) are normal vectors. Then the shape operator A_r of the tube Φ_r is given in terms of its principal vectors by*

- (a) $A_r \left(\frac{\partial}{\partial t_j} \right) = -\cot r \left(\frac{\partial}{\partial t_j} \right)$, $2 \leq j \leq k$;
- (b) $A_r(X_j, 0) = \cot(\Theta_j - r)(X_j, 0)$, $2 \leq j \leq n$;
- (c) $A_r(J\xi, 0) = \cot(2(\Theta - r))(J\xi, 0)$.

For a complex hyperbolic space \mathbf{CH}^n the following analog of Lemma 1 holds:

LEMMA 13 [13]. *Let M be an orientable Hopf hypersurface of \mathbf{CH}^n such that the principal curvature μ in the direction $J\xi$ is constant and equal to $\mu = 2 \coth 2r$. Suppose that Φ_r has constant rank q on M . Then for every point $x_0 \in M$ there exists an open neighborhood U of x_0 such that $\Phi_r U$ is a $q/2$ -dimensional complex submanifold embedded in \mathbf{CH}^n .*

LEMMA 14 [15]. *Let Ω be a Hermitian complex manifold with exact fundamental form $\omega = d\gamma$. Let A be an analytical q -dimensional set with boundary $\partial A \subset \Omega$ such that $A \cup \partial A$ is compact. Then*

$$H^{2q}(A) \leq \frac{1}{q} (\max_{\partial A} |\gamma|) H^{2q-1}(\partial A),$$

where $H^{2q}(A)$, $H^{2q-1}(\partial A)$ are Hausdorff measures, and

$$|\gamma|(z) = \max \{ |\gamma(v)| : v \in T_z \Omega, |v| = 1 \}.$$

LEMMA 15 [8]. *Let M be a Hopf hypersurface of a complex space form $\overline{M}^n(c)$ ($c \neq 0$). If U is an eigenvector of A , then the principal curvature $\mu = g(AU, U)$ is constant.*

3. Proofs of the theorems

Let M_s be the set of points of M such that $\text{rank}(\Phi_r)_*(M_s) = s$, $F_s = \Phi_r(M_s)$, $F = \Phi_r(M)$. From Lemma 4 we obtain that if $X \in T_\alpha \subset T_P M$, where T_α is the eigenspace corresponding to the principal curvature $\alpha = \cot r$, then $JX \in T_\alpha$. Hence s is even and if $s < 2q$ then $s \leq 2q - 2$.

Let

$$E = \bigcup_{s < 2q} F_s \cup F_0,$$

$$F_0 = \left\{ x \in F : x = \Phi_r(L_1) = \Phi_r(L_2), L_1 \neq L_2 \subset M, \right.$$

$$\left. \text{rank}(\Phi_r)_*(P_1) = \text{rank}(\Phi_r)_*(P_2) = 2q \right\},$$

for $P_i \in L_i$, where the L_i are leaves of the distribution $\text{Ker}(\Phi_r)_*$.

Proof of Theorem 1. Let M be a compact Hopf hypersurface in $\mathbf{C}P^n$. This means that the vector $J\xi$ is a principal direction of M , where ξ is the unit normal vector and J is the complex structure in $\mathbf{C}P^n$. From Lemma 2 it follows that the corresponding principal curvature μ is constant and $\mu = 2 \cot 2r$. Let $2q$ be the maximal rank of $(\Phi_r)_*$ on M . Let $P \in M$ be a point such that $\text{rank}(\Phi_r)_*(P) = 2q$ and let M_{2q} be the corresponding connected component of M such that $P \in M_{2q}$ and for $Q \in M_{2q}$, $\text{rank}(\Phi_r)_*(Q) = 2q$. Set $F_{2q} = \Phi_r(M_{2q})$, $\tilde{F} = F_{2q} \cap (\mathbf{C}P^n \setminus E)$. From Lemma 1 we obtain that \tilde{F} is a purely analytic set, $\dim_z \tilde{F} = q$, $z \in \tilde{F}$.

Locally, F_0 is a transversal intersection of two complex submanifolds of dimension q . Hence F_0 is an analytic set of real dimension $\leq 2q - 2$ and Hausdorff measure

$$H^{2q-1}(F_0) = 0.$$

Now apply Lemma 5 to the set $E_1 = \bigcup_{s < 2q} F_s$ and the map Φ_r . Then $\nu \leq 2q - 2$.

If the class of regularity of M is greater or equal to $2(n - q + 1)$, then the class of regularity of Φ_r is $k \geq 2(n - q + 1) - 1$ and

$$\nu + \frac{2n - 1 - \nu}{k} \leq 2q - 2 + \frac{2n - 1}{k} \leq 2q - 1$$

for $k \geq 2n - 1$. From Lemma 5 we have $H^{2q-1}(E_1) = 0$ and so $H^{2q-1}(E) = 0$. From Lemma 6 we obtain that the closure of \tilde{F} is a purely q -dimensional analytic subset of $\mathbf{C}P^n$. Since any analytic subset is $*$ -analytic we obtain from Chow's Theorem (Lemma 7) that $\text{cl } \tilde{F} \subset \mathbf{C}P^n$ is a finite union of algebraic varieties. An analytic set A is irreducible if and only if the set $\text{reg } A$ is connected. From Lemma 8 it follows that $\text{cl } \tilde{F}$ is irreducible as analytic set and we obtain that $\text{cl } \tilde{F} = X$ is an irreducible algebraic variety.

Let S_r be a tube over $X = \text{cl } \tilde{F}$. By Lemma 10 we have $S_r \subset M$ and $S_r = \text{cl } M_{2q}$. We will prove that $\text{cl } M_{2q} = M$. Suppose that $\text{cl } M_{2q} \neq M$. Then in every neighborhood of a point $P \in \partial M_{2q}$ there exist points $Q \in M \setminus \text{cl } M_{2q}$. Let $P \in \partial M_{2q}$. Then $P \in S_r(x, y)$ such that $x \in \text{sng } X$, $y \in \text{sng } \tilde{X}$. Then

$$\partial M_{2q} = \bigcup_{x \in \text{sng } X, y \in \text{sng } \tilde{X}} S_r(x, y).$$

Otherwise some neighborhood of P belongs to $\text{cl } M_{2q}$ and $P \in \text{int cl } M_{2q}$. The set of points

$$\text{sng}(X, \check{X}) = \text{sng } X \times \mathbf{C}P^n \cap \mathbf{C}P^n \times \text{sng } \check{X} \subset V_X \subset \mathbf{C}P^n \times \mathbf{C}P^n$$

is a closed algebraic subvariety of V_X . The dimension of $\text{sng}(X, \check{X})$ is $\leq n - 2$ because the dimension of V_X is equal to $n - 1$. The set ∂M_{2q} is a fiber bundle over $\text{sng}(X, \check{X})$ with the circle S^1 as a leaf. The real dimension of $\text{sng}(X, \check{X})$ is $\leq 2(n - 2)$, whence

$$H_{2n-3}(\text{sng}(X, \check{X}), \mathbf{Z}) = 0.$$

For $E = \partial M_{2q}$, $B = \text{sng}(X, \check{X})$, $F = S^1$ the exact Thom-Gysin sequence has the form [17]

$$\begin{aligned} H_{2n-1}(\text{sng}(X, \check{X}), \mathbf{Z}) &\rightarrow H_{2n-3}(\text{sng}(X, \check{X}), \mathbf{Z}) \rightarrow \\ &\rightarrow H_{2n-2}(\partial M_{2q}, \mathbf{Z}) \rightarrow H_{2n-2}(\text{sng}(X, \check{X}), \mathbf{Z}), \\ 0 &\rightarrow 0 \rightarrow H_{2n-2}(\partial M_{2q}, \mathbf{Z}) \rightarrow 0. \end{aligned}$$

We obtain

$$H_{2n-2}(\partial M_{2q}, \mathbf{Z}) = 0.$$

Next, we apply Lemma 11 with $X = M$, $A = \partial M_{2q}$. Then

$$H_{2n-1}(M, \partial M_{2q}) = H^0(M \setminus \partial M_{2q}).$$

But $M \setminus \partial M_{2q}$ has $m > 1$ connected components and

$$H^0(M \setminus \partial M_{2q}, \mathbf{Z}) = \bigoplus_{i=1}^m \mathbf{Z}$$

is the direct sum of m copies of \mathbf{Z} [17].

For the pair $(M, \partial M_{2q})$ the exact homology sequence has the form

$$\begin{aligned} H_{2n-1}(\partial M_{2q}, \mathbf{Z}) &\rightarrow H_{2n-1}(M, \mathbf{Z}) \rightarrow H_{2n-1}(M, \partial M_{2q}, \mathbf{Z}) \rightarrow \\ &\rightarrow H_{2n-2}(\partial M_{2q}, \mathbf{Z}), \end{aligned}$$

$$H_{2n-1}(\partial M_{2q}, \mathbf{Z}) = H_{2n-2}(\partial M_{2q}, \mathbf{Z}) = 0; \quad H_{2n-1}(M, \mathbf{Z}) = \mathbf{Z}.$$

It follows that $H_{2n-1}(M, \partial M_{2q}, \mathbf{Z}) = \mathbf{Z}$, contradicting the above result. Thus $\text{cl } M_{2q} = M$ and M is a tube over the irreducible algebraic variety $\text{cl } \tilde{F} = X$. □

Proof of Theorem 2. Let S be the hypersphere of minimal radius r_0 such that the hypersurface M is contained in the ball D with boundary $\partial D = S$. Let P be a point of tangency of M and S . Let ξ be the inward unit normal vector at the point P . Then the principal curvature in the direction $J\xi$ is $\mu = 2 \cot 2\rho \geq 2 \cot 2r_0$, and so $\rho \leq r_0 < \pi/2$. Another principal curvature $k_i = \cot \Theta_i$ at the point P satisfies $\cot \Theta_i \geq \cot r_0$, where $2 \cot 2r_0, \cot r_0$

are principal curvatures of the hypersphere S . Then $\Theta_i \leq r_0$. Let $r = \rho - \pi/2$. From Lemma 12 we obtain that the principal curvatures of the tube Φ_r over M are equal to

$$(k_i)_r = \operatorname{tg}(\rho - \Theta_i) \leq \operatorname{tg}(r_0 - \Theta_i) < \infty.$$

Hence $\operatorname{rank}(\Phi_r)_*(P) = 2(n - 1)$, and from Theorem 1 we get that $\Phi_r(M) = \operatorname{cl} \tilde{F} = X$ is an irreducible hypersurface of degree d . Let X_k be a sequence of smooth algebraic hypersurfaces such that $\lim X_k = X$, $\operatorname{degree} X_k = d$ [7], and let \check{X} and \check{X}_k be dual algebraic varieties. Then

$$M = \Phi_{\frac{\pi}{2}-r}(X) = \Phi_r(\check{X}),$$

and from Lemma 9 we obtain $\check{X} = \lim \check{X}_k$. From the above argument we have for $\Phi_{\frac{\pi}{2}-r}(X_k) = M_k$,

$$\lim M_k = M.$$

For large k , M_k is contained in the balls D_k of radius $R < \pi/2$, and M_k does not intersect complex projective space $x_0 = 0$.

Let $f = 0$ be the equation of the algebraic hypersurface X_n where f is a homogeneous polynomial, $\operatorname{grad} f \neq 0$. By Bezout's Theorem [15] the system of equations

$$x_0 = 0, \quad f = 0, \quad f_{x_0} = 0$$

has a nontrivial solution if $n \geq 3$ and the degree of the polynomial f is ≥ 2 . This means that M_k intersects the hyperplane $x_0 = 0$. It follows that f is a linear function and the X_k are all hyperplanes, and the M_k are hyperspheres. Then the hypersurface M is a geodesic hypersphere, too.

For $n = 2$ the equation of the tube has the parametric form

$$z_j = x_j \cos r + \sin r \frac{\frac{\partial f}{\partial x_j}}{|\operatorname{grad} f|} e^{it},$$

where the x_j are coordinates of points of the algebraic variety, $0 \leq t \leq 2\pi$, $0 \leq r \leq \frac{\pi}{2}$, and r is radius of the tube Φ_r ; $j = 0, 1, 2$.

From the real point of view X is a compact two-dimensional manifold.

Let

$$g_1 = |x_0 \cos r|, \quad g_2 = \left| \frac{\frac{\partial f}{\partial x_0}}{|\operatorname{grad} f|} \sin r \right|.$$

If the degree of the polynomial f is ≥ 2 , the zero sets of these regular functions on the manifold X are non empty on the manifold X . Hence there exists a point $P \in X$ such that $g_1 = g_2 = \rho$. Then $z_0 = \rho (e^{i\alpha} + e^{i(\beta+t)})$. Moreover, if $t = \alpha - \beta - \pi$ then $z_0 = 0$. This means that M_k intersects the hyperplane $x_0 = 0$. Thus f is a linear function and M_k and M are geodesic hyperspheres as in the case $n \geq 3$. □

Proof of Theorem 3. Let S be the hypersphere of minimal radius r_0 such that the hypersurface M is contained in the ball D with boundary S . Let P_0 be a point of tangency of M and S . Let ξ be the inward unit normal vector of M at the point P_0 . From Lemma 15 it follows that the principal curvature μ in the direction $J\xi$ is constant. At the point P_0 this curvature satisfies the inequality $\mu \geq 2 \coth 2r_0$ and $\mu = 2 \coth 2r$. We now follow the proof of Theorem 1, using Lemma 13 instead of Lemma 1. Consider the map Φ_r . For a Hopf hypersurface, $\text{rank}(\Phi_r)_*$ is always even. This follows from Lemma 4.

Suppose $2q$ is the maximal rank of $(\Phi_r)_*$ at the points of M . Let $P \in M$ be a point such that $\text{rank}(\Phi_r)_*(P) = 2q$ and M_{2q} is the connected component of M such that for $Q \in M_{2q}$, $\text{rank}(\Phi_r)_*(Q) = 2q$. As in the proof of Theorem 1, set

$$F = \Phi_r(M), \quad F_{2q} = \Phi_r(M_{2q}), \quad F_s = \Phi_r(M_s),$$

$$E = F_0 \bigcup_{s < 2q} F_s, \quad \tilde{F} = F_{2q} \cap \mathbf{C}H^n \setminus E.$$

We obtain that $\text{cl}\tilde{F} = X$ is a compact analytic set in $\mathbf{C}H^n$ with boundary $\partial X \subset E$. The Hausdorff measure $H^{2q-1}(\partial X)$ is equal to 0. From Lemma 14 it follows that $H^{2q}(X) = 0$. This is possible only if $q = 0$ and X is a point. Thus M is a tube over a point and M is a geodesic hypersphere. \square

REFERENCES

- [1] T.E. Cecil and P.J. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc. **269** (1982), 481–499.
- [2] E.M. Chirka, *Complex analytic sets*, Nauka, Moscow, 1985.
- [3] ———, *Complex analytic sets*, Encyclopaedia of Mathematical Sciences. Several complex variables I, A.G. Vitushkin (ed), Springer–Verlag, New York, 1990.
- [4] H. Federer, *Geometric measure theory*, Springer–Verlag, New York, 1969.
- [5] A.T. Fomenko and D.B. Fuchs, *Homotopic topology*, Nauka, Moscow, 1989.
- [6] J. Harris, *Algebraic geometry*, Springer–Verlag, New York, 1996.
- [7] W.V.D. Hodge and D.Pedoe, *Methods of algebraic geometry*, vol. 1, Cambridge, 1947.
- [8] U-Hang Ki and Young Jin Suh, *On real hypersurfaces of a complex space form*, Math. J. Okayama Univ. **32** (1990), 207–221.
- [9] M. Kon, *Pseudo–Einstein real hypersurfaces in complex space forms*, J. Diff. Geometry **14** (1979), 339–354.
- [10] K. Lamotke, *The topology of complex projective varieties after S. Lefschetz*, Topology **20** (1981), 15–51.
- [11] Y. Maeda, *On real hypersurfaces of a complex projective space*, J. Math. Soc. Japan. **28** (1976), 529–540.
- [12] V. Miquel, *Compact Hopf hypersurfaces of constant mean curvature in complex space forms*, Ann. Global Anal. Geom. **12** (1994), 211–218.
- [13] S. Montiel, *Real hypersurfaces of a complex hyperbolic space*, J. Math. Soc. Japan **37** (1985), 515–535.
- [14] D. Mumford, *Algebraic geometry. I*, Grundlehren math. Wiss., vol. 221, Springer–Verlag, New York, 1976.
- [15] I.R. Shafarevich, *Basic algebraic geometry*, Springer–Verlag, New York, 1977.

- [16] B. Shiffman, *On the removal of singularities of analytic sets*, Michigan Math. J. **15** (1968), 111–120.
- [17] E. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.

DEPARTMENT OF MATHEMATICS, KHARKOV STATE UNIVERSITY, SVOBODY SQ. 4, KHARKOV
31007, UKRAINE

E-mail address: Alexander.A.Borisenko@univer.kharkov.ua