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ON THE GLOBAL UNIQUENESS FOR THE  
EINSTEIN-MAXWELL-SCALAR FIELD SYSTEM WITH A  
COSMOLOGICAL CONSTANT

PART 3. MASS INFLATION AND EXTENDIBILITY OF THE  
SOLUTIONS

JOÃO L. COSTA, PEDRO M. GIRÃO, JOSÉ NATÁRIO,  
AND JORGE DRUMOND SILVA

**Abstract**

This paper is the third part of a trilogy dedicated to the following problem: given spherically symmetric characteristic initial data for the Einstein-Maxwell-scalar field system with a cosmological constant  $\Lambda$ , with the data on the outgoing initial null hypersurface given by a subextremal Reissner-Nordström black hole event horizon, study the future extendibility of the corresponding maximal globally hyperbolic development as a “suitably regular” Lorentzian manifold.

In the first part [2] of this series we established the well posedness of the characteristic problem, whereas in the second part [3] we studied the stability of the radius function at the Cauchy horizon.

In this third and final paper we show that, depending on the decay rate of the initial data, mass inflation may or may not occur. When the mass is controlled, it is possible to obtain continuous extensions of the metric across the Cauchy horizon with square integrable Christoffel symbols. Under slightly stronger conditions, we can bound the gradient of the scalar field. This allows the construction of (non-isometric) extensions of the maximal development which are classical solutions of the Einstein equations. Our results provide evidence against the validity of the strong cosmic censorship conjecture when  $\Lambda > 0$ .

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*Key words and phrases*. Einstein equations, black holes, strong cosmic censorship, Cauchy horizon, scalar field, spherical symmetry.

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## CONTENTS

1. Introduction	2
2. Framework and some results from Parts 1 and 2	5
3. Mass inflation	10
4. No mass inflation	11
5. Extensions of the metric beyond the Cauchy horizon	17
6. Extensions of solutions beyond the Cauchy horizon	28
Appendix A. On the choice of the parameters and its consequences	33
Appendix B. Proof of Theorem 3.1	36
Appendix C. Some useful formulas	39
References	42

## 1. INTRODUCTION

This paper is the third part of a trilogy dedicated to the following problem: given spherically symmetric characteristic initial data for the Einstein-Maxwell-scalar field system with a cosmological constant  $\Lambda$ , with the data on the outgoing initial null hypersurface given by a subextremal Reissner-Nordström black hole event horizon, and the remaining data otherwise free, study the future extendibility of the corresponding maximal globally hyperbolic development as a “suitably regular” Lorentzian manifold.

We are motivated by the strong cosmic censorship conjecture and the question of determinism in general relativity. More precisely, the existence of (non-isometric) extensions of the maximal globally hyperbolic development leads to the breakdown of global uniqueness for the Einstein equations. If this phenomenon persists for generic initial conditions then it violates the strong cosmic censorship conjecture. See the Introduction of Part 1 for a more detailed account of the mathematical physics context of this work.

In Part 1, we showed the equivalence (under appropriate regularity conditions for the initial data) between the Einstein equations (2)–(6) and the system of first order PDE (15)–(24). We established existence, uniqueness and identified a breakdown criterion for solutions of this system. In Part 2, we analyzed the properties of the solution up to the Cauchy horizon,<sup>\*</sup> proving, in particular, the stability of the radius function. See Section 2 for a summary of our previous results, as well as for the definitions and notation that will be used henceforth.

In this paper we examine the behavior of the renormalized Hawking mass  $\varpi$  (see (9)) and the scalar field at the Cauchy horizon. Depending on the control that we have on these quantities, we are able to construct extensions of the metric beyond the Cauchy horizon with different degrees of regularity. The quotient

$$\rho := \frac{k_-}{k_+} > 1,$$

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<sup>\*</sup>We recall that when it is possible to isometrically embed the maximal globally hyperbolic development into a larger spacetime, the boundary of the maximal globally hyperbolic development is known as the Cauchy horizon.

of the surface gravities (see (37)) of the Cauchy and event horizons  $r = r_-$  and  $r = r_+$  in the reference Reissner-Nordström black hole plays an important role in our analysis.

We start by briefly recalling the strategy of Dafermos [4, 5] to establish mass inflation (that is, blow-up of  $\varpi$  at the Cauchy horizon), which naturally generalizes to the case of a non-vanishing cosmological constant. This requires the initial field  $\zeta_0$  (see (12) and (25)) to satisfy

$$\zeta_0(u) \geq cu^s \text{ for some } c > 0 \text{ and } 0 < s < \frac{\rho}{2} - 1$$

(Theorem 3.1 whose proof is given in Appendix B). Since the mass is a scalar invariant involving first derivatives of the metric, its blow up excludes the existence of spherically symmetric  $C^1$  extensions.<sup>†</sup> Moreover, using the techniques of [6], one can easily conclude that in this case the Christodoulou-Chruściel inextendibility criterion holds, that is, there is no extension of the metric beyond the Cauchy horizon with Christoffel symbols in  $L^2_{\text{loc}}$ .

The previous approach only allows us to explore a particular subregion of parameter space, corresponding to sufficiently subextremal reference solutions (see the figure below). We proceed by extending the analysis to the full parameter range. First we prove that if the field  $\zeta_0$  satisfies the weaker hypothesis

$$\zeta_0(u) \geq cu^s \text{ for some } c > 0 \text{ and } 0 < s < \rho - 1,$$

then either the renormalized mass  $\varpi$  or the field  $|\frac{\theta}{\lambda}|$  (see (8) and (11)) blow up at the Cauchy horizon (Theorem 3.2). As a consequence, the Kretschmann curvature scalar also blows up (Remark 6.9).

On the other hand, when the initial field  $\zeta_0$  satisfies

$$|\zeta_0(u)| \leq cu^s \text{ for some } c > 0 \text{ and } s > \frac{7\rho}{9} - 1 > 0,$$

we show that the mass remains bounded (Theorem 4.1). This behavior is in contrast with the standard picture of spherically symmetric gravitational collapse.

The case where no mass inflation occurs is then analyzed in further detail. We construct  $C^0$  spherically symmetric extensions of the metric beyond the Cauchy horizon with the second mixed derivatives of  $r$  continuous. There are two natural coordinate choices for the extension, corresponding to either  $\lambda = -1$  or  $\kappa = 1$  (see (8) and (13)) on the outgoing null ray  $u = U$ . Interestingly, these lead to inequivalent  $C^2$  structures for the extended manifolds, a fact that is reflected on the behavior of the Christoffel symbols: when the initial data satisfies

$$c_2 u^{s_2} \leq \zeta_0(u) \leq c_1 u^{s_1} \text{ for some } \frac{7\rho}{9} - 1 < s_1 \leq s_2 < \rho - 1,$$

we prove that one of the Christoffel symbols blows up on  $u = U$  at the Cauchy horizon in the  $\lambda = -1$  coordinates, but not in the  $\kappa = 1$  coordinates. Moreover, for both coordinate systems this Christoffel symbol blows up along almost all outgoing null rays, which excludes the existence of  $C^{0,1}$

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<sup>†</sup>In this paper we will only be concerned with spherically symmetric extensions.

extensions of the metric. Nonetheless, in the  $\kappa = 1$  coordinates the Christoffel symbols are in  $L^2_{\text{loc}}$  and the field  $\phi$  is in  $H^1_{\text{loc}}$  for the whole range of initial data where there is no mass inflation (Corollary 5.11). Therefore, the Christodoulou-Chruściel inextendibility criterion for strong cosmic censorship does not hold in this setting.

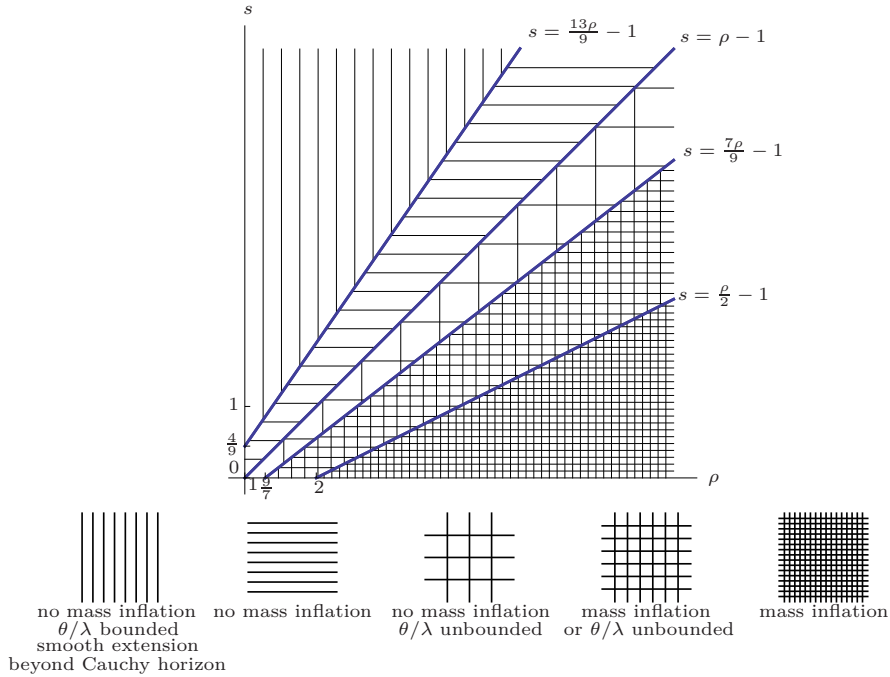
Finally, assuming that

$$|\zeta_0(u)| \leq cu^s \text{ for some } s > \frac{13\rho}{9} - 1,$$

we can bound the field  $\frac{\theta}{\lambda}$  at the Cauchy horizon. This allows us to prove that the solution of the first order system extends, non-uniquely, to a classical solution beyond the Cauchy horizon (Theorem 6.5). We then show that this solution corresponds to a classical solution of the Einstein equations extending beyond the Cauchy horizon (Theorem 6.7). The metric for this solution is  $C^1$  and such that  $r \in C^2$  and  $\partial_u \partial_v \Omega$  (see (1)) exists and is continuous (Remark 6.8). However, we emphasize that the metric does not have to be  $C^2$ , in spite of the Kretschmann curvature scalar being bounded. To the best of our knowledge, these are the first results where the generic existence of extensions as solutions of the Einstein equations is established.

It should be noted that these results, while valid for all signs of the cosmological constant  $\Lambda$ , only provide evidence against the strong cosmic censorship conjecture in the case  $\Lambda > 0$  (see the discussion in the Introduction of Part 1).

In summary, for a given  $\rho$  and  $cu^s \leq \zeta_0(u) \leq Cu^s$ , the behavior of the solution at the Cauchy horizon depends on the value of  $s$  as described in the following figure.



In Appendix A we explain how  $\rho$  depends on the physical parameters  $r_+$ ,  $r_-$  and  $\Lambda$ . In particular,  $\rho$  is a function of  $\frac{r_+}{r_-}$  and  $\frac{\Lambda r_-^2}{3}$ .

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## 2. FRAMEWORK AND SOME RESULTS FROM PARTS 1 AND 2

**The spherically symmetric Einstein-Maxwell-scalar field system with a cosmological constant.** Consider a spherically symmetric space-time with metric

$$g = -\Omega^2(u, v) dudv + r^2(u, v) \sigma_{\mathbb{S}^2}, \quad (1)$$

where  $\sigma_{\mathbb{S}^2}$  is the round metric on the 2-sphere. The Einstein-Maxwell-scalar field system with a cosmological constant  $\Lambda$  and total electric charge  $4\pi e$  reduces to the following system of equations: the wave equation for  $r$ ,

$$\partial_u \partial_v r = \frac{\Omega^2}{2} \frac{1}{r^2} \left( \frac{e^2}{r} + \frac{\Lambda}{3} r^3 - \varpi \right), \quad (2)$$

the wave equation for the massless scalar field  $\phi$ ,

$$\partial_u \partial_v \phi = - \frac{\partial_{ur} \partial_v \phi + \partial_{vr} \partial_u \phi}{r}, \quad (3)$$

the Raychaudhuri equation in the  $u$  direction,

$$\partial_u \left( \frac{\partial_{ur}}{\Omega^2} \right) = -r \frac{(\partial_u \phi)^2}{\Omega^2}, \quad (4)$$

the Raychaudhuri equation in the  $v$  direction,

$$\partial_v \left( \frac{\partial_{vr}}{\Omega^2} \right) = -r \frac{(\partial_v \phi)^2}{\Omega^2}, \quad (5)$$

and the wave equation for  $\ln \Omega$ ,

$$\partial_v \partial_u \ln \Omega = -\partial_u \phi \partial_v \phi - \frac{\Omega^2 e^2}{2r^4} + \frac{\Omega^2}{4r^2} + \frac{\partial_{ur} \partial_{vr}}{r^2}. \quad (6)$$

**The first order system.** Given  $r$ ,  $\phi$  and  $\Omega$ , solutions of the Einstein equations, let

$$\nu := \partial_u r, \quad (7)$$

$$\lambda := \partial_v r, \quad (8)$$

$$\varpi := \frac{e^2}{2r} + \frac{r}{2} - \frac{\Lambda}{6} r^3 + \frac{2r}{\Omega^2} \nu \lambda, \quad (9)$$

$$\mu := \frac{2\varpi}{r} - \frac{e^2}{r^2} + \frac{\Lambda}{3} r^2, \quad (10)$$

$$\theta := r \partial_v \phi, \quad (11)$$

$$\zeta := r \partial_u \phi \quad (12)$$

and

$$\kappa := \frac{\lambda}{1 - \mu}. \quad (13)$$

Notice that we may rewrite (9) as

$$\Omega^2 = - \frac{4\nu\lambda}{1 - \mu} = -4\nu\kappa. \quad (14)$$

The Einstein equations imply the first order system for  $(r, \nu, \lambda, \varpi, \theta, \zeta, \kappa)$

$$\partial_u r = \nu, \quad (15)$$

$$\partial_v r = \lambda, \quad (16)$$

$$\partial_u \lambda = \nu \kappa \partial_r (1 - \mu), \quad (17)$$

$$\partial_v \nu = \nu \kappa \partial_r (1 - \mu), \quad (18)$$

$$\partial_u \varpi = \frac{1}{2} (1 - \mu) \left( \frac{\zeta}{\nu} \right)^2 \nu, \quad (19)$$

$$\partial_v \varpi = \frac{1}{2} \frac{\theta^2}{\kappa}, \quad (20)$$

$$\partial_u \theta = -\frac{\zeta \lambda}{r}, \quad (21)$$

$$\partial_v \zeta = -\frac{\theta \nu}{r}, \quad (22)$$

$$\partial_u \kappa = \kappa \nu \frac{1}{r} \left( \frac{\zeta}{\nu} \right)^2, \quad (23)$$

with the restriction

$$\lambda = \kappa (1 - \mu). \quad (24)$$

Under appropriate regularity conditions for the initial data, the system of first order PDE (15)–(24) also implies the Einstein equations.

**Reissner-Nordström initial data.** We take the initial data on the  $v$  axis to be the data on the event horizon of a subextremal Reissner-Nordström solution with mass  $M$ . So, we choose initial data as follows:

$$\begin{cases} r(u, 0) = r_0(u) = r_+ - u, \\ \nu(u, 0) = \nu_0(u) = -1, \\ \zeta(u, 0) = \zeta_0(u), \end{cases} \quad \text{for } u \in [0, U], \quad (25)$$

and

$$\begin{cases} \lambda(0, v) = \lambda_0(v) = 0, \\ \varpi(0, v) = \varpi_0(v) = M, \\ \theta(0, v) = \theta_0(v) = 0, \\ \kappa(0, v) = \kappa_0(v) = 1, \end{cases} \quad \text{for } v \in [0, \infty[. \quad (26)$$

Here  $r_+ > 0$  is the radius of the event horizon. We assume that  $\zeta_0$  is continuous and  $\zeta_0(0) = 0$ . We will denote  $M$  by  $\varpi_0$ .

**Lemma 2.1.** *Suppose that  $(r, \nu, \lambda, \varpi, \theta, \zeta, \kappa)$  is a solution of the characteristic initial value problem (15)–(24), with initial conditions (25) and (26). Then:*

- $\kappa$  is positive;
- $\nu$  is negative;
- $\lambda$  is negative on  $\mathcal{P} \setminus \{0\} \times [0, \infty[$ ;
- $1 - \mu$  is negative on  $\mathcal{P} \setminus \{0\} \times [0, \infty[$ ;
- $r$  is decreasing with both  $u$  and  $v$ ;
- $\varpi$  is nondecreasing with both  $u$  and  $v$ .

**Well posedness of the first order system and stability of the radius at the Cauchy horizon.**

**Theorem 2.2.** *Consider the characteristic initial value problem (15)–(24) with initial data (25)–(26). Assume  $\zeta_0$  is continuous and  $\zeta_0(0) = 0$ . Then, the problem has a unique solution defined on a maximal past set  $\mathcal{P}$ . Moreover, there exists  $U > 0$  such that  $\mathcal{P}$  contains  $[0, U] \times [0, \infty[$ ,*

$$\inf_{[0, U] \times [0, \infty[} r > 0$$

and

$$\lim_{u \searrow 0} r(u, \infty) = r_-. \quad (27)$$

Here  $r_- > 0$  is the radius of the Cauchy horizon of the Reissner-Nordström reference solution and

$$r(u, \infty) = \lim_{v \rightarrow \infty} r(u, v)$$

(which exists and is decreasing). Similarly, we also define

$$\varpi(u, \infty) = \lim_{v \rightarrow \infty} \varpi(u, v).$$

Theorem 2.2 implies that the spacetime is extendible across the Cauchy horizon with a  $C^0$  metric.

**Two effects of any nonzero field.**

**Theorem 2.3.** *Suppose that there exists a positive sequence  $(u_n)$  converging to 0 such that  $\zeta_0(u_n) \neq 0$ . Then  $r(u, \infty) < r_-$  for all  $u > 0$ .*

**Lemma 2.4.** *Suppose that there exists a positive sequence  $(u_n)$  converging to 0 such that  $\zeta_0(u_n) \neq 0$ . Then*

$$\int_0^\infty \kappa(u, v) dv < \infty \text{ for all } u > 0. \quad (28)$$

This lemma implies that the affine parameter of an outgoing null geodesic is finite at the Cauchy horizon.

**Well posedness for the backwards problem.** In Section 6, we will extend the solutions of Einstein's equations beyond the Cauchy horizon. For this we will need to solve a backwards problem, already discussed in Part 1. The initial conditions will be prescribed as follows:

$$(I_u) \quad \begin{cases} r(u, 0) &= r_0(u), \\ \nu(u, 0) &= \nu_0(u), \\ \zeta(u, 0) &= \zeta_0(u), \end{cases} \quad \text{for } u \in ]0, U],$$

$$(I^v) \quad \begin{cases} \lambda(U, v) &= \lambda_0(v), \\ \varpi(U, v) &= \varpi_0(v), \\ \theta(U, v) &= \theta_0(v), \\ \kappa(U, v) &= \kappa_0(v), \end{cases} \quad \text{for } v \in [0, V],$$

We let

$$\tilde{r}_0(v) = r_0(U) + \int_0^v \lambda_0(v') dv',$$



for  $v \in [0, V]$ . We assume the regularity conditions:

- (h1) the functions  $\nu_0, \zeta_0, \lambda_0, \theta_0$  and  $\kappa_0$  are continuous, and the functions  $r_0$  and  $\varpi_0$  are continuously differentiable.

We assume the sign conditions:

$$(h2) \quad \begin{cases} r_0(u) > 0 & \text{for } u \in ]0, U[, \\ \tilde{r}_0(v) > 0 & \text{for } v \in [0, V], \\ \nu_0(u) < 0 & \text{for } u \in ]0, U[, \\ \kappa_0(v) > 0 & \text{for } v \in [0, V]. \end{cases}$$

We assume the three compatibility conditions:

$$r'_0 = \nu_0, \tag{29}$$

$$(h3) \quad \varpi'_0 = \frac{1}{2} \frac{\theta_0^2}{\kappa_0}, \tag{30}$$

$$\lambda_0 = \kappa_0 \left( 1 - \frac{2\varpi_0}{\tilde{r}_0} + \frac{e^2}{\tilde{r}_0^2} - \frac{\Lambda}{3} \tilde{r}_0^2 \right). \tag{31}$$

**Theorem 2.5.** *The characteristic initial value problem with initial conditions  $(I_u)$  and  $(I^v)$  satisfying (h1)–(h3) has a unique solution defined on a maximal reflected past set  $\mathcal{R}$  containing a neighborhood of  $]0, U[ \times \{0\} \cup \{U\} \times [0, V]$ .*

**Retrieving the Einstein equations from the first order system.** Henceforth, consider the additional regularity condition

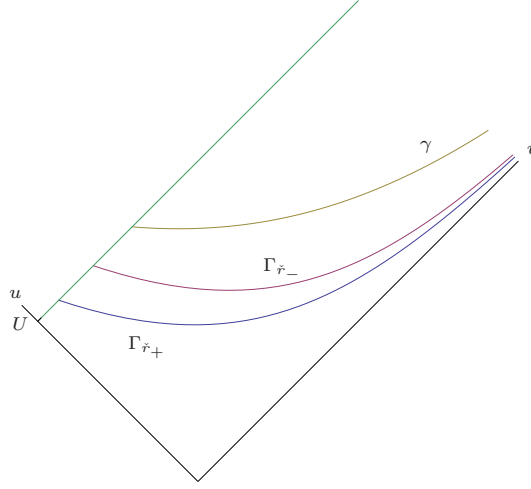
- (h4)  $\nu_0, \kappa_0$  and  $\lambda_0$  are continuously differentiable.

**Lemma 2.6.** *Suppose that  $(r, \nu, \lambda, \varpi, \theta, \zeta, \kappa)$  is the solution of the characteristic initial value problem, or the backwards problem, with initial data satisfying (h1) to (h4). Then the function  $r$  is  $C^2$ , and  $\kappa$  is  $C^1$ .*

**Proposition 2.7.** *The functions  $r, \phi$  and  $\Omega$  satisfy (2), (3), (4) and (5).*

**Proposition 2.8.** *The first order system (15)–(24) implies (6). Since equations (2)–(5) imply the first order system, equations (2)–(5) also imply (6).*

**The partition of spacetime into four regions.** In Part 2, we divide  $[0, U] \times [0, \infty[$  into four disjoint regions, separated by three curves,  $\Gamma_{\tilde{r}_+}$ ,  $\Gamma_{\tilde{r}_-}$  and  $\gamma$ , where different estimates can be obtained (see Appendix C). Next we explain how these curves are constructed.



**The curves  $\Gamma_{\check{r}}$ .** We denote by  $\Gamma_{\check{r}}$  the level sets of the radius function  $r^{-1}(\check{r})$ . These are spacelike curves and consequently may be parameterized by

$$v \mapsto (u_{\check{r}}(v), v) \quad \text{or} \quad u \mapsto (u, v_{\check{r}}(u)).$$

We choose  $\check{r}_+$  and  $\check{r}_-$  sufficiently close to  $r_+$  and  $r_-$  with  $r_- < \check{r}_- < \check{r}_+ < r_+$ .

**The curve  $\gamma = \gamma_{\check{r}_-, \beta}$ .** Given  $\check{r}_-$  as before and  $\beta > 0$ , we define  $\gamma$  to be the curve parametrized by

$$u \mapsto (u, (1 + \beta)v_{\check{r}_-}(u)) =: (u, v_{\gamma}(u)), \quad \text{for } u \in [0, U].$$

**The choice of  $\beta$  so that  $r$  and  $\varpi$  are controlled.** Choose any  $\beta$  such that

$$0 < \beta < \frac{1}{2} \left( \sqrt{1 - 8 \frac{\partial_r(1-\mu)(r_+, \varpi_0)}{\partial_r(1-\mu)(r_-, \varpi_0)}} - 1 \right). \quad (32)$$

Let  $0 < \varepsilon < \varepsilon_0$ . Then there exists  $U_{\varepsilon} > 0$  such that

$$r(u, v) \geq r_- - \varepsilon \quad \text{and} \quad \varpi(u, v) \leq \varpi_0 + \varepsilon \quad (33)$$

for  $(u, v) \in J^-(\gamma) \cap J^+(\Gamma_{\check{r}_-})$  and  $0 < u \leq U_{\varepsilon}$ , provided that the parameters  $\check{r}_+$ ,  $\varepsilon_0$  and  $\delta$  are chosen so that

$$\beta < \frac{1}{2} \left( \sqrt{(1 + \delta)^2 - 8 \frac{(\frac{\check{r}_+}{r_+})^{\delta^2} \min_{r \in [\check{r}_+, r_+]} \partial_r(1-\mu)(r, \varpi_0)}{\partial_r(1-\mu)(r_-, \varpi_0)}} - (1 + \delta) \right). \quad (34)$$

Here  $\hat{\delta}$  is a bound for  $|\frac{\zeta}{\nu}|$  in  $J^-(\check{r}_+)$ . Suppose that there exist positive constants  $C$  and  $s$  such that  $|\zeta_0(u)| \leq Cu^s$ . Then, instead of choosing  $\beta$  according to (32) we may choose

$$0 < \beta < \frac{1}{2} \left( \sqrt{1 - 8 \frac{(1+s)\partial_r(1-\mu)(r_+, \varpi_0)}{\partial_r(1-\mu)(r_-, \varpi_0)}} - 1 \right). \quad (35)$$

In this case, (34) should be replaced by

$$\beta < \frac{1}{2} \left( \sqrt{(1 + \delta)^2 - 8 \frac{[(\frac{\check{r}_+}{r_+})^{\delta^2} + s] \min_{r \in [\check{r}_+, r_+]} \partial_r(1-\mu)(r, \varpi_0)}{\partial_r(1-\mu)(r_-, \varpi_0)}} - (1 + \delta) \right). \quad (36)$$

## 3. MASS INFLATION

We denote the surface gravities of the Cauchy and black hole horizons in the reference subextremal Reissner-Nordström black hole by

$$k_- = -\frac{1}{2}\partial_r(1-\mu)(r_-, \varpi_0), \quad k_+ = \frac{1}{2}\partial_r(1-\mu)(r_+, \varpi_0), \quad (37)$$

and define (see Appendix A)

$$\rho := \frac{k_-}{k_+} > 1. \quad (38)$$

This parameter measures how close the black hole is to being extremal, which corresponds to  $\rho = 1$ .

We start by presenting a sufficient condition for the renormalized mass  $\varpi$  to blow up identically on the Cauchy horizon.

**Theorem 3.1** (Mass inflation). *Suppose that  $\rho > 2$  and*

$$\exists_{c>0} \zeta_0(u) \geq cu^s \text{ for some } 0 < s < \frac{\rho}{2} - 1. \quad (39)$$

*Then*

$$\varpi(u, \infty) = \infty \text{ for each } u > 0. \quad (40)$$

In Appendix A we see how the condition  $\rho > 2$  translates into a relationship between  $r_-$ ,  $r_+$  and  $\Lambda$ .

The proof of Theorem 3.1 generally follows the argument on pages 493–497 of [5], where the  $\Lambda = 0$  case was studied. Nonetheless, the introduction of a cosmological constant requires a different technical approach, in particular the use of a foliation by the level sets of the radius function; moreover, since later on we will need some of the estimates derived in the proof, we present the relevant details in Appendix B.

The previous techniques only allow us to explore the subregion of parameter space determined by (39). The rest of this paper will be dedicated to the analysis of the full parameter range. The first result in that direction is

**Theorem 3.2** (Mass inflation or  $\frac{\theta}{\lambda}$  unbounded). *Suppose that*

$$\exists_{c>0} \zeta_0(u) \geq cu^s \text{ for some } 0 < s < \rho - 1.$$

*If there exists  $U > 0$  such that  $\varpi(U, \infty) < \infty$ , and  $U$  is sufficiently small, then, for each  $0 < \delta < U$ ,  $|\frac{\theta}{\lambda}|(u, v)$  tends to  $+\infty$ , uniformly for  $u \in [\delta, U]$ , as  $v \nearrow \infty$ .*

*Proof.* Suppose that there exists  $U > 0$  such that  $\varpi(U, \infty) < \infty$ . Going through the proof of Theorem 3.1, we see that Case 3.2 must occur. So  $-\lambda$  must be bounded above in  $J^+(\gamma)$  as in (109). Furthermore, the lower estimate on  $\zeta_0$  guarantees the lower bound (112) for  $\theta$  in  $J^+(\gamma)$ . Combining (109) with (112), we obtain

$$\begin{aligned} \left| \frac{\theta}{\lambda} \right|(u, v) &\geq \frac{C}{C(u)} \frac{e^{[-(s+1)\partial_r(1-\mu)(r_+, \varpi_0) - \delta]v}}{e^{[\partial_r(1-\mu)(r_-, \varpi_0) + \delta]v}} \\ &= \frac{1}{C(u)} e^{(2k_+(\rho-s-1) - \delta)v}, \end{aligned} \quad (41)$$

for  $(u, v) \in J^+(\gamma)$ . We choose  $\check{r}_+$  sufficiently close to  $r_+$ ,  $\check{r}_-$  sufficiently close to  $r_-$ ,  $\beta^+$  and  $\beta^-$  sufficiently close to  $\beta$ , and  $U$  sufficiently small so that  $\delta < \rho - s - 1$ . Let  $0 < \delta < U$ . The constant  $C(u)$  is bounded above by  $C(\delta)$  for  $u \in [\delta, U]$ . Then, estimate (41) shows that  $|\frac{\theta}{\lambda}|(u, v)$  tends to  $+\infty$ , uniformly for  $u \in [\delta, U]$ , as  $v \nearrow \infty$ .  $\square$

In Remark 6.9 we will see that under the hypothesis of the previous theorem the Kretschmann scalar always blows up at the Cauchy horizon, as a consequence of either  $\varpi$  or  $\frac{\theta}{\lambda}$  blowing up.

#### 4. NO MASS INFLATION

In this section we will prove that mass inflation does not occur if  $\zeta_0$  decays sufficiently fast as  $u$  tends to zero. We also control the field  $\zeta$  up to the Cauchy horizon.

**Theorem 4.1** (No mass inflation). *Suppose that*

$$\exists_{c>0} |\zeta_0(u)| \leq cu^s \text{ for some nonnegative } s > \frac{7\rho}{9} - 1. \quad (42)$$

Then

$$\varpi(u, \infty) < \infty \text{ for each } 0 < u \leq U,$$

provided that  $U$  is sufficiently small. Furthermore,  $\lim_{u \searrow 0} \varpi(u, \infty) = \varpi_0$ .

Given  $\varepsilon_1 > 0$ , define

$$\mathcal{D} = \mathcal{D}_{\varepsilon_1} = \left\{ (u, v) \in J^+(\gamma) : u \leq U \text{ and } \int_{v_\gamma(u)}^v \left| \frac{\theta^2}{\lambda} \right|(u, \tilde{v}) d\tilde{v} \leq \varepsilon_1 \right\}.$$

The set  $\mathcal{D}$  is connected and contains  $\gamma$ . Our goal is to prove that, for  $U$  small enough,  $\mathcal{D} = J^+(\gamma)$ . This is a consequence of

**Lemma 4.2.** *Assume that  $\rho$  and  $\zeta_0$  are as in Theorem 4.1. Then there exist  $\varepsilon_1 > 0$  and  $U > 0$  such that, for  $(u, v) \in \mathcal{D}$ ,*

$$\int_{v_\gamma(u)}^v \left| \frac{\theta^2}{\lambda} \right|(u, \tilde{v}) d\tilde{v} \leq \frac{\varepsilon_1}{2}.$$

Indeed, for  $\varepsilon_1$  and  $U$  small enough, Lemma 4.2 implies  $\mathcal{D}$  is open in  $J^+(\gamma)$ . Since  $\mathcal{D}$  is also closed in  $J^+(\gamma)$ , we conclude that  $\mathcal{D} = J^+(\gamma)$ .

*Proof of Lemma 4.2.* Our goal is to improve the upper estimate (137) for  $-\lambda$  in  $\mathcal{D}$ , to obtain a lower estimate for  $-\lambda$  in  $\mathcal{D}$ , and to obtain an upper estimate for  $|\theta|$  in  $\mathcal{D}$ . These will allow us to prove that  $\frac{\theta^2}{-\lambda}(u, v)$  decays exponentially in  $v$ , from which the conclusion of the lemma will easily follow. Note that the estimates used in this proof will be sharper than needed here, for use in Section 6.

Integrating (20) as a linear first order ODE for  $\varpi$ , starting from  $\gamma$ , leads to

$$\begin{aligned} \varpi(u, v) &= \varpi(u, v_\gamma(u)) e^{\int_{v_\gamma(u)}^v \left( \frac{\theta^2}{-\lambda} \frac{1}{r} \right)(u, \tilde{v}) d\tilde{v}} \\ &+ \int_{v_\gamma(u)}^v e^{\int_{\tilde{v}}^v \frac{\theta^2}{-\lambda} \frac{1}{r}(u, \tilde{v}) d\tilde{v}} \left( \frac{1}{2} \left( 1 + \frac{e^2}{r^2} - \frac{\Lambda}{3} r^2 \right) \frac{\theta^2}{\lambda} \right)(u, \tilde{v}) d\tilde{v}. \end{aligned} \quad (43)$$

Let  $\tilde{\varepsilon} > 0$ . If  $\varepsilon_1$  and  $U$  are sufficiently small, we have from (33) and (43)

$$|\varpi(u, v) - \varpi_0| < \tilde{\varepsilon}, \quad (44)$$

for  $(u, v) \in \mathcal{D}$ . On the other hand, we have  $r < \check{r}_-$  in  $J^+(\gamma)$  and, using (27), we know  $\lim_{u \searrow 0} r(u, \infty) = r_-$ . Therefore,

$$-C_1 2k_- \leq \partial_r(1 - \mu) \leq -C_1^{-1} 2k_- \quad \text{in } \mathcal{D},$$

with  $C_1 > 1$ . The value of  $C_1$  can be chosen as close to one as desired by decreasing  $\varepsilon_1$ ,  $\check{r}_- - r_-$  and  $U$ . Henceforth,  $C_1$  will denote a constant greater than one, which can be made arbitrarily close to one by a convenient choice of parameters. Similarly,  $\delta$  will denote a positive constant, which can be made arbitrarily small by a convenient choice of parameters.  $C$  will denote a positive constant.

We start by recalling some estimates over  $\gamma$ . Collecting (118), (130), (136) and (42), we get

$$\begin{aligned} \left| \frac{\theta}{\lambda} \right| (u_\gamma(v), v) &\leq C |u_\gamma(v)|^s e^{-2\left(\frac{k_+}{1+\beta} - k_- \beta - \delta\right)v} \\ &\leq C e^{-2\left(\frac{k_+(s+1)}{1+\beta} - k_- \beta - \delta\right)v}. \end{aligned} \quad (45)$$

According to (133) and (134),

$$C e^{-2\left(\frac{k_- \beta}{1+\beta} + \delta\right)v} \leq -\lambda(u_\gamma(v), v) \leq C e^{-2\left(\frac{k_- \beta}{1+\beta} - \delta\right)v}. \quad (46)$$

Combining (45) with (46),

$$|\theta|(u_\gamma(v), v) \leq C e^{-2\left(\frac{k_+(s+1)}{1+\beta} - \frac{k_- \beta^2}{1+\beta} - \delta\right)v}. \quad (47)$$

Finally, according to (135) and (136),

$$\frac{1 + \beta^-}{2k_+} \ln\left(\frac{c}{u}\right) \leq v_\gamma(u) \leq \frac{1 + \beta^+}{2k_+} \ln\left(\frac{C}{u}\right). \quad (48)$$

Recall that  $\beta^- < \beta < \beta^+$  can be chosen arbitrarily close to  $\beta$ .

We now improve the upper estimate (137) for  $-\lambda$  in  $\mathcal{D}$ . Taking into account (131), for  $(u, v) \in \mathcal{D}$ , we have

$$e^{\frac{1}{r(U, \infty)} \int_{v_{\check{r}_-}(\bar{u})}^v \left[ \left| \frac{\theta}{\lambda} \right| |\theta| \right](\bar{u}, \bar{v}) d\bar{v}} \leq 1 + \delta,$$

for  $\bar{u} \in [u_\gamma(v), u]$ . Arguing as in (106),

$$\begin{aligned} \int_{u_\gamma(v)}^u \frac{v}{1 - \mu}(\tilde{u}, v) d\tilde{u} &\geq \frac{1 - \delta}{1 + \delta} \frac{\max_{\Gamma_{\check{r}_-}^{(1-\mu)}}}{\min_{\Gamma_{\check{r}_-}^{(1-\mu)}}} (v_{\check{r}_-}(u_\gamma(v)) - v_{\check{r}_-}(u)) \\ &\geq \frac{C_1^{-1} q}{(1 + \beta)} (v - v_\gamma(u)). \end{aligned}$$

Here,  $0 < q \leq 1$  is a parameter whose importance will become apparent below. Equation (108) together with (46) now show that, for  $(u, v) \in \mathcal{D}$ ,

$$\begin{aligned} -\lambda(u, v) &\leq C e^{-2\left(\frac{k_- \beta}{1+\beta} - \delta\right)v} e^{-\frac{C_1^{-1} q}{(1+\beta)} 2k_- (v - v_\gamma(u))} \\ &= C e^{-2\left(\frac{k_- (\beta + q)}{1+\beta} - \delta\right)v} e^{\frac{C_1^{-1} q}{(1+\beta)} 2k_- v_\gamma(u)}. \end{aligned} \quad (49)$$

The parameter  $q$  makes the second exponential grow slower as  $u \searrow 0$  (at the cost of making the first exponential decay slower). Note that  $q = 0$  corresponds to (137). This is our improved estimate for  $-\lambda$  from above.

We now obtain a lower estimate for  $-\lambda$  in  $\mathcal{D}$ . Arguing as above, we have

$$\begin{aligned} \int_{u_\gamma(v)}^u \frac{\nu}{1-\mu}(\tilde{u}, v) d\tilde{u} &\leq C_1(v_{\tilde{r}_-}(u_\gamma(v)) - v_{\tilde{r}_-}(u)) \\ &\leq \frac{C_1}{(1+\beta)}v. \end{aligned} \quad (50)$$

Using (108) together with (46) once more, for  $(u, v) \in \mathcal{D}$ , we obtain

$$-\lambda(u, v) \geq C e^{-2\left(\frac{k_-\beta}{1+\beta} + \frac{k_-}{1+\beta} + \delta\right)v} = C e^{-2(k_+ + \delta)v}. \quad (51)$$

This is our estimate for  $-\lambda$  from below.

We will now control  $\theta$  in  $\mathcal{D}$ . Integrating (21) and (22) from  $\gamma$  leads to

$$\theta(u, v) = \theta(u_\gamma(v), v) - \int_{u_\gamma(v)}^u \frac{\zeta\lambda}{r}(\tilde{u}, v) d\tilde{u}$$

and

$$\zeta(u, v) = \zeta(u, v_\gamma(u)) - \int_{v_\gamma(u)}^v \frac{\theta\nu}{r}(u, \tilde{v}) d\tilde{v}.$$

It follows that

$$\begin{aligned} \theta(u, v) &= \theta(u_\gamma(v), v) - \int_{u_\gamma(v)}^u \zeta(\tilde{u}, v_\gamma(\tilde{u})) \frac{\lambda}{r}(\tilde{u}, v) d\tilde{u} \\ &\quad + \int_{u_\gamma(v)}^u \frac{\lambda}{r}(\tilde{u}, v) \int_{v_\gamma(\tilde{u})}^v \frac{\theta\nu}{r}(\tilde{u}, \tilde{v}) d\tilde{v} d\tilde{u}. \end{aligned} \quad (52)$$

We fix  $u \leq U$ . Given  $\bar{u} \in [u_\gamma(v), u]$ , since  $r$  is bounded below, from (52) we obtain

$$\begin{aligned} \theta(\bar{u}, v) &\leq |\theta(u_\gamma(v), v)| + C \int_{u_\gamma(v)}^{\bar{u}} |\zeta(\tilde{u}, v_\gamma(\tilde{u}))| |\lambda(\tilde{u}, v)| d\tilde{u} \\ &\quad + C \int_{u_\gamma(v)}^{\bar{u}} |\lambda(\tilde{u}, v)| \int_{v_\gamma(\tilde{u})}^v |\theta(\tilde{u}, \tilde{v})| |\nu(\tilde{u}, \tilde{v})| d\tilde{v} d\tilde{u} \\ &=: |\theta(u_\gamma(v), v)| + I(\bar{u}, v) + II(\bar{u}, v). \end{aligned} \quad (53)$$

In the next two paragraphs we bound  $I$  and  $II$ .

Collecting (118), (129), (136) and (42), we obtain

$$\begin{aligned} \left| \frac{\zeta}{\nu}(\tilde{u}, v_\gamma(\tilde{u})) \right| &\leq C |\tilde{u}|^s e^{-2\left(\frac{k_+}{1+\beta} - k_- \beta - \delta\right)v_\gamma(\tilde{u})} \\ &\leq C e^{-2\left(\frac{k_+(s+1)}{1+\beta} - k_- \beta - \delta\right)v_\gamma(\tilde{u})}. \end{aligned} \quad (54)$$

Using (138), (49) and (54), we have

$$I(\bar{u}, v) \leq C e^{-2\left(\frac{k_-(\beta+q)}{1+\beta} - \delta\right)v} \int_{u_\gamma(v)}^{\bar{u}} e^{-2\left(\frac{k_+(s+1)}{1+\beta} - \frac{k_-(\beta^2+\beta+q)}{(1+\beta)} - \delta\right)v_\gamma(\tilde{u})} \tilde{u}^p d\tilde{u}.$$

Here  $p = \rho\beta - 1 - \delta$ . Using (48), the integral above can be estimated as

$$\begin{aligned}
 & \int_{u_\gamma(v)}^{\bar{u}} e^{-2\left(\frac{k_+(s+1)}{1+\beta} - \frac{k_-(\beta^2+\beta+q)}{(1+\beta)} - \delta\right)v_\gamma(\tilde{u})} \tilde{u}^p d\tilde{u} \\
 & \leq \int_{u_\gamma(v)}^{\bar{u}} e^{-2\left(\frac{k_+(s+1)}{1+\beta} - \frac{k_-(\beta^2+\beta+q)}{(1+\beta)} - \delta\right)\frac{1+\beta}{2k_+} \ln\left(\frac{C}{\tilde{u}}\right)} \tilde{u}^p d\tilde{u} \\
 & \leq \int_0^{\bar{u}} \left(\frac{C}{\tilde{u}}\right)^{-s-1+\rho(\beta^2+\beta+q)+\delta} \tilde{u}^p d\tilde{u} \\
 & \leq C\bar{u}^{s+1-\rho(\beta^2+q)-\delta},
 \end{aligned} \tag{55}$$

if

$$s > \rho(\beta^2 + q) - 1 \tag{56}$$

and if the parameters are chosen so that  $\delta$  is sufficiently small. Therefore, it is possible to bound  $I$  as follows:

$$I(\bar{u}, v) \leq C\bar{u}^{s+1-\rho(\beta^2+q)} e^{-2\left(\frac{k_-(\beta+q)}{1+\beta} - \delta\right)v}. \tag{57}$$

For  $v \geq v_\gamma(u)$ , we define

$$\mathcal{T}_u(v) := \max_{\tilde{u} \in [u_\gamma(v), u]} |\theta(\tilde{u}, v)|. \tag{58}$$

We emphasize that the constants  $C$  will not depend on  $u$ . Using (138), (49) and (58), we see that

$$\begin{aligned}
 II(\bar{u}, v) & \leq C e^{-2\left(\frac{k_-(\beta+q)}{1+\beta} - \delta\right)v} \int_{u_\gamma(v)}^{\bar{u}} e^{\frac{C_1^{-1}q}{(1+\beta)} 2k_- v_\gamma(\tilde{u})} \int_{v_\gamma(\tilde{u})}^v \mathcal{T}_u(\tilde{v}) \tilde{u}^p d\tilde{v} d\tilde{u} \\
 & \leq C e^{-2\left(\frac{k_-(\beta+q)}{1+\beta} - \delta\right)v} \int_{u_\gamma(v)}^{\bar{u}} e^{\frac{C_1^{-1}q}{(1+\beta)} 2k_- v_\gamma(\tilde{u})} \tilde{u}^p d\tilde{u} \int_{v_\gamma(u)}^v \mathcal{T}_u(\tilde{v}) d\tilde{v},
 \end{aligned}$$

where we used the fact that  $v_\gamma(u) \leq v_\gamma(\tilde{u})$ . Again,  $p = \rho\beta - 1 - \delta$ . Using (48), the first integral above can be estimated as

$$\begin{aligned}
 & \int_{u_\gamma(v)}^{\bar{u}} e^{\frac{C_1^{-1}q}{(1+\beta)} 2k_- v_\gamma(\tilde{u})} \tilde{u}^p d\tilde{u} \\
 & \leq \int_{u_\gamma(v)}^{\bar{u}} e^{\frac{C_1^{-1}(1+\beta^+)q}{(1+\beta)} \frac{2k_-}{2k_+} \ln\left(\frac{C}{\tilde{u}}\right)} \tilde{u}^p d\tilde{u} \\
 & \leq \int_0^{\bar{u}} \left(\frac{C}{\tilde{u}}\right)^{C_1^{-1} \frac{1+\beta^+}{1+\beta} \rho q} \tilde{u}^p d\tilde{u} \\
 & \leq C\bar{u}^{\rho(\beta-q)-\delta},
 \end{aligned} \tag{59}$$

if

$$\beta > q \tag{60}$$

and if  $\delta$  is sufficiently small. Therefore it is possible to bound  $II$  as follows:

$$II(\bar{u}, v) \leq C\bar{u}^{\rho(\beta-q)-\delta} e^{-2\left(\frac{k_-(\beta+q)}{1+\beta} - \delta\right)v} \int_{v_\gamma(u)}^v \mathcal{T}_u(\tilde{v}) d\tilde{v}. \tag{61}$$

For all  $v \geq v_\gamma(u)$ , we estimate  $\overline{\mathcal{T}_u(v)}$  using (47), (53), (57) and (61):

$$\begin{aligned} \mathcal{T}_u(v) &\leq C e^{-2\left(\frac{k_+(s+1)}{1+\beta} - \frac{k_-\beta^2}{1+\beta} - \delta\right)v} + C u^{s+1-\rho(\beta^2+q)-\delta} e^{-2\left(\frac{k_-(\beta+q)}{1+\beta} - \delta\right)v} \\ &\quad + C u^{\rho(\beta-q)-\delta} e^{-2\left(\frac{k_-(\beta+q)}{1+\beta} - \delta\right)v} \int_{v_\gamma(u)}^v \mathcal{T}_u(\tilde{v}) d\tilde{v}. \end{aligned} \quad (62)$$

We claim that

$$\begin{aligned} \mathcal{T}_u(v) &\leq C e^{-2\left(\frac{k_+(s+1)}{1+\beta} - \frac{k_-\beta^2}{1+\beta} - \delta\right)v} \\ &\quad + C u^{s+1-\rho(\beta^2+q)-\delta} e^{-2\left(\frac{k_-(\beta+q)}{1+\beta} - \delta\right)v} \end{aligned} \quad (63)$$

for

$$s > \rho(\beta^2 + \beta + q) - 1 \quad (64)$$

and small  $\delta$ .

We impose (64); it can be checked that considering also the opposite inequality will not lead to an improvement of the statement of Theorem 4.1 (for the choice of parameters that we make below).

*Proof of the claim.* Inequality (62) is of the form

$$\mathcal{T}_u(v) \leq C e^{-Av} + C u^b e^{-av} + C e^{-av} \int_{v_\gamma(u)}^v \mathcal{T}_u(\tilde{v}) d\tilde{v},$$

with

$$A = 2\left(\frac{k_+(s+1)}{1+\beta} - \frac{k_-\beta^2}{1+\beta} - \delta\right), \quad (65)$$

$$a = 2\left(\frac{k_-(\beta+q)}{1+\beta} - \delta\right), \quad (66)$$

$$b = s + 1 - \rho(\beta^2 + q) - \delta.$$

Since we impose (64),  $A > a > 0$ , for small  $\delta$ . Let  $\tilde{\mathcal{T}}_u(v) = e^{av} \mathcal{T}_u(v)$ . Then

$$\tilde{\mathcal{T}}_u(v) \leq C e^{-(A-a)v} + C u^b + C \int_{v_\gamma(u)}^v e^{-a\tilde{v}} \tilde{\mathcal{T}}_u(\tilde{v}) d\tilde{v}.$$

Applying Gronwall's inequality, we get

$$\begin{aligned} \tilde{\mathcal{T}}_u(v) &\leq C e^{-(A-a)v} + C u^b + C \int_{v_\gamma(u)}^v (e^{-A\tilde{v}} + u^b e^{-a\tilde{v}}) d\tilde{v} \\ &\leq C e^{-(A-a)v} + C u^b + C e^{-Av_\gamma(u)} + C u^b e^{-av_\gamma(u)} \\ &\leq C e^{-(A-a)v} + C u^b + C u^{A\frac{1+\beta}{2k_+}-\delta} + C u^b u^{a\frac{1+\beta}{2k_+}-\delta} \\ &\leq C e^{-(A-a)v} + C u^b. \end{aligned}$$

To estimate  $e^{-v_\gamma(u)}$  we used (135). We also used  $A\frac{1+\beta}{2k_+}-\delta = s+1-\rho\beta^2-\delta > b = s+1-\rho(\beta^2+q)-\delta$ , for small  $\delta$ .  $\square$

Obviously, for  $(u, v) \in \mathcal{D}$  we have

$$|\theta(u, v)| \leq \mathcal{T}_u(v). \quad (67)$$



Using (51) and (63), we obtain

$$\left| \frac{\theta^2}{\lambda} \right| (u, v) \leq C e^{-2 \left( \frac{2k_+(s+1)}{1+\beta} - \frac{2k_-\beta^2}{1+\beta} - k_- - \delta \right) v} \quad (68)$$

$$+ C u^{2(s+1-\rho(\beta^2+q)-\delta)} e^{-2 \left( \frac{k_-(\beta+2q-1)}{1+\beta} - \delta \right) v}. \quad (69)$$

The exponent in (68) can be made negative if

$$s > \rho \left( \beta^2 + \frac{\beta}{2} + \frac{1}{2} \right) - 1 \quad (70)$$

and the second exponent in (69) can be made negative if

$$\beta > 1 - 2q. \quad (71)$$

Below we will characterize a choice of parameters for which we have

$$\left| \frac{\theta^2}{\lambda} \right| (u, v) \leq C e^{-\Delta v}, \quad (72)$$

for  $(u, v) \in \mathcal{D}$ , with  $\Delta > 0$ . However, before we do that, we note that estimate (72) wraps up the bootstrap argument. Indeed, as  $\lim_{u \searrow 0} v_\gamma(u) = +\infty$ , we can choose  $U$  such that

$$\int_{v_\gamma(u)}^v \left| \frac{\theta^2}{\lambda} \right| (u, \tilde{v}) d\tilde{v} < \frac{\varepsilon_1}{2},$$

for  $(u, v) \in \mathcal{D}$ .

We now bring together the conditions that we must satisfy in order for the above argument to work, and we choose our parameters. The number  $\beta$  is bounded above by (35) and bounded below by (60) and (71); in addition,  $s$  is bounded below by (56), (64) and (70). In fact, the restrictions on  $s$  can be stated in a simpler form: inequality (64) is stricter than (56); inequality (71) implies that (64) is stricter than (70). So, all the restrictions on  $s$  amount to saying that  $s$  is bounded below by (64).

We now select the parameters  $q$  and  $\beta$ . The minimum of the maximum of the lower bounds for  $\beta$  in (60) and (71) is obtained for  $q = \frac{1}{3}$ . This is our choice of  $q$ . Inequality (35) can be satisfied when  $s > \frac{2\rho}{9} - 1$  because

$$\frac{1}{3} = \frac{1}{2} \left( \sqrt{1 + 8\frac{2}{9}} - 1 \right) < \beta < \frac{1}{2} \left( \sqrt{1 + \frac{8(1+s)}{\rho}} - 1 \right).$$

For (64) to be satisfied we impose  $\frac{7\rho}{9} - 1 < s$  because

$$\left[ \rho(\beta^2 + \beta + q) - 1 \right] \Big|_{\substack{\beta = \frac{1}{3} \\ q = \frac{1}{3}}} = \frac{7\rho}{9} - 1 < \left[ \rho(\beta^2 + \beta + q) - 1 \right] \Big|_{q = \frac{1}{3}} < s,$$

Obviously,  $\frac{2\rho}{9} - 1 < \frac{7\rho}{9} - 1$ . Therefore, if  $s > \frac{7\rho}{9} - 1$  and we choose  $\beta = \frac{1}{3} + \varepsilon$ , with  $\varepsilon > 0$  sufficiently small, both (35) and (64) are satisfied.

Therefore, our parameters will be chosen in the following way. Suppose that we are given initial data  $\zeta_0$  satisfying (42). We choose  $\beta > \frac{1}{3}$  (so that (60) and (71) hold with  $q = \frac{1}{3}$ ) and such that (35) and (64) hold. When

$$(\check{r}_+, \check{r}_-, \beta^+, \beta^-, \varepsilon_0, \varepsilon_1, U) \rightarrow (r_+, r_-, \beta, \beta, 0, 0, 0),$$

the parameters  $\delta$  above all converge to 0 (at the cost of increasing the constants  $C$ ). So, we may choose  $\check{r}_+$  sufficiently close to  $r_+$ ,  $\check{r}_-$  sufficiently close

to  $r_-$ ,  $\beta^+$  and  $\beta^-$  sufficiently close to  $\beta$ , and  $\varepsilon_0$ ,  $\varepsilon_1$  and  $U$  sufficiently small so that (36) holds, the exponent in (68) and the second exponent in (69) are negative, the integrals (55) and (59) converge, and, finally, such that the numbers  $A$  in (65) and  $a$  in (66) satisfy  $A > a$ . This will guarantee (72), for a certain positive  $\Delta$  and (a maybe very large but finite value of)  $C$ .

The proof of Lemma 4.2 is complete.  $\square$

Since (44) holds in  $\mathcal{D}$ , the fact that  $\mathcal{D} = J^+(\gamma)$ , established as a consequence of Lemma 4.2, implies Theorem 4.1.

We finish this section by controlling the field  $\zeta$  in the following result.

**Lemma 4.3.** *Suppose that*

$$\exists_{c>0} |\zeta_0(u)| \leq cu^s \text{ for some nonnegative } s > \frac{7\rho}{9} - 1.$$

*Then there exists a constant  $C > 0$  such that*

$$|\zeta(u, v)| \leq Cu^{s-\rho\beta^2-\delta}, \quad (73)$$

*for  $(u, v) \in J^+(\gamma)$ , where  $\delta > 0$  can be chosen arbitrarily close to zero, provided that  $U$  is sufficiently small.*

*Proof.* Integrating (22), we have

$$\zeta(u, v) = \zeta(u, v_\gamma(u)) - \int_{v_\gamma(u)}^v \frac{\theta\nu}{r}(u, \tilde{v}) d\tilde{v}. \quad (74)$$

Collecting (138), (48) and (54), we get

$$\begin{aligned} |\zeta(u, v_\gamma(u))| &\leq Ce^{-2\left(\frac{k_+(s+1)}{1+\beta} - k_- \beta - \delta\right)v_\gamma(u)} u^{\rho\beta-1-\delta} \\ &\leq Cu^{s-\rho\beta^2-\delta}. \end{aligned} \quad (75)$$

On the other hand, from (138), (48), (63) and (67), we obtain

$$\begin{aligned} \int_{v_\gamma(u)}^v \frac{|\theta\nu|}{r}(u, \tilde{v}) d\tilde{v} &\leq Ce^{-2\left(\frac{k_+(s+1)}{1+\beta} - \frac{k_- \beta^2}{1+\beta} - \delta\right)v_\gamma(u)} u^{\rho\beta-1-\delta} \\ &\quad + Cu^{s+1-\rho(\beta^2+q)-\delta} e^{-2\left(\frac{k_-(\beta+q)}{1+\beta} - \delta\right)v_\gamma(u)} u^{\rho\beta-1-\delta} \\ &\leq Cu^{s-\rho\beta^2+\rho\beta-\delta} + Cu^{s-\rho\beta^2+2\rho\beta-\delta} \\ &\leq Cu^{s-\rho\beta^2+\rho\beta-\delta}. \end{aligned} \quad (76)$$

Using (75) and (76) in (74), we obtain (73).  $\square$

## 5. EXTENSIONS OF THE METRIC BEYOND THE CAUCHY HORIZON

In this section we assume that the field  $\zeta_0$  satisfies

$$\exists_{c>0} |\zeta_0(u)| \leq cu^s \text{ for some nonnegative } s > \frac{7\rho}{9} - 1,$$

so that there is no mass inflation, and we examine the possibility of extending the metric beyond the Cauchy horizon. We regard the  $(u, v)$  plane, the domain of our first order system, as a  $C^2$  manifold. Since the Cauchy horizon corresponds to  $v = \infty$ , we must change this coordinate to one with a finite range. There are two natural choices to do so: either resorting to the radius function along the outgoing null ray  $u = U$  for the new coordinate (i.e.

choosing  $\lambda = -1$  on  $u = U$ ), or setting  $\kappa = 1$  on the null ray  $u = U$  (as was done for the initial data along the event horizon).

In the first coordinate system,  $v$  is then replaced by  $\tilde{v} = r(U, 0) - r(U, v)$ . This is the coordinate system that we will later use in Section 6; it transforms the domain  $[0, \infty[$  of  $v$  into a bounded interval for  $\tilde{v}$ , even when the field  $\zeta_0$  is identically zero. In the second coordinate system, which has finite range only when  $\zeta_0$  is not identically zero,  $v$  is replaced by  $\hat{v} := \int_0^v \kappa(U, \bar{v}) d\bar{v}$ .

Both maps  $v \mapsto \tilde{v}$  and  $v \mapsto \hat{v}$  are  $C^2([0, \infty[)$  and have a non-zero derivative, the first one with range  $[0, V[$ , say, and the second one with range  $[0, \hat{V}[$ . So the map  $\hat{v} \mapsto \tilde{v}$  is  $C^2$ . This map extends to a  $C^1$  map from the interval  $[0, \hat{V}]$  to the interval  $[0, V]$  (see (86)). By Remark 5.5, the two coordinate systems  $(u, \tilde{v})$  and  $(u, \hat{v})$  are not equivalent (as  $C^2$  coordinate systems) when  $|\frac{\hat{v}}{\tilde{v}}|$  is unbounded along  $u = U$ .

In both coordinate systems we can extend the metric continuously to the Cauchy horizon, and consequently beyond the Cauchy horizon, with the second mixed derivatives of  $r$  continuous. In the coordinate system  $(u, \hat{v})$  this can be done so that the Christoffel symbols are in  $L^2_{\text{loc}}$  and the field  $\phi$  is in  $H^1_{\text{loc}}$ . Therefore, the Christodoulou-Chruściel inextendibility criterion for strong cosmic censorship does not hold.

**5.1. Coordinates with  $v$  replaced by  $\tilde{v} = r(U, 0) - r(U, v)$ .** If there exists a positive sequence  $(u_n)$  converging to 0 such that  $\zeta_0(u_n) \neq 0$  then we choose  $U$  such that  $(1 - \mu)(U, \infty) < 0$ . In the proof of Lemma 2.4 we showed that such a  $U$  exists; in Proposition 5.2 we will see that under the present assumptions  $(1 - \mu)(U, \infty) < 0$  for any  $U > 0$ , so that actually any choice of  $U$  will do. If  $\zeta_0$  vanishes in a right neighborhood of the origin then the solution is simply Reissner-Nordström and we can choose any  $U$ . We define  $f : [0, \infty[ \rightarrow \mathbb{R}$ , by

$$f(v) = r(U, 0) - r(U, v), \quad (77)$$

so that

$$f'(v) = -\lambda(U, v),$$

and set

$$V = f(\infty) = r(U, 0) - r(U, \infty).$$

We will change the  $v$  coordinate to

$$\tilde{v} = f(v).$$

The functions  $\nu_0$ ,  $\kappa_0$  and  $\lambda_0$  (equal to  $-1$ ,  $1$  and  $0$ , respectively) satisfy hypothesis (h4) (see Section 2). By Lemma 2.6, the function  $r$  is  $C^2$ . Moreover,  $\lambda(U, \cdot) < 0$ . Therefore, the change of coordinates of the previous paragraph is admissible (that is,  $C^2$ ).

We denote by  $\tilde{r}$  the function  $r$  written in the new coordinates, i.e.

$$\tilde{r}(u, \tilde{v}) = \tilde{r}(u, f(v)) = r(u, v).$$

We let  $\tilde{\lambda} = \partial_{\tilde{v}} \tilde{r}$  and  $\tilde{\nu} = \partial_u \tilde{r}$ , whence

$$\tilde{\lambda}(u, \tilde{v}) = \frac{\lambda(u, v)}{f'(v)}$$

and

$$\tilde{\nu}(u, \tilde{v}) = \nu(u, v).$$

In particular,

$$\tilde{\lambda}(U, \tilde{v}) \equiv -1.$$

Similarly, we define

$$-\tilde{\Omega}^2(u, \tilde{v}) \, dud\tilde{v} = -\tilde{\Omega}^2(u, f(v))f'(v) \, dudv = -\Omega^2(u, v) \, dudv.$$

From (9) we then have

$$\tilde{\varpi}(u, \tilde{v}) = \varpi(u, v),$$

and from (13)

$$\tilde{\kappa}(u, \tilde{v}) = \frac{\kappa(u, v)}{f'(v)}.$$

Finally, we also denote by  $\tilde{\phi}$  the function  $\phi$  written in the new coordinates,

$$\tilde{\phi}(u, \tilde{v}) = \tilde{\phi}(u, f(v)) = \phi(u, v),$$

and from (11) and (12)

$$\tilde{\theta}(u, \tilde{v}) = \frac{\theta(u, v)}{f'(v)}, \quad \tilde{\zeta}(u, \tilde{v}) = \zeta(u, v).$$

**Remark 5.1.** *It is obvious that the functions  $\tilde{r}$ ,  $\tilde{v}$ ,  $\tilde{\lambda}$ ,  $\tilde{\varpi}$ ,  $\tilde{\theta}$ ,  $\tilde{\zeta}$  and  $\tilde{\kappa}$  satisfy the first order system (15)–(24), with respect to the new coordinates  $(u, \tilde{v})$ .*

**Proposition 5.2.** *Suppose that*

$$\exists_{c>0} |\zeta_0(u)| \leq cu^s \text{ for some nonnegative } s > \frac{7\rho}{9} - 1.$$

*Then there exists  $U > 0$  such that for all  $0 < \delta < U$ , the functions  $\tilde{r}$ ,  $\tilde{v}$ ,  $\tilde{\lambda}$ ,  $\tilde{\varpi}$ ,  $\tilde{\zeta}$  and  $\tilde{\kappa}$  (but not necessarily  $\tilde{\theta}$ ) admit continuous extensions to the closed rectangle  $[\delta, U] \times [0, V]$ . Equations (15) to (19), (23), and (24) are satisfied on this set. Finally,  $(1 - \mu)(u, V)$  is negative for  $u > 0$ , unless there exists a right neighborhood of the origin where  $\zeta_0$  vanishes.*

*Proof.* If  $\zeta_0$  vanishes in a right neighborhood of the origin, then the conclusion is immediate since the functions are obtained from the Reissner-Nordström solution.

Assume that there exists a positive sequence  $(u_n)$  converging to 0 such that  $\zeta_0(u_n) \neq 0$ . We fix  $0 < \delta < U$ , and proceed in three steps.

*Step 1.* We prove that our functions  $\tilde{r}$ ,  $\tilde{v}$ ,  $\tilde{\lambda}$ ,  $\tilde{\varpi}$ ,  $\tilde{\zeta}$  and  $\tilde{\kappa}$  converge uniformly as functions of  $u \in [\delta, U]$  as  $\tilde{v} \rightarrow V$ . The convergence of  $\tilde{r}(\cdot, \tilde{v})$  to  $\tilde{r}(\cdot, V)$  is uniform on  $[\delta, U]$  because

$$\int_{\tilde{v}}^V |\tilde{\lambda}|(u, \bar{v}) \, d\bar{v} = \int_{f^{-1}(\tilde{v})}^{\infty} |\lambda|(u, \bar{v}) \, d\bar{v} \rightarrow 0$$

as  $\tilde{v} \nearrow V$  (by (137)).

The convergence of  $\tilde{\varpi}(\cdot, \tilde{v})$  to  $\tilde{\varpi}(\cdot, V)$  is also uniform on  $[\delta, U]$  because

$$\int_{\tilde{v}}^V \left| \frac{\tilde{\theta}^2}{\tilde{\lambda}} \right|(u, \bar{v}) \, d\bar{v} = \int_{f^{-1}(\tilde{v})}^{\infty} \left| \frac{\theta^2}{\lambda} \right|(u, \bar{v}) \, d\bar{v} \rightarrow 0$$

as  $\tilde{v} \nearrow V$  (by (68)–(69)).

For  $u \in [\delta, U]$ , using (28),

$$\int_{\tilde{v}}^V \tilde{\kappa}(u, \tilde{v}) d\tilde{v} \leq \int_{\tilde{v}}^V \tilde{\kappa}(\delta, \tilde{v}) d\tilde{v} \rightarrow 0, \text{ as } \tilde{v} \rightarrow \infty. \quad (78)$$

Integrating (18), for  $\tilde{v} \leq \tilde{V} < V$ ,

$$\tilde{\nu}(u, \tilde{v}) - \tilde{\nu}(u, \tilde{V}) = \tilde{\nu}(u, \tilde{v}) \left( 1 - e^{\int_{\tilde{v}}^{\tilde{V}} [\tilde{\kappa} \partial_r (1-\mu)](u, \tilde{v}) d\tilde{v}} \right).$$

Using (138), (78),  $r(U, \infty) > 0$  and  $\varpi(U, \infty) < \infty$ , we conclude that we may define  $\tilde{\nu}(\cdot, V)$ . Also, letting  $\tilde{V} \nearrow V$ , the restriction of  $\tilde{\nu}(\cdot, \tilde{v})$  to  $[\delta, U]$  converges uniformly to  $\tilde{\nu}(\cdot, V)$  as  $\tilde{v} \nearrow V$ . Integrating (18) between  $\tilde{v}$  and  $V$ , we conclude that

$$\tilde{\nu}(u, V) < 0 \quad (79)$$

for each  $u > 0$ .

Integrating (22),

$$\tilde{\zeta}(u, \tilde{V}) = \tilde{\zeta}(u, \tilde{v}) - \int_{\tilde{v}}^{\tilde{V}} \frac{\tilde{\theta} \tilde{\nu}}{\tilde{r}}(u, \tilde{v}) d\tilde{v}.$$

We use (138) and

$$\int_{\tilde{v}}^V |\tilde{\theta}|(u, \tilde{v}) d\tilde{v} = \int_{f^{-1}(\tilde{v})}^{\infty} |\theta|(u, \tilde{v}) d\tilde{v} \rightarrow 0$$

as  $\tilde{v} \nearrow V$  (by (63) and (67)). Note that the last convergence is uniform for  $u \in [\delta, U]$ . Arguing as in the previous paragraph, we may define  $\tilde{\zeta}(\cdot, V)$  as the uniform limit of  $\tilde{\zeta}(\cdot, \tilde{v})$  when  $\tilde{v} \nearrow V$ .

From  $\tilde{\kappa}(U, \tilde{v}) = \frac{-1}{(1-\mu)(U, \tilde{v})}$  and (23), we get

$$\tilde{\kappa}(u, \tilde{v}) = \frac{-1}{(1-\mu)(U, \tilde{v})} e^{-\int_u^U \left(\frac{\tilde{\zeta}^2}{\tilde{r}\tilde{\nu}}\right)(\tilde{u}, \tilde{v}) d\tilde{u}}. \quad (80)$$

Using  $\widetilde{(1-\mu)}(U, V) = (1-\mu)(U, \infty) < 0$ , the uniform convergence of  $\tilde{r}$ ,  $\tilde{\nu}$  and  $\tilde{\zeta}$  as  $\tilde{v} \nearrow V$ , and the fact that  $\tilde{r}$  and  $\tilde{\nu}$  are bounded away from zero, we see that we may define  $\tilde{\kappa}(\cdot, V)$ . Furthermore, since we already proved uniform convergence of  $\tilde{r}$ ,  $\tilde{\varpi}$ ,  $\tilde{\nu}$  and  $\tilde{\zeta}$ , it is clear that  $\tilde{\kappa}(\cdot, V)$  is the uniform limit of  $\tilde{\kappa}(\cdot, \tilde{v})$  when  $\tilde{v} \nearrow V$ . We have

$$\tilde{\kappa}(u, V) \geq \tilde{\kappa}(U, V) = \frac{-1}{\widetilde{(1-\mu)}(U, V)} > 0$$

for  $u \in [\delta, U]$ .

The function  $\tilde{\lambda}$  clearly extends to a continuous function on  $[\delta, U] \times [0, V]$  since  $\tilde{\lambda} = \tilde{\kappa} \widetilde{(1-\mu)}$ .

*Step 2.* The functions  $\tilde{r}$ ,  $\tilde{\nu}$ ,  $\tilde{\lambda}$ ,  $\tilde{\varpi}$ ,  $\tilde{\zeta}$  and  $\tilde{\kappa}$  are continuous in the closed rectangle  $[\delta, U] \times [0, V]$ . Indeed, let  $\tilde{h}$  denote one of these functions. We know  $\tilde{h}(\cdot, V)$  is continuous because it is the uniform limit of continuous functions. Let  $u \in [\delta, U]$  and  $\varepsilon > 0$ . There exists  $\tilde{\delta} > 0$  such that  $|\tilde{u} - u| < \tilde{\delta}$  implies  $|\tilde{h}(\tilde{u}, V) - \tilde{h}(u, V)| < \frac{\varepsilon}{2}$ . Furthermore, again by uniform convergence, there exists  $\hat{\delta} > 0$  such that  $|\tilde{v} - V| < \hat{\delta}$  implies  $|\tilde{h}(\tilde{u}, \tilde{v}) - \tilde{h}(\tilde{u}, V)| < \frac{\varepsilon}{2}$  for

all  $\bar{u} \in [\delta, U]$ . So, if  $|\bar{u} - u| < \tilde{\delta}$  and  $|\tilde{v} - V| < \hat{\delta}$ , then  $|\tilde{h}(\bar{u}, \tilde{v}) - \tilde{h}(u, V)| < \varepsilon$ . This proves continuity of  $\tilde{h}$  at  $(u, V)$ .

*Step 3.* It is clear that the system (15) to (23), except (20), (21) and (22), is satisfied also on the segment  $[\delta, U] \times \{V\}$ . Indeed, to obtain the equations that involve the derivative with respect to  $u$ , we use the fact that if  $\tilde{h}(\cdot, v_n)$  converges uniformly to  $\tilde{h}(\cdot, V)$  and  $\partial_u \tilde{h}(\cdot, v_n)$  converges uniformly to  $\hat{h}(\cdot, V)$  as  $v_n \nearrow V$  then  $\partial_u \tilde{h}(\cdot, V)$  exists and is equal to  $\hat{h}(\cdot, V)$ .

On the other hand, to obtain the equations that involve the derivative with respect to  $v$ , we write these equations in integrated form, say from 0 to  $\tilde{v}_n$ , and let  $\tilde{v}_n \nearrow V$ . From the (trivial) continuity of the indefinite integral of a continuous function and the Fundamental Theorem of Calculus, we deduce that the equations are valid at  $V$ .

Obviously, (24) is satisfied on the segment  $[\delta, U] \times \{V\}$ .

Finally, taking into account

$$\frac{\nu(u, \infty)}{(1 - \mu)(u, \infty)} \leq \frac{\nu(u, 0)}{(1 - \mu)(u, 0)} < \infty$$

(from (114)) and that  $\tilde{\nu}$  is negative on  $[\delta, U] \times \{V\}$  (see (79)), we conclude that  $(1 - \mu)(u, V)$  is uniformly bounded above by a negative constant on  $[\delta, U]$ .  $\square$

*The metric and the field.* Recall that the reason to study our first order system is that its solutions allow the construction of spherically symmetric Lorentzian manifolds  $(\mathcal{M}, g)$  and fields  $\tilde{\phi}$  which solve the Einstein equations. Here  $\mathcal{M} = \mathcal{Q} \times \mathbb{S}^2$ , where  $\mathcal{Q}$  admits the global null coordinate system  $(u, \tilde{v})$  defined on  $[0, U] \times [0, V] \setminus \{(0, V)\}$ , and the metric is

$$g = -\tilde{\Omega}^2(u, \tilde{v}) \, d\tilde{v} \, du + \tilde{r}^2(u, \tilde{v}) \, \sigma_{\mathbb{S}^2},$$

with  $\tilde{\Omega}^2 = -4\tilde{\nu}\tilde{\kappa}$ . We give  $\mathcal{M}$  the structure of a  $C^2$  manifold, i.e. we only allow  $C^2$  changes of coordinates. Although Proposition 5.2 guarantees that  $r$  is  $C^1$  on  $[0, U] \times [0, V] \setminus \{(0, V)\}$ , the regularity of the metric is no better than  $C^0$ , since, as will become apparent in the proof of Proposition 5.3,  $\partial_{\tilde{v}}\tilde{\kappa}$  may blow up on the Cauchy horizon. This allows for  $C^0$  extensions of the metric beyond the Cauchy horizon, by a similar construction as the one that will be used below for the coordinate system  $(u, \hat{v})$ .

The field  $\tilde{\phi}$  is determined, after prescribing  $\tilde{\phi}(0, 0)$ , by integrating (11) and (12). According to [5, Proposition 13.2] (with the choice  $u_1 = v_1 = 0$ ),  $\int_0^v |\theta|(u, \tilde{v}) \, d\tilde{v} + \int_0^u |\zeta|(\tilde{u}, v) \, d\tilde{u} \leq \underline{C}$ . So,  $\tilde{\phi}$  is well defined, bounded and continuous, with continuous partial derivative with respect to  $u$  in  $[0, U] \times [0, V] \setminus \{(0, V)\}$ .

The nonvanishing Christoffel symbols of the metric on  $\mathcal{M}$  are

$$\begin{aligned}
 \tilde{\Gamma}_{AB}^C &, \\
 \tilde{\Gamma}_{AB}^u &= 2\tilde{\Omega}^{-2}\tilde{r}\tilde{\lambda}\sigma_{AB} = -\tilde{r}\frac{\widetilde{1-\mu}}{2\tilde{\nu}}\sigma_{AB}, \\
 \tilde{\Gamma}_{AB}^{\tilde{v}} &= 2\tilde{\Omega}^{-2}\tilde{r}\tilde{\nu}\sigma_{AB} = -\tilde{r}\frac{\widetilde{1-\mu}}{2\tilde{\lambda}}\sigma_{AB}, \\
 \tilde{\Gamma}_{B\tilde{u}}^A &= \tilde{\nu}\tilde{r}^{-1}\delta_B^A, \\
 \tilde{\Gamma}_{B\tilde{v}}^A &= \tilde{\lambda}\tilde{r}^{-1}\delta_B^A, \\
 \tilde{\Gamma}_{uu}^u &= \tilde{\Omega}^{-2}\partial_u(\tilde{\Omega}^2) = \frac{\partial_u\tilde{\nu}}{\tilde{\nu}} + \frac{\partial_u\tilde{\kappa}}{\tilde{\kappa}}, \\
 \tilde{\Gamma}_{\tilde{v}\tilde{v}}^{\tilde{v}} &= \tilde{\Omega}^{-2}\partial_{\tilde{v}}(\tilde{\Omega}^2) = \frac{\partial_{\tilde{v}}\tilde{\nu}}{\tilde{\nu}} + \frac{\partial_{\tilde{v}}\tilde{\kappa}}{\tilde{\kappa}}
 \end{aligned} \tag{81}$$

(see [7, Appendix A]).

**Proposition 5.3.** *Suppose that*

$$\exists_{c_1, c_2 > 0} c_2 u^{s_2} \leq \zeta_0(u) \leq c_1 u^{s_1} \text{ for some } \frac{7\rho}{9} - 1 < s_1 \leq s_2 < \rho - 1,$$

For any  $\delta > 0$  the field  $|\frac{\tilde{\theta}}{\tilde{\lambda}}|(u, \tilde{v})$  tends to  $+\infty$  as  $\tilde{v} \nearrow V$ , uniformly for  $u \in [\delta, U]$ . For all  $u \in [\delta, U]$ , with one possible exception,  $\tilde{\Gamma}_{\tilde{v}\tilde{v}}^{\tilde{v}}(u, \tilde{v})$  is unbounded as  $\tilde{v} \nearrow V$ . Moreover,  $\tilde{\Gamma}_{\tilde{v}\tilde{v}}^{\tilde{v}}(U, \tilde{v})$  tends to  $-\infty$  as  $\tilde{v} \nearrow V$ .

*Proof.* The upper bound on  $\zeta_0$  and Theorem 4.1 imply that  $\varpi(u, \infty) < \infty$  for each  $0 < u \leq U$ , provided that  $U$  is sufficiently small. Fix  $0 < \delta < U$ . Using the lower bound on  $\zeta_0$  together with Theorem 3.2, we know that  $|\frac{\tilde{\theta}}{\tilde{\lambda}}|(u, \tilde{v})$  tends to  $+\infty$ , uniformly for  $u \in [\delta, U]$ , as  $\tilde{v} \nearrow V$ . In particular,

$$|\tilde{\theta}(U, \tilde{v})| \rightarrow +\infty, \text{ as } \tilde{v} \nearrow V. \tag{82}$$

Suppose, by contradiction, that there exist  $u_1 < u_2$  in  $[\delta, U]$  for which  $\tilde{\Gamma}_{\tilde{v}\tilde{v}}^{\tilde{v}}(u_1, \cdot)$  and  $\tilde{\Gamma}_{\tilde{v}\tilde{v}}^{\tilde{v}}(u_2, \cdot)$  are bounded. From Part 1, we recall the wave equation for  $\tilde{\Omega}^2$ ,

$$\partial_u \partial_{\tilde{v}} \log \tilde{\Omega}^2 = -\frac{2\tilde{\theta}\tilde{\zeta}}{\tilde{r}^2} + \frac{4\tilde{\kappa}\tilde{\nu}e^2}{\tilde{r}^4} - \frac{2\tilde{\kappa}\tilde{\nu}}{\tilde{r}^2} + \frac{2\tilde{\lambda}\tilde{\nu}}{\tilde{r}^2}.$$

Using (81) we know that  $\tilde{\Gamma}_{\tilde{v}\tilde{v}}^{\tilde{v}} = \partial_{\tilde{v}} \ln(\tilde{\Omega}^2)$ . Our hypotheses and Lemma B.1 imply that  $\tilde{\theta}$  and  $\tilde{\zeta}$  are positive on  $]0, U] \times [0, V[$ . Thus, there exists a constant  $C > 0$  such that, for  $\delta \leq u_1 < u_2 \leq U$  and  $\tilde{v} \in [0, V[$ ,

$$\int_{u_1}^{u_2} \frac{2\tilde{\theta}\tilde{\zeta}}{\tilde{r}^2}(\tilde{u}, \tilde{v}) d\tilde{u} \leq \tilde{\Gamma}_{\tilde{v}\tilde{v}}^{\tilde{v}}(u_1, \tilde{v}) - \tilde{\Gamma}_{\tilde{v}\tilde{v}}^{\tilde{v}}(u_2, \tilde{v}) + C.$$

According to Proposition 5.2 there exists a positive constant  $c_\delta$  such that

$$\tilde{\lambda}(u, \tilde{v}) \leq -c_\delta < 0 \text{ for } (u, \tilde{v}) \in [\delta, U] \times [0, V[$$

because  $1 - \tilde{\mu}(u, V) < 0$  and  $\tilde{\kappa}(u, V) \geq \tilde{\kappa}(U, V) > 0$  for  $u > 0$ . Hence,

$$\int_{u_1}^{u_2} \frac{2\tilde{\theta}\tilde{\zeta}}{\tilde{r}^2}(\tilde{u}, \tilde{v}) d\tilde{u} \geq \frac{2c_\delta c_2 \delta^{s_2}}{\tilde{r}^2(U, V)} \int_{u_1}^{u_2} \frac{\tilde{\theta}}{|\tilde{\lambda}|}(\tilde{u}, \tilde{v}) d\tilde{u} \rightarrow +\infty \text{ as } \tilde{v} \nearrow V.$$

This shows that  $\tilde{\Gamma}_{\tilde{v}\tilde{v}}^{\tilde{v}}(u_1, \cdot) - \tilde{\Gamma}_{\tilde{v}\tilde{v}}^{\tilde{v}}(u_2, \cdot)$  tends to  $+\infty$ , which is a contradiction. Therefore, there is at most one  $u \in [\delta, U]$  for which  $\tilde{\Gamma}_{\tilde{v}\tilde{v}}^{\tilde{v}}(u, \cdot)$  is bounded.

Arguing as in the proof of Proposition 5.10, we use (81) and (18), and differentiate (80) to obtain

$$\begin{aligned} \tilde{\Gamma}_{\tilde{v}\tilde{v}}^{\tilde{v}}(U, \tilde{v}) &= -\frac{1}{(\widehat{1-\mu})^2(U, \tilde{v})} \frac{2\partial_{\tilde{v}}\tilde{\omega}(U, \tilde{v})}{\tilde{r}(U, \tilde{v})} \frac{1}{\tilde{\kappa}(U, \tilde{v})} \\ &= -\left(\frac{2\tilde{\kappa}\partial_{\tilde{v}}\tilde{\omega}}{\tilde{r}}\right)(U, \tilde{v}) \\ &= -\left(\frac{\tilde{\theta}^2}{\tilde{r}}\right)(U, \tilde{v}). \end{aligned} \quad (83)$$

From (83) we see that  $\tilde{\Gamma}_{\tilde{v}\tilde{v}}^{\tilde{v}}(U, \tilde{v})$  tends to  $-\infty$  as  $\tilde{v} \nearrow V$ .  $\square$

**Remark 5.4.** *Note that:*

- (i) if  $\tilde{\Gamma}_{\tilde{v}\tilde{v}}^{\tilde{v}}(u, \cdot)$  is bounded, then  $\tilde{\Gamma}_{\tilde{v}\tilde{v}}^{\tilde{v}}(u_1, \tilde{v}) \rightarrow +\infty$  as  $\tilde{v} \rightarrow V$  when  $u_1 < u$ , and  $\tilde{\Gamma}_{\tilde{v}\tilde{v}}^{\tilde{v}}(u_2, \tilde{v}) \rightarrow -\infty$  as  $\tilde{v} \rightarrow V$  when  $u_2 > u$ ;
- (ii) if  $\tilde{\Gamma}_{\tilde{v}\tilde{v}}^{\tilde{v}}(u, \tilde{v}) \rightarrow -\infty$ , then  $\tilde{\Gamma}_{\tilde{v}\tilde{v}}^{\tilde{v}}(u_2, \tilde{v}) \rightarrow -\infty$  as  $\tilde{v} \rightarrow V$  when  $u_2 > u$ ;
- (iii) if  $\tilde{\Gamma}_{\tilde{v}\tilde{v}}^{\tilde{v}}(u, \tilde{v}) \rightarrow +\infty$ , then  $\tilde{\Gamma}_{\tilde{v}\tilde{v}}^{\tilde{v}}(u_1, \tilde{v}) \rightarrow +\infty$  as  $\tilde{v} \rightarrow V$  when  $u_1 < u$ .

**5.2. Coordinates with  $v$  replaced by  $\hat{v} := \int_0^v \kappa(U, \bar{v}) d\bar{v}$ .** Assume there exists a positive sequence  $(u_n)$  converging to 0 such that  $\zeta_0(u_n) \neq 0$ . We change the  $v$  coordinate to

$$\hat{v} := \int_0^v \kappa(U, \bar{v}) d\bar{v}. \quad (84)$$

According to (28),  $\hat{V} := \int_0^\infty \kappa(U, \bar{v}) d\bar{v} < \infty$ . From Lemma 2.6,  $\kappa$  is  $C^1$ ; since  $\kappa$  is also positive, this change of coordinates is admissible ( $C^2$ ).

We denote by  $\hat{r}$ ,  $\hat{v}$ ,  $\hat{\lambda}$ ,  $\hat{\omega}$ ,  $\hat{\theta}$ ,  $\hat{\zeta}$  and  $\hat{\kappa}$  the functions written in the coordinates  $(u, \hat{v})$ . In particular,

$$\hat{\kappa}(u, \hat{v}) = \frac{\kappa(u, v)}{\kappa(U, v)} \quad \text{and} \quad \hat{\kappa}(U, \hat{v}) \equiv 1.$$

From  $\hat{\kappa}(U, \hat{v}) \equiv 1$  and (23), we get

$$\hat{\kappa}(u, \hat{v}) = e^{-\int_u^U \left(\frac{\hat{\zeta}^2}{\hat{r}\hat{v}}\right)(\bar{u}, \hat{v}) d\bar{u}}. \quad (85)$$

*Relationship between the  $\tilde{v}$  and the  $\hat{v}$  coordinates.* We now show that when  $|\frac{\theta}{\lambda}|$  is unbounded the change of coordinates from  $\tilde{v}$  to  $\hat{v}$  is not  $C^2$  at the Cauchy horizon. From (77) and (84), we write

$$\frac{d\tilde{v}}{dv}(v) = -\lambda(U, v)$$

and

$$\frac{d\hat{v}}{dv}(v) = \kappa(U, v).$$

So

$$\frac{d\tilde{v}}{d\hat{v}}(\hat{v}) = -(1-\mu)(U, v) = -(\widehat{1-\mu})(U, \hat{v}). \quad (86)$$



Using the chain rule, (10) and (20), we obtain

$$\begin{aligned}
 \frac{d^2\tilde{v}}{d\hat{v}^2}(\hat{v}) &= -\frac{d}{dv}(1-\mu)(U, v)\frac{dv}{d\hat{v}}(\hat{v}) \\
 &= \left(\frac{2}{r}\partial_v\varpi\right)(U, v)\frac{1}{\kappa(U, v)} + \text{bounded terms} \\
 &= \left(\frac{1}{r}\frac{\theta^2}{\kappa^2}\right)(U, v) + \text{bounded terms} \\
 &= \left[\frac{(1-\mu)^2}{r}\left(\frac{\theta}{\lambda}\right)^2\right](U, v) + \text{bounded terms.} \tag{87}
 \end{aligned}$$

According to Proposition 5.2,  $(1-\mu)(U, \cdot)$  is bounded away from zero. So indeed, we have

**Remark 5.5.** *The change of coordinates from  $\hat{v}$  to  $\tilde{v}$  is not  $C^2$  at the Cauchy horizon when  $|\frac{\theta}{\lambda}|$  is unbounded along  $u = U$ .*

The next result is a direct translation of Proposition 5.3 to the new coordinates  $(u, \hat{v})$ .

**Proposition 5.6.** *Suppose that*

$$\exists_{c>0} |\zeta_0(u)| \leq cu^s \text{ for some nonnegative } s > \frac{7\rho}{9} - 1,$$

and suppose there exists a positive sequence  $(u_n)$  converging to 0 such that  $\zeta_0(u_n) \neq 0$ . Then there exists  $U > 0$  such that for all  $0 < \delta < U$ , the functions  $\hat{r}$ ,  $\hat{v}$ ,  $\hat{\lambda}$ ,  $\hat{\varpi}$ ,  $\hat{\zeta}$  and  $\hat{\kappa}$  (but not necessarily  $\hat{\theta}$ ) admit continuous extensions to the closed rectangle  $[\delta, U] \times [0, \hat{V}]$ . Equations (15) to (19), (23), and (24) are satisfied on this set. Finally,  $(1-\mu)(u, \hat{V})$  is negative for  $u > 0$ .

*Proof.* This is a consequence of Proposition 5.2 and the fact that the map  $\hat{v} \mapsto \tilde{v}$  extends to a  $C^1$  map from  $[0, \hat{V}]$  to  $[0, V]$ . For example, to check (16) at the Cauchy horizon, note that from  $\partial_{\tilde{v}}\tilde{r}(u, \tilde{v}) = \tilde{\lambda}(u, \tilde{v})$  we conclude that

$$\begin{aligned}
 \partial_{\hat{v}}\hat{r}(u, \hat{v}) &= \partial_{\tilde{v}}\tilde{r}(u, \tilde{v})\frac{d\tilde{v}}{d\hat{v}}(\hat{v}) = \tilde{\lambda}(u, \tilde{v})[-(1-\mu)(U, v)] \\
 &= \frac{\lambda(u, v)}{\lambda(U, v)}(1-\mu)(U, v) = \frac{\lambda(u, v)}{\kappa(U, v)} = \hat{\lambda}(u, \hat{v}).
 \end{aligned}$$

□

The spherically symmetric Lorentzian manifold  $\mathcal{M}$  is now  $\hat{\mathcal{Q}} \times \mathbb{S}^2$ , where  $\hat{\mathcal{Q}}$  admits the global null coordinate system  $(u, \hat{v})$  defined on  $[0, U] \times [0, \hat{V}] \setminus \{(0, \hat{V})\}$ , and the metric is

$$g = -\hat{\Omega}^2(u, \hat{v}) dud\hat{v} + \hat{r}^2(u, \hat{v}) \sigma_{\mathbb{S}^2},$$

with  $\hat{\Omega}^2 = -4\hat{v}\hat{\kappa}$ . The field  $\hat{\phi}(u, \hat{v})$  equals  $\phi(u, v)$  and so  $\tilde{\phi}(u, \tilde{v})$ . The nonvanishing Christoffel symbols of the metric on  $\mathcal{M}$  are written as the ones above, with tildes replaced by hats. For example, instead of (81), we have

$$\hat{\Gamma}_{\hat{v}\hat{v}}^{\hat{v}} = \hat{\Omega}^{-2}\partial_{\hat{v}}(\hat{\Omega}^2) = \frac{\partial_{\hat{v}}\hat{\nu}}{\hat{\nu}} + \frac{\partial_{\hat{v}}\hat{\kappa}}{\hat{\kappa}}. \tag{88}$$

**Proposition 5.7.** *Suppose that*

$$\exists_{c_1, c_2 > 0} c_2 u^{s_2} \leq \zeta_0(u) \leq c_1 u^{s_1} \text{ for some } \frac{7\rho}{9} - 1 < s_1 \leq s_2 < \rho - 1.$$

For any  $\delta > 0$  the field  $|\frac{\hat{\theta}}{\lambda}|(u, \hat{v})$  tends to  $+\infty$  as  $\hat{v} \nearrow \hat{V}$ , uniformly for  $u \in [\delta, U]$ . Moreover,  $\hat{\Gamma}_{\hat{v}\hat{v}}^{\hat{v}}(u, \hat{v}) \rightarrow +\infty$  as  $\hat{v} \nearrow \hat{V}$  for each  $u \in [\delta, U]$ .

*Proof.* The fact that  $|\frac{\hat{\theta}}{\lambda}|(u, \hat{v})$  tends to  $+\infty$ , uniformly for  $u \in [\delta, U]$ , as  $\hat{v} \nearrow \hat{V}$  follows from Proposition 5.3 because this quantity is invariant under changes of coordinates.

The coordinate transformation from  $\tilde{v}$  to  $\hat{v}$  has regularity  $C^1([0, V]) \cap C^2([0, V[)$ . As a coordinate transformation is a diffeomorphism, we have that both  $\frac{d\hat{v}}{d\tilde{v}}$  and  $\frac{d\tilde{v}}{d\hat{v}}$  are always different from zero. Hence, the fact that  $\tilde{\lambda}$  is bounded and bounded away from zero in  $[\delta, U] \times [0, V]$  implies that  $\hat{\lambda}$  is bounded and bounded away from zero  $[\delta, U] \times [0, \hat{V}]$ . So, obviously,  $\hat{\kappa}$  is bounded in  $[\delta, U] \times [0, \hat{V}]$ . Therefore, the proof of Proposition 5.3 applies to the present case and for all  $u \in [\delta, U]$ , with one possible exception,  $\hat{\Gamma}_{\hat{v}\hat{v}}^{\hat{v}}(u, \hat{v})$  is unbounded as  $\hat{v} \nearrow \hat{V}$ .

In this case we have that  $\hat{\Gamma}_{\hat{v}\hat{v}}^{\hat{v}}(U, \hat{v})$  is bounded because  $\hat{\kappa}(U, \hat{v}) \equiv 1$ . It follows from Remark 5.4 that

$$\hat{\Gamma}_{\hat{v}\hat{v}}^{\hat{v}}(u, \hat{v}) \rightarrow +\infty \quad \text{as } \hat{v} \rightarrow \hat{V}$$

for each  $u \in [\delta, U]$ . □

**Remark 5.8.** *Examining the proof of Proposition 5.7, we see that we only used the specific form of the coordinates  $\hat{v}$  in the last paragraph. So, if  $\tilde{v} \mapsto \hat{v}$  is any coordinate transformation with regularity  $C^1([0, V]) \cap C^2([0, V[)$ , we conclude that there is at most one value  $u > 0$  for which  $\hat{\Gamma}_{\hat{v}\hat{v}}^{\hat{v}}(u, \hat{v})$  is bounded. If  $\hat{\Gamma}_{\hat{v}\hat{v}}^{\hat{v}}(\bar{u}, \hat{v})$  is bounded, then  $\hat{\Gamma}_{\hat{v}\hat{v}}^{\hat{v}}(u, \hat{v}) \rightarrow -\infty$  as  $\hat{v} \rightarrow \hat{V}$  for  $u > \bar{u}$ , and  $\hat{\Gamma}_{\hat{v}\hat{v}}^{\hat{v}}(u, \hat{v}) \rightarrow +\infty$  as  $\hat{v} \rightarrow \hat{V}$  for  $u < \bar{u}$ . This excludes the existence of  $C^{0,1}$  extensions of the metric using these coordinates.*

**Remark 5.9.** *Suppose that the hypotheses of Proposition 5.3 hold. Then  $\tilde{\Gamma}_{\tilde{v}\tilde{v}}^{\tilde{v}}(U, \tilde{v})$  tends to  $-\infty$  as  $\tilde{v} \rightarrow V$ , and  $\hat{\Gamma}_{\hat{v}\hat{v}}^{\hat{v}}(U, \hat{v})$  is bounded. From elementary Riemannian geometry we have*

$$\hat{\Gamma}_{\hat{v}\hat{v}}^{\hat{v}} = \frac{d\tilde{v}}{d\hat{v}} \tilde{\Gamma}_{\tilde{v}\tilde{v}}^{\tilde{v}} + \frac{d\hat{v}}{d\tilde{v}} \frac{d^2\tilde{v}}{d\hat{v}^2},$$

and so  $\frac{d^2\tilde{v}}{d\hat{v}^2}$  must blow up at the Cauchy horizon (as was already shown in (87) by direct computation). This again shows that the two coordinate systems  $(u, \tilde{v})$  and  $(u, \hat{v})$  are not  $C^2$  compatible. More generally, the same reasoning can be applied to show the  $C^2$  incompatibility of any two coordinate systems whose Christoffel symbols  $\Gamma_{\tilde{v}\tilde{v}}^{\tilde{v}}$  have different asymptotic behavior at the Cauchy horizon. In particular, different choices of  $U$  yield incompatible  $(u, \hat{v})$  coordinates (when  $\frac{\theta}{\lambda}$  is unbounded).

It turns out that, although unbounded, the Christoffel symbols of the  $(u, \hat{v})$  coordinates are in  $L^2$ .

**Proposition 5.10.** *Suppose that*

$$\exists_{c>0} |\zeta_0(u)| \leq cu^s \text{ for some nonnegative } s > \frac{7\rho}{9} - 1.$$

For any  $0 < \delta < U$ , the Christoffel symbols  $\hat{\Gamma}_{AB}^C$ ,  $\hat{\Gamma}_{AB}^u$ ,  $\hat{\Gamma}_{AB}^{\hat{v}}$ ,  $\hat{\Gamma}_{B\hat{u}}^A$ ,  $\hat{\Gamma}_{B\hat{v}}^A$  and  $\hat{\Gamma}_{uu}^u$  are bounded in  $[\delta, U] \times [0, \hat{V}]$ . Furthermore,  $\int_0^{\hat{V}} |\hat{\Gamma}_{\hat{v}\hat{v}}^{\hat{v}}|^2(u, \hat{v}) d\hat{v}$  and  $\int_0^{\hat{V}} |\hat{\theta}|^2(u, \hat{v}) d\hat{v}$  are bounded for  $u \in [\delta, U]$ . Consequently, the Christoffel symbols and  $\hat{\theta}$  (and also  $\hat{\zeta}$ ) belong to  $L^2(\mathcal{M}_\delta)$ , with  $\mathcal{M}_\delta$  the preimage of  $[\delta, U] \times [0, \hat{V}]$  by the double null coordinate system  $(u, \hat{v})$ .

*Proof.* In the proof of Proposition 5.6 we showed that all the functions in the first order system except  $\hat{\theta}$ , i.e. the functions  $\hat{r}$ ,  $\hat{v}$ ,  $\hat{\lambda}$ ,  $\hat{\omega}$ ,  $\hat{\zeta}$  and  $\hat{\kappa}$ , extend to continuous functions in  $[\delta, U] \times [0, \hat{V}]$ , with  $\hat{r} > 0$ ,  $\hat{v} < 0$  and  $\hat{\kappa} > 0$ . In addition, we proved that all the equations of the first order system (15)–(24), except (20), (21) and (22), are satisfied in  $[\delta, U] \times [0, \hat{V}]$ ; in particular (23) (the equation for  $\partial_u \hat{\kappa}$ ) is satisfied in this rectangle; moreover, the expression for  $\partial_u \hat{v}$  is obtained from

$$\hat{v}(u, \hat{v}) = -e^{-\int_0^{\hat{v}} \left( 2\hat{\kappa} \frac{1}{\hat{r}^2} \left( \frac{\hat{\omega}^2}{\hat{r}} + \frac{4}{3} \hat{r}^3 - \hat{\omega} \right) \right) (u, \bar{v}) d\bar{v}}.$$

Therefore,  $\hat{\Gamma}_{AB}^C$ ,  $\hat{\Gamma}_{AB}^u$ ,  $\hat{\Gamma}_{AB}^{\hat{v}}$ ,  $\hat{\Gamma}_{B\hat{u}}^A$ ,  $\hat{\Gamma}_{B\hat{v}}^A$ , and  $\hat{\Gamma}_{uu}^u$  are bounded in  $[\delta, U] \times [0, \hat{V}]$ .

By (20), we know that  $\int_0^{\hat{V}} |\hat{\theta}|^2(u, \hat{v}) d\hat{v}$  is bounded for  $u \in [\delta, U]$ . Differentiating both sides of (85) with respect to  $\hat{v}$ , and using (18) and (22), we get

$$\begin{aligned} \frac{\partial_{\hat{v}} \hat{\kappa}}{\hat{\kappa}}(u, \hat{v}) &= -2 \int_u^U \frac{\hat{\zeta} \partial_{\hat{v}} \hat{\zeta}}{\hat{r} \hat{v}}(\bar{u}, \hat{v}) d\bar{u} + \int_u^U \frac{\hat{\zeta}^2 \hat{\lambda}}{\hat{r}^2 \hat{v}}(\bar{u}, \hat{v}) d\bar{u} \\ &\quad + \int_u^U \frac{\hat{\zeta}^2 \partial_{\hat{v}} \hat{v}}{\hat{r} \hat{v}^2}(\bar{u}, \hat{v}) d\bar{u} \\ &= 2 \int_u^U \frac{\hat{\zeta} \hat{\theta}}{\hat{r}^2}(\bar{u}, \hat{v}) d\bar{u} + \int_u^U \frac{\hat{\zeta}^2 \hat{\lambda}}{\hat{r}^2 \hat{v}}(\bar{u}, \hat{v}) d\bar{u} \\ &\quad + \int_u^U \frac{\hat{\zeta}^2 \hat{\kappa} \partial_{\hat{r}}(\widehat{1 - \mu})}{\hat{r} \hat{v}}(\bar{u}, \hat{v}) d\bar{u} \end{aligned}$$

Let  $0 < \delta < U$ . From the proof of Theorem 2.3 (see Part 2) we have  $\hat{v}(u, \hat{V}) \leq -c < 0$  for  $u \in [\delta, U]$ . The previous equality, (18) and (88) then imply that there exists a  $C > 0$  such that

$$|\hat{\Gamma}_{\hat{v}\hat{v}}^{\hat{v}}|(u, \hat{v}) \leq C \left( 1 + \int_u^U |\hat{\theta}|(\bar{u}, \hat{v}) d\bar{u} \right)$$

for  $(u, \hat{v}) \in [\delta, U] \times [0, \hat{V}]$ , and so, using Hölder's inequality,

$$|\hat{\Gamma}_{\hat{v}\hat{v}}^{\hat{v}}|^2(u, \hat{v}) \leq C \left( 1 + \int_u^U |\hat{\theta}|^2(\bar{u}, \hat{v}) d\bar{u} \right).$$

Therefore,

$$\begin{aligned} \int_0^{\hat{V}} |\hat{\Gamma}_{\hat{v}\hat{v}}|^2(u, \hat{v}) d\hat{v} &\leq C \left( 1 + \int_0^{\hat{V}} \int_u^U |\hat{\theta}|^2(\bar{u}, \hat{v}) d\bar{u} d\hat{v} \right) \\ &= C \left( 1 + \int_u^U \int_0^{\hat{V}} |\hat{\theta}|^2(\bar{u}, \hat{v}) d\hat{v} d\bar{u} \right) \\ &\leq C, \end{aligned}$$

for  $u \in [\delta, U]$ .

Finally, note that the square of the  $L^2$  norm of a function  $\hat{h}$  on  $\mathcal{M}_\delta$  is given by

$$\int_{\mathcal{M}_\delta} \hat{h}^2 dV_4 = 4\pi \int_{[\delta, U] \times [0, \hat{V}]} \left[ \hat{r}^2 \frac{\hat{\Omega}^2}{2} \hat{h}^2 \right] (u, \hat{v}) dud\hat{v}.$$

Since the functions  $\hat{r}$  and  $\hat{\Omega}^2 = -4\hat{\nu}\hat{\kappa}$  are bounded in  $[\delta, U] \times [0, \hat{V}]$ , we conclude that the Christoffel symbols and  $\hat{\theta}$  are in  $L^2(\mathcal{M}_\delta)$ .  $\square$

So, in our framework the Christodoulou-Chruściel formulation of strong cosmic censorship (see [1]) does not hold:

**Corollary 5.11.** *Suppose that*

$$\exists_{c>0} |\zeta_0(u)| \leq cu^s \text{ for some nonnegative } s > \frac{7\rho}{9} - 1.$$

*Then  $(\mathcal{M}, g)$  and  $\hat{\phi}$  extend across the Cauchy horizon (in a non-unique way) to spherically symmetric  $(\check{\mathcal{M}}, \check{g})$  and  $\check{\phi}$ , with  $\check{\mathcal{M}} = \check{\mathcal{Q}} \times \mathbb{S}^2$  a  $C^2$  manifold and*

$$\check{g} = -\check{\Omega}^2(u, \hat{v}) dud\hat{v} + \check{r}^2(u, \hat{v}) \sigma_{\mathbb{S}^2}$$

*a  $C^0$  metric on  $\check{\mathcal{M}}$ . Here  $\check{\mathcal{Q}}$  has a global null coordinate system  $(u, \hat{v})$  defined on  $[0, U] \times [0, \hat{V}] \setminus \{(0, \hat{V})\} \cup \mathcal{V}$ , with  $\mathcal{V}$  a neighborhood of  $]0, U] \times \{\hat{V}\}$ . Furthermore,*

$$\check{\Gamma} \in L_{\text{loc}}^2 \quad \text{and} \quad \check{\phi} \in H_{\text{loc}}^1. \quad (89)$$

*Proof.* For  $(u, \hat{v})$  with  $u > 0$  and  $\hat{v} > \hat{V}$ , define

$$\check{\Omega}^2(u, \hat{v}) = \hat{\Omega}^2(u, \hat{V}), \quad \check{\phi}(u, \hat{v}) = \hat{\phi}(u, \hat{V}),$$

and

$$\check{r}(u, \hat{v}) = \hat{r}(u, \hat{V}) + \hat{\lambda}(u, \hat{V})(\hat{v} - \hat{V}).$$

Choose a neighborhood  $\mathcal{V}$  of  $]0, U] \times \{\hat{V}\}$  such that  $\check{r} > 0$  on  $[0, U] \times [0, \hat{V}] \setminus \{(0, \hat{V})\} \cup \mathcal{V}$ . The extensions  $\check{\Omega}^2$ ,  $\check{\phi}$  and  $\check{r}$  of  $\hat{\Omega}^2$ ,  $\hat{\phi}$  and  $\hat{r}$  are continuous. For  $u > 0$  and  $\hat{v} > \hat{V}$ , we get

$$\begin{aligned} \partial_u \check{\Omega}^2(u, \hat{v}) &= \partial_u \hat{\Omega}^2(u, \hat{V}), & \partial_{\hat{v}} \check{\Omega}^2(u, \hat{v}) &= 0, \\ \partial_u \check{\phi}(u, \hat{v}) &= \partial_u \hat{\phi}(u, \hat{V}), & \partial_{\hat{v}} \check{\phi}(u, \hat{v}) &= 0, \\ \check{\nu}(u, \hat{v}) &= \hat{\nu}(u, \hat{V}) + \partial_u \hat{\lambda}(u, \hat{V})(\hat{v} - \hat{V}) \end{aligned}$$

and

$$\check{\lambda}(u, \hat{v}) = \hat{\lambda}(u, \hat{V}).$$

Clearly,  $\partial_u \check{\Omega}^2$ ,  $\check{\lambda}$  and  $\check{\nu}$  are also continuous. Therefore,  $\check{\Gamma}_{AB}^C$ ,  $\check{\Gamma}_{AB}^u$ ,  $\check{\Gamma}_{AB}^{\hat{v}}$ ,  $\check{\Gamma}_{B\hat{u}}^A$ ,  $\check{\Gamma}_{B\hat{v}}^A$ , and  $\check{\Gamma}_{uu}^u$  are continuous, and so is the field  $\check{\zeta}$ . Finally,  $\check{\Gamma}_{\hat{v}\hat{v}}^{\hat{v}}$  and  $\check{\theta}$  are

zero for  $\hat{v} > \hat{V}$ . It would be easy to construct other extensions of  $(\mathcal{M}, g)$  and  $\hat{\phi}$  satisfying (89).  $\square$

Note that there is no guarantee that the extensions above satisfy the Einstein equations. Moreover, the function  $\hat{\theta}$  may not admit a continuous extension to the Cauchy horizon.

**Remark 5.12.** *Since in the previous extension*

$$\partial_{\hat{v}}\check{\nu}(u, \hat{v}) = \partial_u\check{\lambda}(u, \hat{v}) = \partial_u\hat{\lambda}(u, \hat{V}),$$

for  $\hat{v} > \hat{V}$ , we constructed a  $C^0$  extension of the metric such that  $(\check{\Gamma} \in L^2_{\text{loc}}$ ,  $\check{\phi} \in H^1_{\text{loc}}$  and) the second mixed derivatives of  $\check{r}$  are continuous. This would not be possible if  $\hat{\omega}(\cdot, \hat{V})$  were  $+\infty$  (see (17) and (18)). In [5, Theorem 11.1] M. Dafermos constructs  $C^0$  extensions of the metric without assuming any restriction on the continuous function  $\zeta_0$ , so without any control on  $\hat{\omega}(\cdot, \hat{V})$ .

## 6. EXTENSIONS OF SOLUTIONS BEYOND THE CAUCHY HORIZON

It is clear that in order to improve on the results of the previous section we need to control the field  $\frac{\theta}{\lambda}$ . In view of Proposition 5.3, this requires a stronger restriction on the exponent  $s$ . Once the field is controlled, it turns out to be possible to construct smooth extensions of our spacetime which in fact are solutions of the Einstein equations.

More precisely, in the main part of this section we assume that

$$\exists_{c>0} |\zeta_0(u)| \leq cu^s \text{ for some } s > \frac{13\rho}{9} - 1.$$

In Lemma 6.1, we obtain the desired bound for  $\frac{\theta}{\lambda}$  in  $J^+(\gamma)$ . We then start by proving that our solution of the first order system (15)–(24) can be extended to the closed rectangle  $[\delta, U] \times [0, V]$ , for any  $0 < \delta < U$ , while still satisfying (15)–(24). By taking the values of the functions at the Cauchy horizon as initial data on  $[\delta, U] \times \{V\}$ , and choosing new initial data on  $\{U\} \times [V, V + \varepsilon]$ , we can build (non-unique) extensions of the solution beyond the Cauchy horizon. The new initial data can be chosen with the required regularity so that we obtain classical solutions of the Einstein equations. We finish the section by analyzing the behavior of the Kretschmann scalar at the Cauchy horizon, under the hypotheses used in this and in the previous sections.

**Lemma 6.1** (Bounding  $\frac{\theta}{\lambda}$ ). *Suppose that*

$$\exists_{c>0} |\zeta_0(u)| \leq cu^s \text{ for some } s > \frac{13\rho}{9} - 1.$$

*Then there exists a constant  $C > 0$  such that*

$$\left| \frac{\theta}{\lambda} \right|(u, v) \leq C, \tag{90}$$

*for  $(u, v) \in J^+(\gamma)$ , provided  $U$  is sufficiently small. Furthermore,*

$$\lim_{(u,v) \rightarrow (0,\infty)} \left| \frac{\theta}{\lambda} \right|(u, v) = 0. \tag{91}$$

*Proof.* Integrating (115), we obtain

$$\begin{aligned} \frac{\theta}{\lambda}(u, v) &= \frac{\theta}{\lambda}(u_\gamma(v), v) e^{-\int_{u_\gamma(v)}^u \left[\frac{\nu}{1-\mu} \partial_r(1-\mu)\right](\tilde{u}, v) d\tilde{u}} \\ &\quad - \int_{u_\gamma(v)}^u \frac{\zeta}{r}(\tilde{u}, v) e^{-\int_{\tilde{u}}^u \left[\frac{\nu}{1-\mu} \partial_r(1-\mu)\right](\tilde{u}, v) d\tilde{u}} d\tilde{u}. \end{aligned} \quad (92)$$

By Theorem 4.1, we know that we have  $|\partial_r(1-\mu) + 2k_-| < \delta$  in  $J^+(\gamma)$  for sufficiently small  $U$ . Using (45) and (50),

$$\begin{aligned} \left| \frac{\theta}{\lambda} \right| (u_\gamma(v), v) e^{-\int_{u_\gamma(v)}^u \left[\frac{\nu}{1-\mu} \partial_r(1-\mu)\right](\tilde{u}, v) d\tilde{u}} \\ \leq C e^{-2\left(\frac{k_+(s+1)}{1+\beta} - k_- \beta - \delta\right)v} e^{2\left(\frac{k_-}{1+\beta} + \delta\right)v} \\ \leq C e^{-\frac{2k_+}{1+\beta}(s+1-\rho(\beta^2+\beta+1)-\delta)v}. \end{aligned} \quad (93)$$

This exponent can be made negative for

$$s > \rho(\beta^2 + \beta + 1) - 1. \quad (94)$$

Now, according to (121)

$$\frac{\nu}{1-\mu}(\tilde{u}, v) \leq \frac{1+\delta}{2k_+\tilde{u}},$$

due to the monotonicity of  $\frac{\nu}{1-\mu}$ . Thus,

$$e^{-\int_{\tilde{u}}^u \left[\frac{\nu}{1-\mu} \partial_r(1-\mu)\right](\tilde{u}, v) d\tilde{u}} \leq e^{(\rho+\delta) \ln\left(\frac{u}{\tilde{u}}\right)} = \left(\frac{u}{\tilde{u}}\right)^{\rho+\delta} \leq \left(\frac{U}{\tilde{u}}\right)^{\rho+\delta}. \quad (95)$$

Combining this with (73), if  $s > \rho(\beta^2 + 1) - 1$  and if the parameters are chosen appropriately, we get

$$\begin{aligned} \int_{u_\gamma(v)}^u \frac{|\zeta|}{r}(\tilde{u}, v) e^{-\int_{\tilde{u}}^u \left[\frac{\nu}{1-\mu} \partial_r(1-\mu)\right](\tilde{u}, v) d\tilde{u}} d\tilde{u} \\ \leq C \int_{u_\gamma(v)}^u \tilde{u}^{s-\rho\beta^2-\delta} \tilde{u}^{-\rho-\delta} d\tilde{u} \\ \leq C u^{s+1-\rho(\beta^2+1)-\delta}. \end{aligned} \quad (96)$$

Using (93) and (96) in (92), taking into account that the right-hand side of (94) would be  $\frac{13\rho}{9} - 1$  if  $\beta$  were  $\frac{1}{3}$ , and recalling that we can choose  $\beta = \frac{1}{3} + \varepsilon$ , we obtain (90).

To prove the last assertion, notice that for  $(u, v) \in J^-(\gamma) \cap J^+(\Gamma_{\tilde{r}_-})$  the estimate on the right-hand side of (45) applies since  $u \leq u_\gamma(v)$ . Also, recall (122). All this information, together with (92) and the bounds (93) and (96), implies (91).  $\square$

**Theorem 6.2** (Extending the solution of the first order system up to the Cauchy horizon). *Suppose that*

$$\exists_{c>0} |\zeta_0(u)| \leq cu^s \text{ for some } s > \frac{13\rho}{9} - 1.$$

*Then there exists  $U > 0$  such that for all  $0 < \delta < U$ , the functions  $\tilde{r}, \tilde{v}, \tilde{\lambda}, \tilde{\omega}, \tilde{\theta}, \tilde{\zeta}$  and  $\tilde{\kappa}$  satisfy the first order system (15)–(24) on the closed rectangle  $[\delta, U] \times [0, V]$ .*

**Remark 6.3.** Theorem 3.2 and Proposition 5.3 imply that there is no hope of lowering the constant  $\frac{13}{9}$  below 1.

*Proof of Theorem 6.2.* We fix  $0 < \delta < U$ . We already did most of the work in Proposition 5.2. So, we just need to prove the assertion for  $\tilde{\theta}$  and that (20), (21) and (22) are satisfied on  $[\delta, U] \times [0, V]$ . As before, we proceed in three steps.

*Step 1.* We prove that  $\tilde{\theta}$  satisfies

$$\forall \varepsilon > 0 \exists \tilde{\delta} > 0 \forall u \in [\delta, U] \quad |\tilde{v} - V| < \tilde{\delta} \Rightarrow |\tilde{\theta}(u, \tilde{v}) - \tilde{\theta}(u, V)| < \varepsilon. \quad (97)$$

We let  $v \nearrow \infty$  in (92). Taking into account the estimate (93) for the first term on the right-hand side, and using Lebesgue's Dominated Convergence Theorem and (96) for the second term on the right-hand side, we conclude that

$$\frac{\theta}{\lambda}(u, \infty) = - \int_0^u \frac{\zeta}{r}(\tilde{u}, \infty) e^{-\int_{\tilde{u}}^u [\frac{\nu}{1-\mu} \partial_r(1-\mu)](\tilde{u}, \infty) d\tilde{u}} d\tilde{u}.$$

Hence

$$\left(\frac{\tilde{\theta}}{\tilde{\lambda}}\right)(u, V) = \frac{\theta}{\lambda}(u, \infty)$$

is well defined and  $\tilde{\theta}(u, V) = \left(\frac{\tilde{\theta}}{\tilde{\lambda}}\right)(u, V) \tilde{\lambda}(u, V)$  is also well defined. We now wish to prove uniform convergence of  $\frac{\theta}{\lambda}(\cdot, v)$  to  $\frac{\theta}{\lambda}(\cdot, \infty)$ , as  $v \nearrow \infty$ . We write

$$\begin{aligned} \frac{\theta}{\lambda}(u, v) - \frac{\theta}{\lambda}(u, \infty) &= \frac{\theta}{\lambda}(u_{\gamma}(v), v) e^{-\int_{u_{\gamma}(v)}^u [\frac{\nu}{1-\mu} \partial_r(1-\mu)](\tilde{u}, v) d\tilde{u}} \\ &\quad - \int_{\hat{\delta}}^u \frac{\zeta}{r}(\tilde{u}, v) e^{-\int_{\tilde{u}}^u [\frac{\nu}{1-\mu} \partial_r(1-\mu)](\tilde{u}, v) d\tilde{u}} d\tilde{u} \\ &\quad + \int_{\hat{\delta}}^u \frac{\zeta}{r}(\tilde{u}, \infty) e^{-\int_{\tilde{u}}^u [\frac{\nu}{1-\mu} \partial_r(1-\mu)](\tilde{u}, \infty) d\tilde{u}} d\tilde{u} \\ &\quad - \int_{u_{\gamma}(v)}^{\hat{\delta}} \frac{\zeta}{r}(\tilde{u}, v) e^{-\int_{\tilde{u}}^u [\frac{\nu}{1-\mu} \partial_r(1-\mu)](\tilde{u}, v) d\tilde{u}} d\tilde{u} \\ &\quad + \int_0^{\hat{\delta}} \frac{\zeta}{r}(\tilde{u}, \infty) e^{-\int_{\tilde{u}}^u [\frac{\nu}{1-\mu} \partial_r(1-\mu)](\tilde{u}, \infty) d\tilde{u}} d\tilde{u} \\ &=: I + II + III + IV + V. \end{aligned}$$

Suppose that we are given  $\varepsilon > 0$ . Notice the upper limits of the integrals in  $IV$  and  $V$ : the outer integrals have upper limit  $\hat{\delta}$ , while the inner integrals have upper limit  $u$ . Nevertheless, we may do computations similar to (96), using (95), to conclude that we may choose  $\hat{\delta} > 0$  so that  $|IV| + |V| < \frac{\varepsilon}{3}$ , for all  $u \in [\delta, U]$ . We fix such a  $\hat{\delta}$ . By (93), there exists  $\tilde{V}_\varepsilon > 0$  such that for  $v \geq \tilde{V}_\varepsilon$  we have  $|I| < \frac{\varepsilon}{3}$ , again for all  $u \in [\delta, U]$ . When estimating  $|II + III|$  we replace the upper limit of integration  $u$  by  $U$ . Finally, by uniform convergence of the functions in the integral  $II$  to the functions in the integral  $III$ , in  $[\hat{\delta}, U]$ , there exists  $V_\varepsilon \geq \tilde{V}_\varepsilon$  such that  $|II + III| < \frac{\varepsilon}{3}$ , for  $v \geq V_\varepsilon$ . So for  $v \geq V_\varepsilon$  and for all  $u \in [\delta, U]$ , we have

$$\left| \frac{\theta}{\lambda}(u, v) - \frac{\theta}{\lambda}(u, \infty) \right| < \varepsilon.$$

This establishes the desired uniform convergence.

*Step 2.* As in Step 2 of the proof of Proposition 5.2, we conclude that  $\tilde{\theta}$  is continuous in the closed rectangle  $[\delta, U] \times [0, V]$ .

*Step 3.* As in Step 3 of the proof of Proposition 5.2, we conclude that (20), (21) and (22) are satisfied also on the segment  $[\delta, U] \times \{V\}$ .  $\square$

**Remark 6.4.** *The function  $\tilde{\omega}(U, \cdot)$  is continuously differentiable on  $[0, V]$  due to (20).*

*On the choice of initial data beyond the Cauchy horizon.* Fix  $0 < \hat{\varepsilon} < \tilde{r}(U, V)$ , and consider the continuous extension  $\tilde{\lambda}(U, \cdot) \equiv -1$  to the interval  $[0, V + \hat{\varepsilon}]$ . According to this choice, define

$$\tilde{r}(U, \tilde{v}) = \tilde{r}(U, V) + \int_V^{\tilde{v}} \tilde{\lambda}(U, \bar{v}) d\bar{v} = \tilde{r}(U, V) - (\tilde{v} - V),$$

for  $\tilde{v} \in ]V, V + \hat{\varepsilon}]$ . The upper bound on  $\hat{\varepsilon}$  is imposed to guarantee that

$$\tilde{r}(U, V + \hat{\varepsilon}) = \tilde{r}(U, V) - \hat{\varepsilon} > 0.$$

Choose a continuously differentiable extension of  $\tilde{\omega}(U, \cdot)$  to the interval  $[0, V + \hat{\varepsilon}]$ , with  $\partial_{\tilde{v}} \tilde{\omega}(U, \cdot) \geq 0$ , for  $\tilde{v} \in ]V, V + \hat{\varepsilon}]$ . Since  $\widetilde{(1 - \mu)}(U, V) < 0$ , by continuity, there exists  $0 < \varepsilon \leq \hat{\varepsilon}$  such that

$$\widetilde{(1 - \mu)}(U, \tilde{v}) = \left( 1 - \frac{2\tilde{\omega}}{\tilde{r}} + \frac{e^2}{\tilde{r}^2} - \frac{\Lambda}{3}\tilde{r}^2 \right) (U, \tilde{v}) < 0,$$

for  $\tilde{v} \in ]V, V + \varepsilon]$ . For  $\tilde{v} \in ]V, V + \varepsilon]$ , define

$$\tilde{\kappa}(U, \tilde{v}) = \frac{-1}{\widetilde{(1 - \mu)}(U, \tilde{v})} \quad (98)$$

and

$$\tilde{\theta}(U, \tilde{v}) = \text{sign } \tilde{\theta}(U, V) \sqrt{2\tilde{\kappa}\partial_{\tilde{v}}\tilde{\omega}}(U, \tilde{v}). \quad (99)$$

Take the sign of  $\tilde{\theta}(U, V)$  to be +1 if  $\tilde{\theta}(U, V) \geq 0$ , and -1 if  $\tilde{\theta}(U, V) < 0$ . These choices guarantee (30) and (31). Together with the values of  $\tilde{r}(u, V)$ ,  $\tilde{\nu}(u, V)$  and  $\tilde{\zeta}(u, V)$ , they provide initial data for the first order system (15)–(24) on  $]0, U] \times \{V\} \cup \{U\} \times [V, V + \varepsilon]$ .

**Theorem 6.5** (Extending the solution of the first order system beyond the Cauchy horizon). *Suppose that*

$$\exists_{c>0} |\zeta_0(u)| \leq cu^s \text{ for some } s > \frac{13\rho}{9} - 1.$$

*Then there exist (non-unique) extensions of the solution of the first order system (15)–(24) beyond the Cauchy horizon, which are still solutions of (15)–(24).*

*Proof.* Choose any continuously differentiable extension of  $\tilde{\omega}(U, \cdot)$ , with  $\partial_{\tilde{v}} \tilde{\omega}(U, \cdot) \geq 0$ . As described above, this determines initial data for the first order system (15)–(24) on  $]0, U] \times \{V\} \cup \{U\} \times [V, V + \varepsilon]$ , for some  $\varepsilon > 0$ . According to Theorem 2.5, there exists a unique solution defined on a maximal reflected past set  $\mathcal{R}$  containing a neighborhood of  $]0, U] \times \{V\} \cup \{U\} \times [V, V + \varepsilon]$ . This is an extension of the original solution beyond the Cauchy horizon: as explained in Part 1, solutions of (15)–(24) can be glued



along a common edge of two rectangles provided that all functions coincide on that edge, since the extended functions are clearly continuous and the equations imply the continuity of the relevant partial derivatives.  $\square$

**Remark 6.6.** *The original version of the existence and uniqueness theorem in Part 1 could have been used here instead of Theorem 2.5, by defining our coordinate  $\tilde{v}$  using the values of  $r(\delta, \cdot)$  instead of the values  $r(U, \cdot)$ , i.e. we could have replaced (77) by*

$$\tilde{v} = f(v) = r(\delta, 0) - r(\delta, v).$$

*In this case, we should consider the first order system with initial data on  $[\delta, U] \times \{V\} \cup \{\delta\} \times [V, V + \varepsilon]$ . We would then obtain an extension of the solution to  $[\delta, U] \times [V, V + \tilde{\varepsilon}]$  for some  $0 < \tilde{\varepsilon} < \varepsilon$ . However, our approach above, using Theorem 2.5, guarantees that the domain of our extended solution contains a neighborhood of the whole Cauchy horizon  $]0, U] \times \{V\}$ . If we had insisted on using the original existence and uniqueness theorem in Part 1, we would only have known that there existed a solution whose domain contained a neighborhood of  $]0, U] \times \{V\}$ , for  $\delta$  arbitrarily small; but if  $\delta$  changed, the solution might change, because we would have to change the initial data.*

We now wish to see that the solution of our first order system corresponds to a solution of the Einstein equations. Using Propositions 2.7 and 2.8, we know that this is the case provided that the regularity hypothesis (h4) (see Section 2) is satisfied, which it is. Indeed, the extended solution is a solution of the backward problem where  $\tilde{\lambda}(U, \tilde{v}) \equiv -1$  and  $\tilde{\kappa}(U, \tilde{v})$  are  $C^1$  on  $[0, V + \varepsilon]$  by our choice of initial data. On the other hand,  $\tilde{\nu}(u, 0) = \nu_0(u) \equiv -1$ . Hence, we proved

**Theorem 6.7** (Extending the solution of the Einstein equations beyond the Cauchy horizon). *Under the hypotheses of Theorem 6.5, there exists a neighborhood  $\mathcal{V}$  of  $]0, U] \times \{V\}$  such that the extended functions  $\check{r}$ ,  $\check{\phi}$  and  $\check{\Omega}$  are (classical) solutions of the Einstein equations (2), (3), (4), (5) and (6) in  $[0, U] \times [0, V] \setminus \{(0, V)\} \cup \mathcal{V}$ .*

**Remark 6.8.** *By Lemma 2.6 we conclude that  $\check{r}$  is  $C^2$ , and  $\check{\nu}$  and  $\check{\kappa}$  are  $C^1$ . Therefore,  $\check{\Omega}^2$  is  $C^1$ , and so the metric is also  $C^1$ . The field  $\check{\phi}$  is also  $C^1$  because  $\check{\theta}$  and  $\check{\zeta}$  are continuous. Furthermore,  $\partial_u \partial_{\tilde{v}} \check{\Omega}^2$  exists and is continuous in this  $(u, \tilde{v})$  chart. We emphasize that  $\check{\Omega}^2$  does not have to be  $C^2$  in this  $(u, \tilde{v})$  chart. Indeed,*

$$\check{\Omega}^2(u, 0) = -4\nu_0(0)\kappa(u, 0) = 4e^{-\int_0^u \frac{\zeta_0^2(u')}{r(u', 0)} du'}.$$

*This implies*

$$\partial_u \check{\Omega}^2(u, 0) = -4 \frac{\zeta_0^2(u)}{r(u, 0)} e^{-\int_0^u \frac{\zeta_0^2(u')}{r(u', 0)} du'}, \quad (100)$$

*and*

$$\partial_u^2 \check{\Omega}^2(u, 0) = 4 \left( -\frac{\zeta_0^2(u)}{r^2(u, 0)} + \frac{\zeta_0^4(u)}{r^2(u, 0)} - \frac{(\zeta_0^2)'(u)}{r(u, 0)} \right) e^{-\int_0^u \frac{\zeta_0^2(u')}{r(u', 0)} du'},$$

with  $r(u, 0) = r_+ - u$ . So, if  $0 \leq u \leq U$  is a point where  $\zeta_0^2$  is not differentiable, then  $\partial_u^2 \check{\Omega}^2(u, 0)$  does not exist.

*The Kretschmann scalar.* Consider  $\mathcal{M}$  as a  $C^3$  manifold. We finish with some remarks about the behavior of the Kretschmann scalar

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta},$$

whose blowup prevents the existence of  $C^2$  extensions of the metric across the Cauchy horizon. A straightforward, though lengthy, computation shows that

$$\begin{aligned} R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} &= \frac{16}{r^6} \left[ \left( \varpi - \frac{3e^2}{2r} + \frac{\Lambda}{6}r^3 \right) + \frac{r(1-\mu)}{2} \left( \frac{\zeta}{\nu} \right) \left( \frac{\theta}{\lambda} \right) \right]^2 \\ &\quad + \frac{16}{r^6} \left( \varpi - \frac{e^2}{2r} + \frac{\Lambda}{6}r^3 \right)^2 + \frac{16}{r^6} \left( \varpi - \frac{e^2}{r} - \frac{\Lambda}{3}r^3 \right)^2 \\ &\quad + 4 \frac{(1-\mu)^2}{r^4} \left( \frac{\zeta}{\nu} \right)^2 \left( \frac{\theta}{\lambda} \right)^2 \end{aligned}$$

(see [9, Section 2], [4, Section 2] and [8, Appendix A]). Note that if  $e = \Lambda = 0$ , the Kretschmann scalar reduces to  $\frac{48\varpi^2}{r^6}$ , the well known value for the Schwarzschild metric.

**Remark 6.9** (Kretschmann scalar).

(i) Under the hypotheses of Theorem 3.2, for each  $0 < u \leq U$ ,

$$(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})(u, \tilde{v}) \rightarrow \infty, \text{ as } \tilde{v} \nearrow V.$$

(ii) Under the hypotheses of Theorem 6.2,

$$\exists C > 0 \quad |R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}| \leq C.$$

*Proof.* In case (i), the conclusion is immediate if  $\tilde{\varpi}(u, V) = \infty$ . When  $\tilde{\varpi}(u, V) < \infty$ , we know that  $\tilde{\varpi}(u, V)$  is close to  $\varpi_0$  for small  $u$ . We have estimates (138) and (111), for  $-\nu$  from above and for  $\zeta$  from below, respectively, and also that  $(1-\mu)(u, \cdot)$  is bounded from above by a negative constant (see the proof of Proposition 5.2; it applies to the present situation because we only need  $\tilde{\varpi}(u, V)$  to be close to  $\varpi_0$  to show that  $\tilde{\nu}(u, V) < 0$ ). Therefore, the result follows from  $|\frac{\tilde{\theta}}{\lambda}|(u, \tilde{v}) \rightarrow +\infty$ , as  $\tilde{v} \nearrow V$ , for  $u > 0$ .

In case (ii), the renormalized mass  $\varpi$  and  $|\frac{\tilde{\theta}}{\lambda}|$  are bounded (see (90)).  $\square$

#### APPENDIX A. ON THE CHOICE OF THE PARAMETERS AND ITS CONSEQUENCES

The objective of this appendix is to study the behavior of  $\rho$  (defined in (38), the quotient of the surface gravities at  $r_-$  and at  $r_+$  of the reference subextremal Reissner-Nordström black hole) as a function of the parameters  $\Lambda$ ,  $\varpi_0$  and  $e$ . It turns out that it is easiest to express  $\rho$  in terms of the new parameters  $\sigma$  and  $\Upsilon$ , defined in (101). The formula for  $\rho$  in terms of  $\sigma$  and  $\Upsilon$  is given in (103). At the end of this appendix, the reader can find a figure showing the behavior of  $\rho$  in the  $(\sigma, \Upsilon)$  plane.

We consider the fourth order polynomial

$$p(r) := r^2(1 - \mu)(r, \varpi_0) = -\frac{\Lambda}{3}r^4 + r^2 - 2\varpi_0 r + e^2.$$

Since we assume  $p$  has zeros at  $r_-$  and  $r_+$ , it can be factored as

$$p(r) = [r^2 - (r_+ + r_-)r + r_-r_+] \left[ -\frac{\Lambda}{3}r^2 + cr + \frac{e^2}{r_-r_+} \right].$$

The constant  $c$  can be computed by imposing that the coefficient of  $p$  in  $r^3$  is equal to zero. We obtain  $c = -\frac{\Lambda}{3}(r_- + r_+)$ . Hence,  $p$  can be factored as

$$p(r) = [r^2 - (r_+ + r_-)r + r_-r_+] \left[ -\frac{\Lambda}{3}r^2 - \frac{\Lambda}{3}(r_- + r_+)r + \frac{e^2}{r_-r_+} \right].$$

Since the coefficient of  $p$  in  $r$  is equal to  $-2\varpi_0$ , we must have

$$\varpi_0 = \frac{e^2}{2r_-} + \frac{e^2}{2r_+} + \frac{\Lambda}{6}(r_- + r_+)r_-r_+.$$

On the other hand, since the coefficient of  $p$  in  $r^2$  is equal to 1, we must have

$$\frac{e^2}{r_-r_+} = 1 - \frac{\Lambda}{3}(r_-^2 + r_-r_+ + r_+^2).$$

We define

$$\sigma := \frac{r_+}{r_-} \quad \text{and} \quad \Upsilon := \frac{\Lambda r_-^2}{3}. \quad (101)$$

Then

$$\frac{e^2}{r_-r_+} = 1 - \Upsilon(\sigma^2 + \sigma + 1).$$

A simple computation shows that

$$\frac{\varpi_0}{r_-} = \frac{1}{2}(\sigma + 1)[1 - \Upsilon(\sigma^2 + 1)].$$

Of course, we could think of  $\Lambda$ ,  $\varpi_0$  and  $e$  as the independent parameters, and use the equation  $p(r) = 0$  to determine  $r_-$  and  $r_+$ . Instead, we think of  $r_-$ ,  $r_+$  and  $\Lambda$  as the independent parameters, and  $\varpi_0$  and  $e$  as the dependent ones. More precisely, we regard  $r_-$ ,  $\sigma$  and  $\Upsilon$  as the independent parameters and  $\frac{e^2}{r_-r_+}$  and  $\frac{\varpi_0}{r_-}$  as the dependent ones. Clearly,  $\sigma > 1$ .

When  $\Lambda > 0$ , the polynomial  $p$  has a third positive root  $r_c$ , the radius of the Reissner-Nordström de Sitter cosmological event horizon. This is the positive solution of

$$r^2 + (r_- + r_+)r - \frac{3e^2}{\Lambda r_-r_+} = 0.$$

The value of  $r_c$  is given by

$$r_c = \frac{-(r_- + r_+) + \sqrt{(r_- + r_+)^2 + \frac{12e^2}{\Lambda r_-r_+}}}{2}.$$

The fact that  $r_+ < r_c$  imposes a restriction on our independent parameters, namely

$$\frac{3e^2}{\Lambda r_-r_+} > 2r_+^2 + r_-r_+.$$

In terms of  $\sigma$  and  $\Upsilon$ , this can be written as

$$\frac{1 - \Upsilon(\sigma^2 + \sigma + 1)}{\Upsilon} > 2\sigma^2 + \sigma,$$

or

$$\Upsilon < \frac{1}{3\sigma^2 + 2\sigma + 1}. \quad (102)$$

If  $\Lambda \leq 0$ , condition (102) is also trivially satisfied. We say that a choice of parameters  $(\sigma, \Upsilon)$  is admissible if  $\sigma > 1$  and (102) holds.

Now we compute  $\rho$  as defined in (38), obtaining

$$\begin{aligned} \rho &= \left(\frac{r_+}{r_-}\right)^2 \frac{\frac{e^2}{r_-} + \frac{\Lambda}{3}r_-^3 - \varpi_0}{-\frac{e^2}{r_+} - \frac{\Lambda}{3}r_+^3 + \varpi_0} \\ &= \left(\frac{r_+}{r_-}\right)^2 \frac{\frac{e^2}{r_-r_+} \frac{r_+}{r_-} + \frac{\Lambda r_-^2}{3} - \frac{\varpi_0}{r_-}}{-\frac{e^2}{r_-r_+} - \frac{\Lambda r_-^2}{3} \frac{r_+^3}{r_-^3} + \frac{\varpi_0}{r_-}} \\ &= \sigma^2 \frac{(1 - \Upsilon(\sigma^2 + \sigma + 1))\sigma + \Upsilon - \frac{1}{2}(\sigma + 1)[1 - \Upsilon(\sigma^2 + 1)]}{-(1 - \Upsilon(\sigma^2 + \sigma + 1)) - \Upsilon\sigma^3 + \frac{1}{2}(\sigma + 1)[1 - \Upsilon(\sigma^2 + 1)]} \\ &= \sigma^2 \frac{1 - \Upsilon(\sigma^2 + 2\sigma + 3)}{1 - \Upsilon(3\sigma^2 + 2\sigma + 1)}. \end{aligned} \quad (103)$$

Taking into account (102), in the region of interest, the condition  $\rho > 1$  is equivalent to

$$\Upsilon < \frac{1}{3\sigma^2 + 2\sigma + 1} \quad \text{and} \quad \Upsilon < \frac{1}{(\sigma + 1)^2}.$$

As the first upper bound is smaller than the second, we conclude that for all admissible choices of parameters we have  $\rho > 1$ , that is

$$-\partial_r(1 - \mu)(r_-, \varpi_0) > \partial_r(1 - \mu)(r_+, \varpi_0).$$

We prove mass inflation in the region  $\rho > 2$ . Using (102) and (103), the condition  $\rho > 2$  is equivalent to

$$\frac{\sigma^2 - 2}{\sigma^4 + 2\sigma^3 - 3\sigma^2 - 4\sigma - 2} < \Upsilon < \frac{1}{3\sigma^2 + 2\sigma + 1}$$

if

$$\sigma < \sigma_0 := \frac{1}{2} \left( -1 + \sqrt{9 + 4\sqrt{6}} \right) \approx 1.66783.$$

The value  $\sigma_0$  is the only positive solution of  $\sigma^4 + 2\sigma^3 - 3\sigma^2 - 4\sigma - 2 = 0$ . For  $\sigma \geq \sigma_0$ , the condition  $\rho > 2$ , with the restriction (102), is always satisfied. Indeed, for  $\sigma > \sigma_0$ , we have

$$\frac{\sigma^2 - 2}{\sigma^4 + 2\sigma^3 - 3\sigma^2 - 4\sigma - 2} > \frac{1}{3\sigma^2 + 2\sigma + 1}$$

because the difference

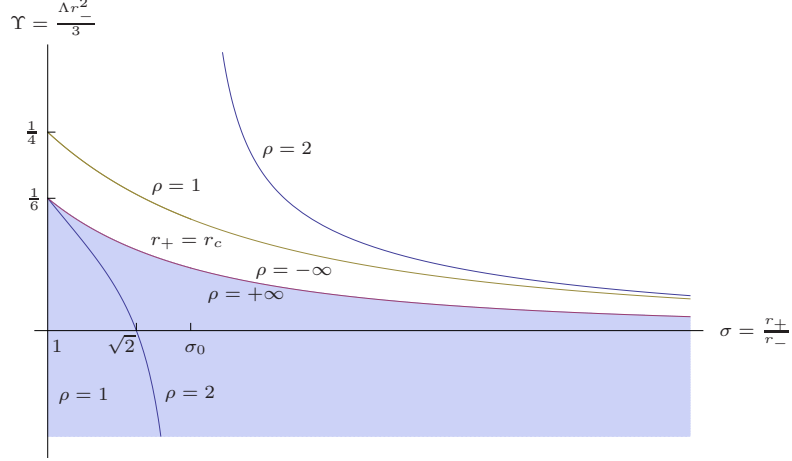
$$\frac{\sigma^2 - 2}{\sigma^4 + 2\sigma^3 - 3\sigma^2 - 4\sigma - 2} - \frac{1}{(\sigma + 1)^2}$$

is equal to

$$\frac{2\sigma^2}{(\sigma^4 + 2\sigma^3 - 3\sigma^2 - 4\sigma - 2)(\sigma + 1)^2},$$

and this is positive for  $\sigma > \sigma_0$ .

In the next figure we sketch part of the  $(\sigma, \Upsilon)$ -plane. As we just saw, the restriction  $r_+ < r_c$  translates into (102) and this region (shaded in the figure) is the only relevant one for our purposes. We remark that on the line  $\sigma = 1$  the value of  $\rho$  is equal to one.



## APPENDIX B. PROOF OF THEOREM 3.1

We start by establishing the following useful result.

**Lemma B.1.** *Assume that  $\zeta_0(u) > 0$  for  $u > 0$ . Then  $\theta > 0$  and  $\zeta > 0$  in  $\mathcal{P} \setminus \{0\} \times [0, \infty[$ .*

*Proof.* The proof proceeds in three steps.

*Step 1.* If  $\theta_0 > 0$  and  $\zeta_0 > 0$ , then  $\theta > 0$  and  $\zeta > 0$  in  $\mathcal{P}$ . Otherwise, there would exist a point  $(u, v) \in \mathcal{P}$  such that  $\theta(u, v) = 0$  or  $\zeta(u, v) = 0$  but  $\theta > 0$  and  $\zeta > 0$  in  $J^-(u, v)$ . Integrating (21) and (22), we obtain a contradiction.

*Step 2.* Since in Part 1 we proved continuous dependence of the solution on  $\theta_0$  and  $\zeta_0$ , if  $\theta_0 \geq 0$  and  $\zeta_0 \geq 0$ , then  $\theta \geq 0$  and  $\zeta \geq 0$ .

*Step 3.* Suppose that  $(u, v) \in \mathcal{P} \setminus \{0\} \times [0, \infty[$ . Since  $\zeta_0(u) > 0$  for  $u > 0$ , (22) implies that  $\zeta(u, v) > 0$ , because, from the previous step,  $\theta \geq 0$ . So  $\zeta > 0$  in  $\mathcal{P} \setminus \{0\} \times [0, \infty[$ . Now (21) implies that  $\theta$  is positive on  $\mathcal{P} \setminus \{0\} \times [0, \infty[$  because  $\lambda$  is negative on this set.  $\square$

*Proof of Theorem 3.1.* We follow the argument on pages 493–497 of [5]. We consider the same three cases as in the proof of Lemma 2.4 presented in Part 2.

*Case 1.* If (40) holds, there is nothing to prove.

*Case 2.* If  $\lim_{u \searrow 0} \varpi(u, \infty) > \varpi_0$ , then (40) holds (see page 494 of [5]).

*Case 3.* Suppose now that  $\lim_{u \searrow 0} \varpi(u, \infty) = \varpi_0$ . As in the proof of Lemma 2.4, we have  $\left(\frac{e^2}{r} + \frac{\Lambda}{3}r^3 - \varpi\right)(u, v) \geq 0$ . Then, from (17) it follows that  $\partial_u(-\lambda) \leq 0$  in  $J^+(\Gamma_{\tilde{r}_-})$ , whereas from Lemma B.1 it follows  $\partial_u \theta \geq 0$ . As a consequence, the integral

$$I(u) := \int_{v_{\tilde{r}_-}(u)}^{\infty} \left[ \frac{\theta^2}{-\lambda} \right] (u, \tilde{v}) d\tilde{v}$$

is a nondecreasing function of  $u$ .

*Case 3.1.*  $I(u) = +\infty$  for all small  $u$ , say  $0 < u \leq U$ . Consider such a  $u$ . We observe that the following limit exists and is finite:

$$\lim_{v \nearrow \infty} (1 - \mu)(u, v) =: (1 - \mu)(u, \infty) = 1 - \frac{2\varpi(u, \infty)}{r(u, \infty)} + \frac{e^2}{r^2(u, \infty)} - \frac{\Lambda}{3} r^2(u, \infty).$$

Equation (18) and  $\left(\frac{e^2}{r} + \frac{\Lambda}{3} r^3 - \varpi\right)(u, v) \geq 0$  imply that  $v \mapsto \nu(u, v)$  is a nondecreasing function in  $J^+(\gamma)$ . So we may define

$$\nu(u, \infty) = \lim_{v \nearrow +\infty} \nu(u, v).$$

Integrating (114) we get  $\lim_{v \nearrow \infty} \frac{\nu}{1-\mu}(u, v) = 0$ . Therefore,  $\nu(u, \infty) = 0$ . Let  $0 < \delta < u \leq U$ . Clearly,

$$r(u, v) = r(\delta, v) + \int_{\delta}^u \nu(s, v) ds.$$

Thus, by Lebesgue's Monotone Convergence Theorem,

$$r(u, \infty) = r(\delta, \infty) + \int_{\delta}^u \nu(s, \infty) ds = r(\delta, \infty).$$

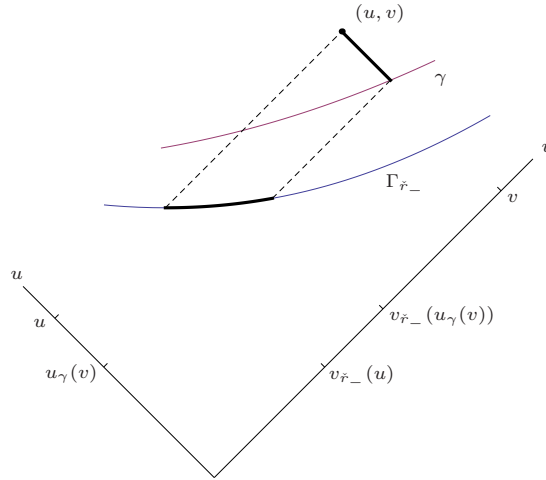
Letting  $\delta$  decrease to zero, due to (27), we obtain  $r(u, \infty) \equiv r_-$ . This contradicts Theorem 2.3.

*Case 3.2.*  $I(u) < +\infty$  for all small  $u$ , say  $0 < u \leq U$ . Arguing as in pages 495–496 of [5], we know  $\lim_{u \searrow 0} I(u) = 0$ . We will use this information to improve our upper bound on  $-\lambda$  in the region  $J^+(\gamma)$ . Then we will obtain a lower bound for  $\theta$  in this region. Finally, we use these bounds to arrive at the contradiction that  $I(u) = +\infty$ .

Let  $\varepsilon > 0$ . As  $\lim_{u \searrow 0} I(u) = 0$ , we may choose  $U > 0$  sufficiently small, so that for all  $(u, v) \in J^+(\gamma)$  with  $0 < u \leq U$ ,

$$e^{\frac{1}{r(U, \infty)} \int_{v_{\tilde{r}_-}(\bar{u})}^v \left[ \left| \frac{\theta}{\lambda} \right| |\theta| \right](\bar{u}, \bar{v}) d\bar{v}} \leq 1 + \varepsilon, \quad (104)$$

for  $\bar{u} \in [u_{\gamma}(v), u]$ .



Next we use (114), (126) and (104). We may bound the integral of  $\nu$  along  $\Gamma_{\check{r}_-}$  in terms of the integral of  $\frac{\nu}{1-\mu}$  on the segment  $[u_\gamma(v), u] \times \{v\}$  in the following way:

$$\begin{aligned} - \int_{u_\gamma(v)}^u \nu(\tilde{u}, v_{\check{r}_-}(\tilde{u})) d\tilde{u} &\leq \\ &- \min_{\Gamma_{\check{r}_-}}(1-\mu) \int_{u_\gamma(v)}^u \frac{\nu}{1-\mu}(\tilde{u}, v_{\check{r}_-}(\tilde{u})) d\tilde{u} \leq \\ &-(1+\varepsilon) \min_{\Gamma_{\check{r}_-}}(1-\mu) \int_{u_\gamma(v)}^u \frac{\nu}{1-\mu}(\tilde{u}, v) d\tilde{u}. \end{aligned} \quad (105)$$

Applying successively (105), (117), (123), and (130),

$$\begin{aligned} \int_{u_\gamma(v)}^u \frac{\nu}{1-\mu}(\tilde{u}, v) d\tilde{u} &\geq \frac{1}{-(1+\varepsilon) \min_{\Gamma_{\check{r}_-}}(1-\mu)} \int_{u_\gamma(v)}^u -\nu(\tilde{u}, v_{\check{r}_-}(\tilde{u})) d\tilde{u} \\ &= \frac{1}{-(1+\varepsilon) \min_{\Gamma_{\check{r}_-}}(1-\mu)} \int_{v_{\check{r}_-}(u)}^{v_{\check{r}_-}(u_\gamma(v))} -\lambda(u_{\check{r}_-}(\tilde{v}), \tilde{v}) d\tilde{v} \\ &\geq \frac{\max_{\Gamma_{\check{r}_-}}(1-\mu)}{(1+\varepsilon) \min_{\Gamma_{\check{r}_-}}(1-\mu)} \int_{\frac{v_\gamma(u)}{1+\beta}}^{\frac{v}{1+\beta}} \kappa(u_{\check{r}_-}(\tilde{v}), \tilde{v}) d\tilde{v} \\ &\geq \frac{(1-\varepsilon) \max_{\Gamma_{\check{r}_-}}(1-\mu)}{(1+\varepsilon) \min_{\Gamma_{\check{r}_-}}(1-\mu)} \left( \frac{v-v_\gamma(u)}{1+\beta} \right). \end{aligned} \quad (106)$$

Thus (see (132)),

$$\begin{aligned} e^{\int_{u_\gamma(v)}^u \left[ \frac{\nu}{1-\mu} \partial_r(1-\mu) \right] (\tilde{u}, v) d\tilde{u}} &\leq e^{\left[ \max_{J+(\gamma)} \partial_r(1-\mu) \right] \int_{u_\gamma(v)}^u \frac{\nu}{1-\mu}(\tilde{u}, v) d\tilde{u}} \\ &\leq e^{\left[ \max_{J+(\gamma)} \partial_r(1-\mu) \right] \frac{(1-\varepsilon) \max_{\Gamma_{\check{r}_-}}(1-\mu)}{(1+\varepsilon) \min_{\Gamma_{\check{r}_-}}(1-\mu)} \left( \frac{v-v_\gamma(u)}{1+\beta} \right)} \\ &\leq e^{\left[ \partial_r(1-\mu)(\check{r}_-, \varpi_0) + \max_{J+(\gamma)} \frac{2(\varpi - \varpi_0)}{r^2} \right] \frac{(1-\varepsilon) \max_{\Gamma_{\check{r}_-}}(1-\mu)}{(1+\varepsilon) \min_{\Gamma_{\check{r}_-}}(1-\mu)} \left( \frac{v-v_\gamma(u)}{1+\beta} \right)} \\ &\leq e^{\left[ \partial_r(1-\mu)(\check{r}_-, \varpi_0) + \frac{\varepsilon}{(r_- - \varepsilon_0)^2} \right] \frac{(1-\varepsilon) \max_{\Gamma_{\check{r}_-}}(1-\mu)}{(1+\varepsilon) \min_{\Gamma_{\check{r}_-}}(1-\mu)} \left( \frac{v-v_\gamma(u)}{1+\beta} \right)}. \end{aligned} \quad (107)$$

We integrate (17) and we use (134) and (107) to obtain

$$-\lambda(u, v) = -\lambda(u_\gamma(v), v) e^{\int_{u_\gamma(v)}^u \left[ \frac{\nu}{1-\mu} \partial_r(1-\mu) \right] (\tilde{u}, v) d\tilde{u}} \quad (108)$$

$$\begin{aligned} &\leq C e^{(1-\tilde{\delta}) \partial_r(1-\mu)(\check{r}_-, \varpi_0) \left( \frac{\beta v}{1+\beta} + \frac{v}{1+\beta} - \frac{v_\gamma(u)}{1+\beta} \right)} \\ &= C(u) e^{(1-\tilde{\delta}) \partial_r(1-\mu)(\check{r}_-, \varpi_0) v}. \end{aligned} \quad (109)$$

The value of  $\tilde{\delta}$  can be made small by choosing  $U$  sufficiently small. Here  $C(u) = C e^{-(1-\tilde{\delta}) \partial_r(1-\mu)(\check{r}_-, \varpi_0) \frac{v_\gamma(u)}{1+\beta}}$ . This is the desired upper estimate for  $-\lambda$ .

Now we turn to obtaining the lower estimate for  $\theta$ . Combining (119) with (120), for  $(u, v) \in \Gamma_{r_+ - \delta}$  we have

$$\begin{aligned} -\lambda(u, v) &\geq \left(\frac{r_+ - \delta}{r_+}\right) \frac{\partial_r(1 - \mu)(r_+, \varpi_0)}{1 + \varepsilon} u e^{[\partial_r(1 - \mu)(r_+, \varpi_0) - \varepsilon]v} \\ &\geq C u e^{[\partial_r(1 - \mu)(r_+, \varpi_0) - \varepsilon]v}. \end{aligned} \quad (110)$$

Note that  $C$  can be chosen independently of  $\delta$ . Using (22), Lemma B.1 and (39),

$$\zeta(u, v) \geq c u^s \text{ for all } (u, v). \quad (111)$$

We take into account that (110) and (111) are valid for arbitrary  $\delta$ , small, and that  $J^-(\Gamma_{\check{r}_+})$  is foliated by curves  $\Gamma_{r_+ - \delta}$  for  $0 < \delta < r_+ - \check{r}_+$ . Therefore, integrating (21), for  $(u, v) \in \Gamma_{\check{r}_+}$  we have

$$\begin{aligned} \theta(u, v) &\geq C u^{s+2} e^{[\partial_r(1 - \mu)(r_+, \varpi_0) - \varepsilon]v} \\ &\geq C e^{[-(s+1)\partial_r(1 - \mu)(r_+, \varpi_0) - \varepsilon]v}. \end{aligned} \quad (112)$$

For the last inequality, we used (120). The constant  $C$  depends on  $\check{r}_+$ . The value of  $\varepsilon$  can be made small by choosing  $\check{r}_+$  sufficiently close to  $r_+$ . We know that  $\partial_u \theta \geq 0$ . Thus, (112) also holds in  $J^+(\Gamma_{\check{r}_+})$ . This is the desired lower estimate for  $\theta$ .

We can now obtain a lower bound for  $I(u)$  using (109) and (112):

$$\begin{aligned} I(u) &\geq \int_{v_\gamma(u)}^\infty \left[ \frac{\theta^2}{-\lambda} \right] (u, \tilde{v}) d\tilde{v} \\ &\geq C(u) \int_{v_\gamma(u)}^\infty \frac{e^{[-2(s+1)\partial_r(1 - \mu)(r_+, \varpi_0) - 2\varepsilon]\tilde{v}}}{e^{(1 - \tilde{\delta})\partial_r(1 - \mu)(\check{r}_-, \varpi_0)\tilde{v}}} d\tilde{v}. \end{aligned}$$

This integral is infinite if

$$-2(s+1)\partial_r(1 - \mu)(r_+, \varpi_0) - \partial_r(1 - \mu)(\check{r}_-, \varpi_0) > 0,$$

or, equivalently,

$$s < \frac{1 - \partial_r(1 - \mu)(\check{r}_-, \varpi_0)}{2 \partial_r(1 - \mu)(r_+, \varpi_0)} - 1, \quad (113)$$

provided that  $\tilde{\varepsilon}$  and  $\tilde{\delta}$  are chosen sufficiently small (which we can achieve by decreasing  $U$  and  $\delta$ , if necessary). To complete the proof of Theorem 3.1 we just have to note that given  $s < \frac{\ell}{2} - 1$  we can always choose  $\check{r}_-$  so that (113) holds, contradicting  $I(u) < \infty$ .  $\square$

### APPENDIX C. SOME USEFUL FORMULAS

Here we collect some formulas that were obtained in Part 2 and that are needed to study the behavior of the solution at the Cauchy horizon.

**The Raychaudhuri equations written in terms of  $\kappa$  and  $\frac{\nu}{1 - \mu}$ .** Using equations (16), (18), (20) and (24), we get

$$\partial_v \left( \frac{\nu}{1 - \mu} \right) = \frac{\nu}{1 - \mu} \left( \frac{\theta}{\lambda} \right)^2 \frac{\lambda}{r}. \quad (114)$$

The equations (23) and (114) are the Raychaudhuri equations.



**Evolution equations for  $\frac{\theta}{\lambda}$  and  $\frac{\zeta}{\nu}$ .**

$$\partial_u \frac{\theta}{\lambda} = -\frac{\zeta}{r} - \frac{\theta}{\lambda} \frac{\nu}{1-\mu} \partial_r (1-\mu), \quad (115)$$

$$\partial_v \frac{\zeta}{\nu} = -\frac{\theta}{r} - \frac{\zeta}{\nu} \frac{\lambda}{1-\mu} \partial_r (1-\mu). \quad (116)$$

**The integrals of  $\nu$  and  $\lambda$  along a curve  $\Gamma_{\check{r}}$ .**

$$\int_{u_{\check{r}(v)}^u} \nu(\tilde{u}, v_{\check{r}(\tilde{u})}) d\tilde{u} = \int_{v_{\check{r}(u)}^v} \lambda(u_{\check{r}(\tilde{v})}, \tilde{v}) d\tilde{v}. \quad (117)$$

**Estimates in  $J^-(\Gamma_{\check{r}_+})$ .**

$$\left| \frac{\zeta}{\nu} \right| (u, v) \leq C \max_{\tilde{u} \in [0, u]} |\zeta_0|(\tilde{u}).$$

$$\begin{aligned} -\frac{\lambda}{1-\mu} \partial_r (1-\mu) &\leq -\left(\frac{\check{r}_+}{r_+}\right)^{\hat{\delta}^2} \min_{r \in [\check{r}_+, r_+]} \partial_r (1-\mu)(r, \varpi_0) \\ &= -\alpha < 0, \end{aligned} \quad (118)$$

where  $\hat{\delta}$  is a bound for  $\left| \frac{\zeta}{\nu} \right|$  in  $J^-(\check{r}_+)$ .

**Estimates for  $(u, v) \in \Gamma_{r_+-\delta}$ .**

$$\begin{aligned} -\left(\frac{r_+}{r_+-\delta}\right) \left( \frac{\partial_r (1-\mu)(r_+, \varpi_0)}{1-\varepsilon} + \frac{4\tilde{\delta}}{r_+^2} \right) \delta &\leq \lambda \\ &\leq -\left(\frac{r_+-\delta}{r_+}\right) \frac{\partial_r (1-\mu)(r_+, \varpi_0)}{1+\varepsilon} \delta, \end{aligned} \quad (119)$$

$$\delta e^{-[\partial_r (1-\mu)(r_+, \varpi_0) + \varepsilon]v} \leq u \leq \delta e^{-[\partial_r (1-\mu)(r_+, \varpi_0) - \varepsilon]v}. \quad (120)$$

**Estimate in  $J^-(\Gamma_{\check{r}_-}) \cap J^+(\Gamma_{\check{r}_+})$ .**

$$\left(\frac{\check{r}_-}{r_+}\right)^{\hat{\delta}^2} \frac{1-\varepsilon}{\partial_r (1-\mu)(r_+, \varpi_0) u} \leq \frac{\nu}{1-\mu}(u, v) \leq \frac{1+\varepsilon}{\partial_r (1-\mu)(r_+, \varpi_0) u}. \quad (121)$$

**Estimate in  $J^-(\Gamma_{\check{r}_-})$ .**

$$\lim_{\substack{(u,v) \rightarrow (0,\infty) \\ (u,v) \in J^-(\check{r}_-)}} \left| \frac{\theta}{\lambda} \right| (u, v) = 0. \quad (122)$$

**Relation between the integrals of  $\lambda$  and  $\kappa$  along the curve  $\Gamma_{\check{r}_-}$ .**

$$-\max_{\Gamma_{\check{r}_-}} (1-\mu) \int_{v_{\check{r}_-(u)}^v} \kappa(u_{\check{r}_-(\tilde{v})}, \tilde{v}) d\tilde{v} \quad (123)$$

$$\begin{aligned} &\leq -\int_{v_{\check{r}_-(u)}^v} \lambda(u_{\check{r}_-(\tilde{v})}, \tilde{v}) d\tilde{v} \leq \\ &\quad -\min_{\Gamma_{\check{r}_-}} (1-\mu) \int_{v_{\check{r}_-(u)}^v \kappa(u_{\check{r}_-(\tilde{v})}, \tilde{v}) d\tilde{v}. \end{aligned} \quad (124)$$

**Relation between the integrals of  $\nu$  and  $\frac{\nu}{1-\mu}$  along the curve  $\Gamma_{\check{r}_-}$ .**

$$-\max_{\Gamma_{\check{r}_-}}(1-\mu) \int_{u_{\check{r}_-}(v)}^u \frac{\nu}{1-\mu}(\tilde{u}, v_{\check{r}_-}(\tilde{u})) d\tilde{u} \quad (125)$$

$$\leq - \int_{u_{\check{r}_-}(v)}^u \nu(\tilde{u}, v_{\check{r}_-}(\tilde{u})) d\tilde{u} \leq -\min_{\Gamma_{\check{r}_-}}(1-\mu) \int_{u_{\check{r}_-}(v)}^u \frac{\nu}{1-\mu}(\tilde{u}, v_{\check{r}_-}(\tilde{u})) d\tilde{u}. \quad (126)$$

**Relation between the integrals of  $\frac{\nu}{1-\mu}$  and  $\kappa$  along the curve  $\Gamma_{\check{r}_-}$ .**

$$\frac{\max_{\Gamma_{\check{r}_-}}(1-\mu)}{(1+\varepsilon)\min_{\Gamma_{\check{r}_-}}(1-\mu)} \int_{v_{\check{r}_-}(u)}^v \kappa(u_{\check{r}_-}(\tilde{v}), \tilde{v}) d\tilde{v} \quad (127)$$

$$\leq \int_{u_{\check{r}_-}(v)}^u \frac{\nu}{1-\mu}(\tilde{u}, v) d\tilde{u} \leq \frac{\min_{\Gamma_{\check{r}_-}}(1-\mu)}{\max_{\Gamma_{\check{r}_-}}(1-\mu)} \int_{v_{\check{r}_-}(u)}^v \kappa(u_{\check{r}_-}(\tilde{v}), \tilde{v}) d\tilde{v}. \quad (128)$$

**Estimates in  $J^-(\gamma) \cap J^+(\Gamma_{\check{r}_-})$ .**

$$\left| \frac{\zeta}{\nu} \right|(u, v) \leq C \sup_{[0, u]} |\zeta_0| e^{-\left(\frac{\alpha}{1+\beta^+} + \partial_r(1-\mu)(r_- - \varepsilon_0, \varpi_0)\beta\right)v}, \quad (129)$$

$$\left| \frac{\theta}{\lambda} \right|(u, v) \leq C \sup_{[0, u]} |\zeta_0| e^{-\left(\frac{\alpha}{1+\beta^+} + \partial_r(1-\mu)(r_- - \varepsilon_0, \varpi_0) \frac{\min_{\Gamma_{\check{r}_-}}(1-\mu)}{\max_{\Gamma_{\check{r}_-}}(1-\mu)} \beta\right)v}, \quad (130)$$

$$e^{\frac{1}{r_- - \varepsilon_0} \int_{v_{\check{r}_-}(\bar{u})}^v \left[ \left| \frac{\theta}{\lambda} \right| |\theta| \right](\bar{u}, \bar{v}) d\bar{v}} \leq 1 + \varepsilon. \quad (131)$$

**Estimates for  $(u, v) \in \gamma$ .**

$$e^{\int_{u_{\check{r}_-}(v)}^u \left[ \frac{\nu}{1-\mu} \partial_r(1-\mu) \right](\tilde{u}, v) d\tilde{u}} \leq e^{\left[ \partial_r(1-\mu)(\check{r}_-, \varpi_0) + \frac{\varepsilon}{(r_- - \varepsilon_0)^2} \right] \frac{(1-\varepsilon)}{(1+\varepsilon)} \frac{\max_{\Gamma_{\check{r}_-}}(1-\mu)}{\min_{\Gamma_{\check{r}_-}}(1-\mu)} \frac{\beta}{1+\beta} v}, \quad (132)$$

$$\tilde{c} e^{(1+\delta)\partial_r(1-\mu)(r_- - \varepsilon_0, \varpi_0) \frac{\beta}{1+\beta} v} \quad (133)$$

$$\leq -\lambda(u, v) \leq \tilde{C} e^{(1-\delta)\partial_r(1-\mu)(\check{r}_-, \varpi_0) \frac{\beta}{1+\beta} v}, \quad (134)$$

$$c e^{-\partial_r(1-\mu)(r_+, \varpi_0) \frac{v}{1+\beta^-}} \quad (135)$$

$$\leq u \leq C e^{-\partial_r(1-\mu)(r_+, \varpi_0) \frac{v}{1+\beta^+}}. \quad (136)$$

The bound (136) is actually valid in  $J^-(\gamma) \cap J^+(\Gamma_{\check{r}_-})$ .

**Estimates in  $J^+(\gamma)$ .**

$$-\lambda(u, v) \leq C e^{(1-\delta)\partial_r(1-\mu)(\check{r}_-, \varpi_0)^{\frac{\beta}{1+\beta}}} v, \quad (137)$$

$$-\nu(u, v) \leq C u^{-\frac{1+\beta^-}{1+\beta^+} \frac{\partial_r(1-\mu)(\check{r}_-, \varpi_0)}{\partial_r(1-\mu)(r_+, \varpi_0)} \beta} - 1. \quad (138)$$

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