QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXXI, NUMBER 4
DECEMBER 2013, PAGES 661–677
S 0033-569X(2013)01303-3
Article electronically published on August 28, 2013

ON THE GLOBAL WEAK SOLUTION TO A GENERALIZED TWO-COMPONENT CAMASSA-HOLM SYSTEM

By

WENKE TAN (*Department of Mathematics, Sun Yat-sen University, Guangzhou 510275, People's Republic of China*),

YUE LIU (*Department of Mathematics, University of Texas, Arlington, Texas 76019-0408*),

AND

ZHAOYANG YIN (*Department of Mathematics, Sun Yat-sen University, Guangzhou 510275, People's Republic of China*)

Abstract. Considered herein is a generalized two-component Camassa-Holm system modeling the shallow water waves moving over a linear shear flow. The existence of the global weak solutions to the generalized two-component Camassa-Holm system is established and the solution is obtained as a limit of approximate global strong solutions.

1. Introduction. In this paper we consider the following generalized two-component Camassa-Holm system:

$$\begin{cases} m_t - Au_x + \sigma(2u_x m + um_x) + 3(1-\sigma)uu_x + \rho\rho_x = 0, & t > 0,\ x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0,\ x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $m = u - u_{xx}$ and σ is a real parameter. System (1.1) was recently derived in [7] following Ivanov's modeling approach [38]. It is a model from the shallow water theory with nonzero constant vorticity, where $u(t,x)$ is the horizontal velocity and $\rho(t,x)$ is related to the free surface elevation from equilibrium (or scalar density) with the boundary assumptions $u \to 0$, $\rho \to 1$ as $|x| \to \infty$. The scalar $A > 0$ characterizes a linear underlying shear flow, and hence the system in (1.1) models wave-current interactions. It is noted that flows with constant vorticity are ubiquitous in nature since tidal currents are of this type [18]. The real dimensionless constant σ is a parameter which provides the competition, or balance, in fluid convection between nonlinear steepening and amplification due

Received October 7, 2011.
2010 *Mathematics Subject Classification.* Primary 35D30, 35G25, 35A01.
Key words and phrases. Two-component Camassa-Holm system, global weak solution.
E-mail address: tanwenkezyq@163.com
E-mail address: yliu@uta.edu
E-mail address: mcsyzy@mail.sysu.edu.cn

to stretching. System (1.1) can be written in terms of u and ρ:

$$\begin{cases} u_t - u_{txx} - Au_x + 3uu_x - \sigma(2u_x u_{xx} + uu_{xxx}) + \rho\rho_x & = 0, \\ & t > 0,\ x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0,\ x \in \mathbb{R}, \end{cases} \tag{1.2}$$

with $u \to 0$, $\rho \to 1$ as $\mid x \mid \to \infty$. System (1.2) has two Hamiltonians in the following:

$$H_1 = \frac{1}{2}\int_{\mathbb{R}} (mu + (\rho - 1)^2)dx, \tag{1.3}$$

$$H_2 = \frac{1}{2}\int_{\mathbb{R}} (u^3 + \sigma u u_x^2 + 2u(\rho - 1) + u(\rho - 1)^2 - Au^2)dx. \tag{1.4}$$

In the case $\rho = 0$, (1.2) becomes

$$u_t - u_{xxt} - Au_x + 3uu_x = \sigma(2u_x u_{xx} + uu_{xxx}),$$

which models finite length, small amplitude radial deformation waves in cylindrical hyperelastic rods [27]. In particular, when $\sigma = 1$, it is a standard Camassa-Holm (C-H) equation; that is,

$$u_t - u_{xxt} - Au_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}.$$

The standard Camassa-Holm equation models the unidirectional propagation of the shallow water waves over a flat bottom. Here $u(t, x)$ stands for the fluid velocity at time t in the spatial x direction [4, 23, 39]. It has a bi-Hamiltonian structure [33] and is completely integrable [4, 11]. Also there is a geometric interpretation of (1.1) in terms of geodesic flow on the diffeomorphism group of the circle [22, 41]. Its solitary waves are peaked [5]. They are orbitally stable and interact like solitons [1, 25]. The peaked traveling waves replicate a characteristic for the waves of great height – waves of largest amplitude that are exact solutions of the governing equations for water waves; cf. [12, 17, 51]. Recently, it was claimed in [43] that the equation might be relevant to the modeling of tsunami; see also the discussion in [21].

The Cauchy problem and initial-boundary value problem for the Camassa-Holm equation have been studied extensively [15, 28, 31, 32, 44, 48, 54]. It has been shown that this equation is locally well-posed [14, 15, 28, 44, 48] for initial data $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. More interestingly, it has global strong solutions [10, 14, 15] and also finite time blow-up solutions [10, 13, 14, 15, 16, 28, 44, 48]. On the other hand, it has global weak solutions in $H^1(\mathbb{R})$ [2, 3, 24, 53]. The advantage of the Camassa-Holm equation in comparison with the KdV equation lies in the fact that the Camassa-Holm equation has peaked solitons and models wave breaking [5, 13] (by wave breaking we understand that the wave profile remains bounded while its slope becomes unbounded in finite time [52]).

Moreover, if $\sigma = 1$, the system in (1.2) recovers the standard two-component C-H system,

$$\begin{cases} m_t - Au_x + 2u_x m + um_x + \rho\rho_x = 0, & t > 0,\ x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0,\ x \in \mathbb{R}. \end{cases} \tag{1.5}$$

System (1.5) was first derived in [47] (also see [49]), which is formally integrable. Recently, Constantin and Ivanov [20] and Ivanov [38] showed a rigorous justification of the derivation of the system in (1.5). Mathematical properties of this system have been studied in many works; cf. [6, 30, 34, 45, 46]. Chen, Liu and Zhang [6] established a reciprocal transformation between the two-component Camassa-Holm system and the first negative flow of the AKNS hierarchy. Escher, Lechtenfeld and Yin [30] argued the well-posedness for the two-component periodic Camassa-Holm system in the Sobolev space $H^s \times H^{s-1}$ with $s \geq 2$ by applying Kato's theory [40] and provided some precise blow-up scenarios for strong solutions to the system. Guan and Yin [34] studied the wave-breaking criterion, the global existence and blow-up phenomena of strong solutions in $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}), s \geq 2$. The local well-posedness is improved by Gui and Liu [36] to the Besov spaces (especially in the Sobolev spaces $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > \frac{3}{2}$). The blow-up criterion is made more precise in [55], where the authors showed that the wave breaking in finite time only depends on the slope of u. This blow-up criterion is further improved in [37]. Guan and Yin [35] recently obtained the result of the existence of global weak solutions to (1.5) by approximation techniques.

Chen and Liu [7, 8] recently studied (1.2) and established the blow-up criterion and determined the exact blow-up rate of solutions. In addition, They gave a sufficient condition for global solutions in $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}), s > \frac{3}{2}$ with $0 \leq \sigma < 2$. However, the existence and uniqueness of global weak solutions to system (1.2) have not yet been discussed.

Our main aim of the present paper is to establish existence of a global weak solution to (1.2) with $0 \leq \sigma < 2$. The main result of this paper can be stated in the following.

THEOREM 1.1. Let $(u_0, \rho_0 - 1) \in (H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})) \times (L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}))$. If $ess\inf_{x\in\mathbb{R}} \rho_0(x) > 0$ and $0 \leq \sigma < 2$, then (1.2) has an admissible weak solution (u, ρ) with the initial value (u_0, ρ_0). Moreover, we have

$$\int_{\mathbb{R}} (u^2 + u_x^2 + (\rho - 1)^2) dx = \int_{\mathbb{R}} (u_0^2 + u_{0,x}^2 + (\rho_0 - 1)^2) dx. \tag{1.6}$$

Furthermore, we have

$$(u(t,\cdot), \rho(t,\cdot) - 1) \in C(\mathbb{R}^+, H^1(\mathbb{R}) \times L^2(\mathbb{R})).$$

REMARK 1.1. To establish the result of the existence of the global weak solution of (1.2), we need the global strong solutions of (1.2) as the approximate solutions. As we will see in Theorem 2.1 showed in [7, 8], the existence of global strong solutions is obtained under the condition $0 \leq \sigma < 2$.

The motivation to obtain the global weak solution of (1.2) is inspired by the work in [19, 53]. To prove the existence of a global weak solution, we first mollify the initial data and get a sequence of approximate solutions in $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s \geq 3$. Then we prove that the limit of the approximate solutions in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ is a weak solution of (1.2). The difficulty in the proof is the interaction between two components of solution u and ρ and the low integrability of u and ρ. To overcome this problem, we derive a condition on ρ to improve the integrability of u and ρ so that we can choose an entropy function to cancel the interaction.

The paper is organized as follows. In Section 2, we recall some useful properties for the initial-value problem to the strong solution of (1.2). In Section 3, we prove the global existence of the approximate solutions. Finally, we establish necessary properties of compactness in Section 4. Using the obtained compactness results, we prove that the limit of the approximate solution is a global weak solution of (1.2).

NOTATION. In the following, we denote by $*$ the spatial convolution. Given a Banach space X, we denote its norm by $\|\cdot\|_X$.

2. Preliminaries.

In this section, we will recall and present some useful lemmas which will be used in the sequel.

Notice that in system (1.2) it is required that $u(t,x)\to 0$ and $\rho(t,x)\to 1$ as $|x|\to\infty$, at any instant t. Note also that if $p(x) := \frac{1}{2}e^{-|x|}, x\in\mathbb{R}$, then $(1-\partial_x^2)^{-1}f = p*f$ for all $f\in L^2(\mathbb{R})$. Then, we can rewrite the system (1.2) as follows:

$$\begin{cases} u_t+\sigma uu_x = -\partial_x p*(-Au+\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2 & +\frac{1}{2}(\rho-1)^2+(\rho-1)), \\ & t>0,\ x\in\mathbb{R}, \\ (\rho-1)_t+(u(\rho-1))_x = -u_x, & t>0,\ x\in\mathbb{R}, \\ u(0,x)=u_0(x), & x\in\mathbb{R}, \\ \rho(0,x)-1=\rho_0(x)-1, & x\in\mathbb{R}. \end{cases} \tag{2.1}$$

We now give some useful results of (2.1).

LEMMA 2.1 ([8]). Let $\sigma=0$ and (u,ρ) be the solution of the system (2.1) with initial data $(u_0,\rho_0-1)\in H^s(\mathbb{R})\times H^{s-1}(\mathbb{R})$, $s>3/2$, and T the maximal time of existence. Then

$$\sup_{x\in\mathbb{R}} u_x(t,x) \le \sup_{x\in\mathbb{R}} u_{0,x}(x)+\frac{1}{2}\left(\sup_{x\in\mathbb{R}}\rho_0^2(x)+C_1^2\right)t,\quad t\le T, \tag{2.2}$$

$$\inf_{x\in\mathbb{R}} u_x(t,x) \ge \inf_{x\in\mathbb{R}} u_{0,x}(x)+\frac{1}{2}\left(\inf_{x\in\mathbb{R}}\rho_0^2(x)-C_2^2\right)t,\quad t\le T, \tag{2.3}$$

where the constants above are defined as follows:

$$C_1=\sqrt{\frac{3+A^2}{2}}\|(u_0,\rho_0-1)\|_{H^1\times L^2}, \tag{2.4}$$

$$C_2=\sqrt{2+C_1^2}. \tag{2.5}$$

LEMMA 2.2 ([7]). Let $0<\sigma<2$ and (u,ρ) be the solution of (2.1) with initial data $(u_0,\rho_0-1)\in H^s(\mathbb{R})\times H^{s-1}(\mathbb{R})$, $s>\frac{3}{2}$, and T the maximal time of existence. Assume that $\inf_{x\in\mathbb{R}}\rho_0(x)>0$.

(1) If $0<\sigma\le 1$, then

$$|\inf_{x\in\mathbb{R}} u_x(t,x)| \le \frac{1}{\inf_{x\in\mathbb{R}}\rho_0(x)}C_2e^{C_1t}, \tag{2.6}$$

$$|\sup_{x\in\mathbb{R}} u_x(t,x)| \le \frac{1}{\inf_{x\in\mathbb{R}}\rho_0^{\frac{\sigma}{2-\sigma}}(x)}C_2^{\frac{1}{2-\sigma}}e^{\frac{C_1t}{2-\sigma}},\quad t\in[0,T). \tag{2.7}$$

(2) If $1 \leq \sigma < 2$, then

$$| \inf_{x\in\mathbb{R}} u_x(t,x) | \leq \frac{1}{\inf_{x\in\mathbb{R}} \rho_0^{\frac{\sigma}{2-\sigma}}(x)} C_2^{\frac{1}{2-\sigma}} e^{\frac{C_1 t}{2-\sigma}}, \tag{2.8}$$

$$| \sup_{x\in\mathbb{R}} u_x(t,x) | \leq \frac{1}{\inf_{x\in\mathbb{R}} \rho_0(x)} C_2 e^{C_1 t}, \quad t \in [0,T). \tag{2.9}$$

The constant C_1 and C_2 are defined as follows, where

$$C_1 = 2 + \frac{2 + A^2 + | \sigma | + 2 | 3 - \sigma |}{4} \| (u_0, \rho_0 - 1) \|^2_{H^1(\mathbb{R})\times L^2(\mathbb{R})}, \tag{2.10}$$

$$C_2 = 1 + \| u_{0,x} \|^2_{L^\infty(\mathbb{R})} + \| \rho_0 \|^2_{L^\infty(\mathbb{R})} . \tag{2.11}$$

Lemma 2.3 ([7]). Let $\sigma \neq 0$ and (u, ρ) be the solution of (2.1) with initial data $(u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > \frac{3}{2}$, and let T be the maximal time of existence. Assume that there is an $M \geq 0$ such that

$$\inf_{(t,x)\in[0,T)\times\mathbb{R}} \sigma u_x \geq -M. \tag{2.12}$$

(1) If $\sigma > 0$, then

$$\| \rho(t,\cdot) \|_{L^\infty(\mathbb{R})} \leq \| \rho_0 \|_{L^\infty(\mathbb{R})} e^{\frac{Mt}{\sigma}}, \quad t \leq T. \tag{2.13}$$

(2) If $\sigma < 0$, then

$$\| \rho(t,\cdot) \|_{L^\infty(\mathbb{R})} \leq \| \rho_0 \|_{L^\infty(\mathbb{R})} e^{Nt}, \quad t \leq T, \tag{2.14}$$

where

$$N = \| u_{0,x} \|_{L^\infty(\mathbb{R})} + (\frac{C_3}{| \sigma |^{\frac{1}{2}}}),$$

$$C_3 = (2 + \frac{5 + A^2 - 2\sigma}{2})^{\frac{1}{2}} \| (u_0, \rho_0 - 1) \|_{H^1(\mathbb{R})\times L^2(\mathbb{R})}, \quad \text{for} \quad \sigma < 0.$$

We are now in the position to state a global existence theorem of [7, 8].

Theorem 2.1 ([7, 8]). Let $0 \leq \sigma < 2$ and (u, ρ) be the solution of (2.1) with initial data $(u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > \frac{3}{2}$, and T the maximal time existence. If

$$\inf_{x\in\mathbb{R}} \rho_0(x) > 0, \tag{2.15}$$

then $T = \infty$ and the solution (u, ρ) is global.

3. The approximate solutions. In this section, we construct the approximate solution sequence $(u_n(t,x), \rho_n(t,x))$ as a solution to system (2.1) with initial data

$$(u_0(x), \rho_0(x) - 1) \in (H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})) \times (L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})).$$

Additionally, the initial data satisfies the condition $ess \inf_{x\in\mathbb{R}} \rho_0(x) > 0$.

In the following, we denote by $j_\varepsilon(x)$ the standard mollifiers. We first define $(u_0^n(x), \rho_0^n(x))$ as follows:

$$u_0^n(x) = j_{\frac{1}{n}} * u_0(x), \tag{3.1}$$

$$\rho_0^n(x) = j_{\frac{1}{n}} * \rho_0(x), \quad n \in \mathbb{N}. \tag{3.2}$$

Since $ess\inf_{x\in\mathbb{R}}\rho_0(x)>0$, we obtain that

$$\inf_{x\in\mathbb{R}}\rho_0^n(x)\geq ess\inf_{x\in\mathbb{R}}\rho_0(x)>0,\quad n\in\mathbb{N}. \tag{3.3}$$

It is clear that

$$(u_0^n,\rho_0^n-1)\to(u_0,\rho_0-1)\ \text{ in }\ H^1(\mathbb{R})\times L^2(\mathbb{R}), \tag{3.4}$$

$$\| u_0^n \|_{H^1(\mathbb{R})}\leq\| u_0 \|_{H^1(\mathbb{R})}, \tag{3.5}$$

$$\| \rho_0^n-1 \|_{L^2(\mathbb{R})}\leq\| \rho_0-1 \|_{L^2(\mathbb{R})},\quad n\in\mathbb{N}, \tag{3.6}$$

$$\| u_{0\,x}^n \|_{L^\infty(\mathbb{R})}\leq\| u_{0x} \|_{L^\infty(\mathbb{R})}, \tag{3.7}$$

$$\| \rho_0^n \|_{L^\infty(\mathbb{R})}\leq\| \rho_0 \|_{L^\infty(\mathbb{R})},\quad n\in\mathbb{N}. \tag{3.8}$$

Now, we can state the main result for the approximate solutions.

THEOREM 3.1. Assume $0\leq\sigma<2$. Let $(u_0(x),\rho_0(x)-1)\in(H^1(\mathbb{R})\cap W^{1,\infty}(\mathbb{R}))\times(L^2(\mathbb{R})\cap L^\infty(\mathbb{R}))$ with the condition $ess\inf_{x\in\mathbb{R}}\rho_0(x)>0$, and let (u_0^n,ρ_0^n) be defined as in (3.1) and (3.2). Then, given any $T>0$, there exists a sequence of solutions $(u^n,\rho^n-1)\in C([0,T],H^s(\mathbb{R})\times H^{s-1}(\mathbb{R}))$ to the Cauchy problem (2.1) with the initial data (u_0^n,ρ_0^n-1). Furthermore, these solutions satisfy the following properties:

(1) There exists a constant $M(T)$ such that

$$\| u_x^n(t,\cdot) \|_{L^\infty(\mathbb{R})}\leq M(T), \tag{3.9}$$

$$\| \rho^n(t,\cdot)-1 \|_{L^\infty(\mathbb{R})}\leq M(T),\quad n\in\mathbb{N}. \tag{3.10}$$

(2)

$$\begin{aligned}&\| u^n(t,\cdot) \|_{H^1(\mathbb{R})}^2+\| \rho^n(t,\cdot)-1 \|_{L^2(\mathbb{R})}^2\\ =&\| u_0^n(x) \|_{H^1(\mathbb{R})}^2+\| \rho_0^n(x)-1 \|_{L^2(\mathbb{R})}^2\\ \leq&\| u_0(x) \|_{H^1(\mathbb{R})}^2+\| \rho_0(x)-1 \|_{L^2(\mathbb{R})}^2 .\end{aligned} \tag{3.11}$$

Proof. First, by (3.3) and Theorem 2.1, we deduce that there exists a sequence of global solutions $(u^n(t,x),\rho^n(t,x))\in H^s(\mathbb{R})\times H^{s-1}(\mathbb{R})$, $s\geq 3$. Second, notice that the second equation of system (2.1) has characteristic

$$\begin{cases}\frac{\partial q}{\partial t}=u(t,q),0<t<T,\\ q(0,x)=x,x\in\mathbb{R}.\end{cases}$$

We have

$$\frac{d\rho(t,q)}{dt}=u_x(t,q)\rho(t,q).$$

Using this equation, (3.7)-(3.8) and Lemmas 2.1-2.3, we get (3.9) and (3.10). In view of (1.3) and (3.5)-(3.6), we get (3.11). □

4. Precompactness. With the basic energy estimates and uniform a priori estimates in Section 3, we are now ready to obtain the necessary compactness of approximate solutions $(u^n(t,x),\rho^n(t,x))$. We first recall two useful lemmas.

LEMMA 4.1 ([42]). Let X be a reflexive Banach space and let f_n be bounded in $L^\infty(0,T;X)$ for some $T \in (0,\infty)$. We assume that $f_n \in C(0,T;Y)$, where Y is a Banach space such that $X \hookrightarrow Y$, Y' is separable and dense in X'. Furthermore $(\phi, f_n)_{Y' \times Y}$ is uniform continuous in $t \in [0,T]$ and uniform in $n \geq 1$. Then, f_n is relative compact in $C^w(0,T;X)$, the space of continuous functions from $[0,T]$ with values in X when the latter space is equipped with its weak topology.

LEMMA 4.2 ([42]). Let $f \in W^{1,p}(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ with $1 \leq q \leq \infty$. Then

$$\| j_\varepsilon * \partial_x(fg) - \partial_x(fj_\varepsilon * g) \|_{L^r(\mathbb{R})} \leq C \| f \|_{W^{1,p}(\mathbb{R})} \| g \|_{L^q(\mathbb{R})}, \tag{4.1}$$

$$j_\varepsilon * \partial_x(fg) - \partial_x(fj_\varepsilon * g) \to 0 \ \text{ in } \ L^r(\mathbb{R}) \ \ as \ \ \varepsilon \to 0, \tag{4.2}$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Let us denote $P^n(t,x) = p * (\frac{3-\sigma}{2}u_n^2 + \frac{\sigma}{2}u_{nx}^2 + \frac{1}{2}(\rho_n - 1)^2 + (\rho_n - 1))$ in the following text.

LEMMA 4.3. Let $0 \leq \sigma < 2$. Then there exist subsequences $\{(u^{n_k}, \rho^{n_k} - 1)\} \subset \{(u^n, \rho^n - 1)\}$ and $\{P^{n_k}\} \subset \{P^n\}$ and a pair of functions $(u, \rho - 1) \in L^\infty(\mathbb{R}^+; H^1(\mathbb{R}) \times L^2(\mathbb{R}))$ and $\bar{P} \in L^\infty(\mathbb{R}^+; H^1(\mathbb{R}))$ such that

$$(u^{n_k}, \rho^{n_k} - 1) \rightharpoonup (u, \rho - 1) \ \text{ in } \ H^1((0,T) \times \mathbb{R}) \times L^2((0,T) \times \mathbb{R}), \ \ \forall T > 0, \tag{4.3}$$

$$u^{n_k} \to u \ \text{ uniformly on each compact subset of } \mathbb{R}^+ \times \mathbb{R}, \tag{4.4}$$

$$P^{n_k} \to \bar{P} \ \text{ uniformly on each compact subset of } \mathbb{R}^+ \times \mathbb{R}. \tag{4.5}$$

Proof. By (2.1), we have

$$\| u_t^n \|_{L^2(\mathbb{R})} \leq \sigma \| u^n u_x^n \|_{L^2(\mathbb{R})} + | A | \| \partial_x p * u^n \|_{L^2(\mathbb{R})} + \| \partial_x P^n \|_{L^2(\mathbb{R})} . \tag{4.6}$$

Using (3.11), Sobolev's inequality and Young's inequality, we get

$$\| u^n u_x^n \|_{L^2(\mathbb{R})} \leq \| u^n \|_{L^\infty(\mathbb{R})} \| u_x^n \|_{L^2(\mathbb{R})} \leq \| u_0 \|_{H^1(\mathbb{R})}^2 + \| \rho_0 - 1 \|_{L^2(\mathbb{R})}^2, \tag{4.7}$$

$$\begin{aligned} \| P_x^n \|_{L^2(\mathbb{R})} \leq & \| p_x \|_{L^2(\mathbb{R})} (\| u^n \|_{H^1(\mathbb{R})}^2 + \| \rho^n - 1 \|_{L^2(\mathbb{R})}^2) \\ & + \| p_x \|_{L^1(\mathbb{R})} \| \rho^n - 1 \|_{L^2(\mathbb{R})} \\ \leq & (\| u_0 \|_{H^1(\mathbb{R})}^2 + \| \rho_0 - 1 \|_{L^2(\mathbb{R})}^2) + (\| u_0 \|_{H^1(\mathbb{R})} \\ & + \| \rho_0 - 1 \|_{L^2(\mathbb{R})}), \end{aligned} \tag{4.8}$$

$$\begin{aligned} \| \partial_x p * u^n \|_{L^2(\mathbb{R})} & \leq \| p_x \|_{L^1(\mathbb{R})} \| u^n \|_{L^2(\mathbb{R})} \\ & \leq \| u_0 \|_{H^1(\mathbb{R})} + \| \rho_0 - 1 \|_{L^2(\mathbb{R})}, \end{aligned} \tag{4.9}$$

where we used the fact that $\| p_x \|_{L^1(\mathbb{R})} \leq 1$ and $\| p_x \|_{L^2(\mathbb{R})} \leq 1$. From (4.6)-(4.9), we can obtain that

$$\| u_t^n \|_{L^2(\mathbb{R})} \leq 3(\| u_0 \|_{H^1(\mathbb{R})}^2 + \| u_0 \|_{H^1(\mathbb{R})} + \| \rho_0 - 1 \|_{L^2(\mathbb{R})}^2 + \| \rho_0 - 1 \|_{L^2(\mathbb{R})}). \tag{4.10}$$

Using (3.11) and (4.10), we get that for any $T > 0$, there exists a pair of functions $(u, \rho - 1) \in L^\infty(0,T; H^1(\mathbb{R}) \times L^2(\mathbb{R}))$ such that (4.3) holds.

Next, we turn to the compactness of u^n. It follows from (3.11) that $\{u^n(t,x)\}$ is uniformly bounded in $L^\infty(\mathbb{R}^+; H^1(\mathbb{R}))$. Also, $\{u_t^n(t,x)\}$ is uniformly bounded in

$L^2(0,T;L^2(\mathbb{R}))$ for any $T>0$, due to (4.10). Thus, by the classical Lions-Aubin lemma, there exist $u\in L^\infty(\mathbb{R}^+;H^1(\mathbb{R}))$ and a subsequence $\{u^{n_k}\}$ that is weakly compact in $L^\infty(\mathbb{R}^+;H^1(\mathbb{R}))$; furthermore, $\{u^{n_k}\}$ converges to $u(t,x)$ uniformly on each compact subset of $\mathbb{R}^+\times\mathbb{R}$ as $k\to\infty$. In addition, $u(t,x)$ is a continuous function.

Finally, we show the compactness of $\{P^n\}$. As in the proof of (4.8), the similar computation shows that $\| P^n \|_{L^2(\mathbb{R})}$ is uniformly bounded. Thus, we have that $\{P^n\}$ is uniformly bounded in $L^\infty(\mathbb{R}^+;H^1(\mathbb{R}))$. Next, we give the estimate of $\| P^n_t \|_{L^2(\mathbb{R})}$. Notice that

$$\begin{aligned}\partial_t P^n =&(3-\sigma)p*u^n\partial_t u^n+\sigma p*u^n_x\partial_t u^n_x\\ &+p*(\rho^n-1)\partial_t(\rho^n-1)+p*\partial_t(\rho^n-1).\end{aligned}\tag{4.11}$$

Differentiating the first equation in (2.1), we obtain

$$\begin{aligned}u^n_{tx}+\sigma u^n u^n_{xx}+\frac{\sigma}{2}(u^n_x)^2=&\frac{1}{2}(\rho^n-1)^2+(\rho^n-1)+\frac{3-\sigma}{2}(u^n)^2\\ &+A\partial_x^2 p*u^n-P^n.\end{aligned}\tag{4.12}$$

Using the identity $(1-\partial_x^2)p*f=f$ and $u^n u^n_x u^n_{xx}+\frac{1}{2}(u^n_x)^3=\frac{1}{2}(u^n(u^n_x)^2)_x$, we have

$$\begin{aligned}&p*(u^n_x\partial_t u^n_x)\\ =&-\sigma p*(u^n u^n_x u^n_{xx}+\frac{1}{2}(u^n_x)^3)+\frac{1}{2}p*(u^n_x(\rho^n-1)^2)-p*(u^n_x P^n)\\ &+\frac{3-\sigma}{2}p*((u^n)^2u^n_x)+Ap*(u^n_x p*u^n)-Ap*(u^n u^n_x)\\ =&-\frac{\sigma}{2}p_x*(u^n(u^n_x)^2)+\frac{1}{2}p*(u^n_x(\rho^n-1)^2)-p*(u^n_x P^n)\\ &+\frac{3-\sigma}{2}p*((u^n)^2u^n_x)+Ap*(u^n_x p*u^n)-Ap*(u^n u^n_x).\end{aligned}\tag{4.13}$$

By (3.9)-(3.11), Sobolev's inequality and Young's inequality, we get

$$\begin{aligned}\| p*u^n_x\partial_t u^n_x \|_{L^2(\mathbb{R})}\le&CM(T)(\| u_0 \|^2_{H^1(\mathbb{R})}+\| \rho_0-1 \|^2_{L^2(\mathbb{R})}\\ &+\| u_0 \|_{H^1(\mathbb{R})}+\| \rho_0-1 \|_{L^2(\mathbb{R})}),\quad \forall T>0.\end{aligned}\tag{4.14}$$

The similar computations show that

$$\begin{aligned}&\| \partial_t P^n \|_{L^2(\mathbb{R})}\\ \le&C(\| p*(u^n\partial_t u^n) \|_{L^2(\mathbb{R})}+\| p*\partial_t(\rho^n-1) \|_{L^2(\mathbb{R})}\\ &+\| p*((\rho^n-1)\partial_t(\rho^n-1)) \|_{L^2(\mathbb{R})}+\| p*u^n_x\partial_t u^n_x \|_{L^2(\mathbb{R})})\\ \le&CM(T)(\| u_0 \|^2_{H^1(\mathbb{R})}+\| \rho_0-1 \|^2_{L^2(\mathbb{R})}\\ &+\| u_0 \|_{H^1(\mathbb{R})}+\| \rho_0-1 \|_{L^2(\mathbb{R})}),\quad \forall T>0.\end{aligned}\tag{4.15}$$

Thus, $\{P^n_t\}$ is uniformly bounded in $L^2([0,T]\times\mathbb{R})$ for any $T>0$. Using the Lions-Aubin lemma, there exists $\bar{P}\in L^\infty(\mathbb{R}^+,H^1(\mathbb{R}))$ such that $\{P^{n_k}\}$ converges to $\bar{P}(t,x)$ uniformly on each compact subset of $\mathbb{R}^+\times\mathbb{R}$ as $k\to\infty$. This completes the proof of the lemma. □

Now we can consider the pair of functions $(u, \rho - 1)$ which is the weak limit of $(u^{n_k}, \rho^{n_k} - 1)$. By Theorem 3.1 and Lemma 4.3, we have for given any $T > 0$ that

$$u^{n_k} u_x^{n_k} \rightharpoonup u u_x \quad \text{in} \quad L^2([0,T] \times \mathbb{R}), \tag{4.16}$$

$$u^{n_k}(\rho^{n_k} - 1) \rightharpoonup u(\rho - 1) \quad \text{in} \quad L^2([0,T] \times \mathbb{R}). \tag{4.17}$$

In addition, by Theorem 3.1 and the interpolation theory, we obtain that for any $T > 0$ and $1 < p < \infty$,

$$\| (u_x^n)^2 \|_{L^p([0,T]\times\mathbb{R})} + \| (\rho^n - 1)^2 \|_{L^p([0,T]\times\mathbb{R})} \leq C(T). \tag{4.18}$$

Thus, there exists a pair of functions $(\overline{u_x^2}, \overline{(\rho-1)^2})$ such that

$$(u_x^{n_k})^2 \rightharpoonup \overline{u_x^2} \quad \text{and} \quad (\rho^{n_k} - 1)^2 \rightharpoonup \overline{(\rho - 1)^2} \quad \text{in} \quad L^p([0,T] \times \mathbb{R}), \tag{4.19}$$

where $1 < p < \infty$. Furthermore, we have that

$$u_x^2(t,x) \leq \overline{u_x^2} \quad \text{and} \quad (\rho(t,x) - 1)^2 \leq \overline{(\rho - 1)^2}(t,x) \quad \text{a.e. on} \quad \mathbb{R}^+ \times \mathbb{R}. \tag{4.20}$$

In the following, if there is no ambiguity, we still write the superscript $\{n_k\}$ as $\{n\}$. Now we give the system which $(u_x, \rho - 1)$ satisfies.

Lemma 4.4. If $\sigma \in [0, 2)$, then we have

$$\begin{aligned} \partial_t u_x + \sigma \partial_x (u u_x) =& \frac{\sigma}{2}\overline{u_x^2} + \frac{1}{2}\overline{(\rho-1)^2} + (\rho - 1) \\ &+ \frac{3-\sigma}{2} u^2 + A\partial_x^2 p * u - \bar{P}, \end{aligned} \tag{4.21}$$

$$\partial_t(\rho - 1) + \partial_x(u(\rho - 1)) = - u_x \tag{4.22}$$

in the sense of distributions on $\mathbb{R}^+ \times \mathbb{R}$.

Proof. In view of (2.1) and (4.12), we deduce that

$$\begin{aligned} \partial_t u_x^n + \sigma \partial_x (u^n u_x^n) =& \frac{\sigma}{2}(u_x^n)^2 + \frac{1}{2}(\rho^n - 1)^2 + (\rho^n - 1) \\ &+ \frac{3-\sigma}{2}(u^n)^2 + A\partial_x^2 p * u^n - P^n, \end{aligned} \tag{4.23}$$

$$\partial_t(\rho^n - 1) + \partial_x(u^n(\rho^n - 1)) = - u_x^n. \tag{4.24}$$

Using Lemma 4.3, (4.19) and (4.16)-(4.17), we get (4.21) and (4.22). ☐

The next lemma contains renormalized formulations of (4.21) and (4.22).

Lemma 4.5. Let $\sigma \in [0, 2)$. For any $b(z) \in C^1(\mathbb{R})$ and $b(0) = 0$, we have that

$$\begin{aligned} &\partial_t b(u_x) + \sigma \partial_x (u b(u_x)) \\ =& \sigma u_x b(u_x) - \sigma u_x^2 b'(u_x) + \frac{\sigma}{2} b'(u_x)\overline{u_x^2} + \frac{1}{2} b'(u_x)\overline{(\rho-1)^2} - b'(u_x)\bar{P} \\ &+ \frac{3-\sigma}{2} b'(u_x) u^2 + b'(u_x)(\rho - 1) + A b'(u_x)\partial_x^2 p * u \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} &\partial_t b(\rho - 1) + \partial_x (u b(\rho - 1)) \\ =& u_x b(\rho - 1) - u_x b'(\rho - 1) - u_x(\rho - 1) b'(\rho - 1) \end{aligned} \tag{4.26}$$

hold in the sense of distributions on $\mathbb{R}^+ \times \mathbb{R}$.

Proof. Denote $\langle f\rangle_\varepsilon = j_\varepsilon * f$. Mollifying (4.21) and (4.22), we get

$$\begin{aligned}\partial_t\langle u_x\rangle_\varepsilon + \sigma\partial_x(u\langle u_x\rangle_\varepsilon) =& \frac{\sigma}{2}\langle\overline{u_x^2}\rangle_\varepsilon + \frac{1}{2}\langle\overline{(\rho-1)^2}\rangle_\varepsilon + \langle(\rho-1)\rangle_\varepsilon \\ &+ \frac{3-\sigma}{2}\langle u^2\rangle_\varepsilon + A\langle\partial_x^2 p * u\rangle_\varepsilon - \langle\bar{P}\rangle_\varepsilon + r_\varepsilon^1, \end{aligned} \tag{4.27}$$

$$\partial_t\langle(\rho-1)\rangle_\varepsilon + \partial_x(u\langle\rho-1\rangle_\varepsilon) = - \langle u_x\rangle_\varepsilon + r_\varepsilon^2, \tag{4.28}$$

where

$$r_\varepsilon^1 = \sigma\partial_x(u\langle u_x\rangle_\varepsilon) - \sigma\langle\partial_x(uu_x)\rangle_\varepsilon, \tag{4.29}$$

$$r_\varepsilon^2 = \partial_x(u\langle\rho-1\rangle_\varepsilon) - \langle\partial_x(u(\rho-1))\rangle_\varepsilon. \tag{4.30}$$

Multiplying (4.27) by $b'(\langle u_x\rangle_\varepsilon)$ and taking $\varepsilon\to 0$, we get (4.25) due to Lemma 4.2. Multiplying (4.28) by $b'(\langle\rho-1\rangle_\varepsilon)$ and taking $\varepsilon\to 0$, we have (4.26) due to Lemma 4.2. We should point out that since u_x and $\rho-1$ are uniformly bounded in $L^\infty([0,T]\times\mathbb{R})$ for any given $T>0$, the boundedness of $b'(z)$ is not necessary. This completes the proof of Lemma 4.5. □

The next lemma is important to cancel the interaction between u_x^n and $\rho^n - 1$ in the process of taking the weak limit.

Lemma 4.6. If $\sigma\in[0,2)$, then we have

$$\begin{aligned}&\partial_t(\overline{u_x^2} + \overline{(\rho-1)^2}) + \partial_x(\sigma u\overline{u_x^2} + u\overline{(\rho-1)^2}) \\ =& (3-\sigma)u^2u_x + 2Au_x\partial_x^2 p * u - 2u_x\bar{P}\end{aligned} \tag{4.31}$$

in the sense of distributions on $\mathbb{R}^+\times\mathbb{R}$.

Proof. Since $(u^n, \rho^n - 1)$ is a solution of the system (2.1) in $H^s(\mathbb{R})\times H^{s-1}(\mathbb{R})$ with $s>\frac{3}{2}$, we have

$$\begin{aligned}&\partial_t u_x^n + \sigma u^n u_{xx}^n + \frac{\sigma}{2}(u_x^n)^2 \\ =& \frac{1}{2}(\rho^n-1)^2 + \frac{3-\sigma}{2}(u^n)^2 + (\rho^n-1) + A\partial_x^2 p * u^n - P^n,\end{aligned} \tag{4.32}$$

$$\partial_t(\rho^n-1) + \partial_x(u^n(\rho^n-1)) = -u_x^n. \tag{4.33}$$

In view of the boundedness of u_x and $\rho-1$, multiplying (4.32) by $2u_x^n$, we get

$$\begin{aligned}&\partial_t(u_x^n)^2 + \sigma\partial_x(u^n(u_x^n)^2) \\ =& u_x^n(\rho^n-1)^2 + (3-\sigma)u_x^n(u^n)^2 + 2u_x^n(\rho^n-1) \\ &+ 2Au_x^n\partial_x^2 p * u^n - 2u_x^n P^n.\end{aligned} \tag{4.34}$$

Multiplying (4.33) by $2(\rho^n-1)$, we have

$$\partial_t(\rho^n-1)^2 + \partial_x(u^n(\rho^n-1)^2) = -u_x^n(\rho^n-1)^2 - 2u_x^n(\rho^n-1). \tag{4.35}$$

Adding (4.34) and (4.35), we obtain

$$\begin{aligned}&\partial_t((u_x^n)^2+(\rho^n-1)^2)+\partial_x(\sigma u^n(u_x^n)^2+u^n(\rho^n-1)^2)\\=&(3-\sigma)u_x^n(u^n)^2+2Au_x^n\partial_x^2p*u^n-2u_x^nP^n.\end{aligned}\tag{4.36}$$

Using Lemma 4.3 and (4.19), and then taking $n\to\infty$, we get (4.31). □

LEMMA 4.7. If $\sigma\in[0,2)$, there hold

$$\lim_{t\to0^+}\int_{\mathbb{R}}u_x^2dx=\lim_{t\to0^+}\int_{\mathbb{R}}\overline{u_x^2}dx=\int_{\mathbb{R}}u_{0,x}^2dx,\tag{4.37}$$

$$\lim_{t\to0^+}\int_{\mathbb{R}}(\rho-1)^2dx=\lim_{t\to0^+}\int_{\mathbb{R}}\overline{(\rho-1)^2}dx=\int_{\mathbb{R}}(\rho_0-1)^2dx.\tag{4.38}$$

Proof. By Theorem 3.1 and (4.10), for any $T>0$, we have that u^n is uniformly bounded in $L^\infty(0,T;H^1(\mathbb{R}))$ and u_t^n is uniformly bounded in $L^\infty(0,T;L^2(\mathbb{R}))$. Using Lemma 4.1 and Lemma 4.3, we get

$$u^n\to u\ \text{ in }\ C^w([0,T],H^1(\mathbb{R}))\ \text{ as }\ n\to\infty.\tag{4.39}$$

Similarly, by Theorem 3.1 and (4.33), we have that ρ^n-1 is uniformly bounded in $L^\infty([0,T],L^2(\mathbb{R}))$ and $(\rho^n-1)_t$ is uniformly bounded in $L^\infty([0,T],H^{-1}(\mathbb{R}))$. Thus, we obtain that

$$\rho^n-1\to\rho-1\ \text{ in }\ C^w([0,T],L^2(\mathbb{R}))\ \text{ as }\ n\to\infty\tag{4.40}$$

due to Lemma 4.1 and Lemma 4.3.

From (4.39) and (4.40), we get

$$u_x\rightharpoonup u_{0,x}\ \text{ and }\ \rho-1\rightharpoonup\rho_0-1\ \text{ in }\ L^2(\mathbb{R})\ \text{ as }\ t\to0^+.\tag{4.41}$$

It then follows that

$$\liminf_{t\to0^+}\int_{\mathbb{R}}u_x^2dx\ge\int_{\mathbb{R}}u_{0,x}^2dx,\tag{4.42}$$

$$\liminf_{t\to0^+}\int_{\mathbb{R}}(\rho-1)^2dx\ge\int_{\mathbb{R}}(\rho_0-1)^2dx.\tag{4.43}$$

Therefore, we deduce that

$$\begin{aligned}\liminf_{t\to0^+}\int_{\mathbb{R}}(u_x^2+(\rho-1)^2)dx&\ge\liminf_{t\to0^+}\int_{\mathbb{R}}u_x^2dx+\liminf_{t\to0^+}\int_{\mathbb{R}}(\rho-1)^2dx\\&\ge\int_{\mathbb{R}}u_{0,x}^2+(\rho_0-1)^2dx.\end{aligned}\tag{4.44}$$

On the other hand, from (3.11) we have that

$$\begin{aligned}&\int_{\mathbb{R}}(u^2+\overline{u_x^2}+\overline{(\rho-1)^2})dx \\ \leq&\liminf_{n\to\infty}\int_{\mathbb{R}}((u^n)^2+(u_x^n)^2+(\rho^n-1)^2)dx \\ =&\liminf_{n\to\infty}\int_{\mathbb{R}}((u_0^n)^2+(u_{0,x}^n)^2+(\rho_0^n-1)^2)dx \\ =&\int_{\mathbb{R}}((u_0)^2+(u_{0,x})^2+(\rho_0-1)^2)dx.\end{aligned} \tag{4.45}$$

Using the continuity of u and $\lim_{t\to 0^+}\int_{\mathbb{R}}u^2dx=\int_{\mathbb{R}}u_0^2dx$, we have

$$\limsup_{t\to 0^+}\int_{\mathbb{R}}(\overline{u_x^2}+\overline{(\rho-1)^2})dx\leq\int_{\mathbb{R}}(u_{0,x}^2+(\rho-1)^2)dx. \tag{4.46}$$

In view of (4.42)-(4.44) and (4.46), we get (4.37) and (4.38). □

Now we state the main theorem of this section.

THEOREM 4.1. There hold

$$\overline{u_x^2}=u_x^2 \quad \text{and} \quad \overline{(\rho-1)^2}=(\rho-1)^2, \quad \text{a.e. on } \mathbb{R}^+\times\mathbb{R}. \tag{4.47}$$

Proof. Taking $b(z)=z^2$ in Lemma 4.5 and adding (4.25) and (4.26), we get

$$\begin{aligned}&\partial_t(u_x^2+(\rho-1)^2)+\partial_x(\sigma u u_x^2+u(\rho-1)^2) \\ =&\sigma u_x(\overline{u_x^2}-u_x^2)+u_x(\overline{(\rho-1)^2}-(\rho-1)^2) \\ &+(3-\sigma)u_xu^2+2Au_x\partial_x^2p*u-2u_x\bar{P}.\end{aligned} \tag{4.48}$$

Subtracting (4.48) from (4.31), we get

$$\begin{aligned}&\partial_t(\overline{u_x^2}-u_x^2)+\partial_t(\overline{(\rho-1)^2}-(\rho-1)^2) \\ &-\partial_x(\sigma u(\overline{u_x^2}-u_x^2)+u(\overline{(\rho-1)^2}-(\rho-1)^2)) \\ =&\sigma(-u_x)(\overline{u_x^2}-u_x^2)+(-u_x)(\overline{(\rho-1)^2}-(\rho-1)^2).\end{aligned} \tag{4.49}$$

Using (3.9) and (4.3), we have that for any $T>0$,

$$u_x(t,x)\leq M(T) \quad \text{on} \quad [0,T]\times\mathbb{R}.$$

Then, integrating (4.49) by parts we obtain

$$\begin{aligned}&\int_{\mathbb{R}}(\overline{u_x^2}-u_x^2)+(\overline{(\rho-1)^2}-(\rho-1)^2)dx \\ \leq&2M(T)\int_0^t\int_{\mathbb{R}}(\overline{u_x^2}-u_x^2)+(\overline{(\rho-1)^2}-(\rho-1)^2)dx.\end{aligned} \tag{4.50}$$

Using Gronwall's inequality and Lemma 4.7, we conclude that

$$\int_{\mathbb{R}}(\overline{u_x^2}-u_x^2)+(\overline{(\rho-1)^2}-(\rho-1)^2)dx\leq 0. \tag{4.51}$$

On the other hand, it follows from (4.20) that

$$0 \leq \int_{\mathbb{R}} (\overline{u_x^2} - u_x^2) + (\overline{(\rho-1)^2} - (\rho-1)^2) dx \leq 0. \tag{4.52}$$

Thus,

$$\int_{\mathbb{R}} (\overline{u_x^2} - u_x^2) dx = \int_{\mathbb{R}} (\overline{(\rho-1)^2} - (\rho-1)^2) dx = 0. \tag{4.53}$$

This implies (4.47). □

5. Global weak solutions. Before giving the precise statement of the main result, we first introduce the definition of an admissible weak solution to the Cauchy problem (2.1).

DEFINITION 5.1. Let $(u_0, \rho_0 - 1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$. If there is a pair of functions $(u, \rho - 1) \in L^\infty([0,\infty); H^1(\mathbb{R}) \times L^2(\mathbb{R}))$ such that the system in (2.1) holds in the sense of distributions and $(u(t,x), \rho(t,x) - 1) \to (u_0, \rho_0 - 1)$ as $t \to 0^+$ in the sense of distributions, and if the energy inequality

$$\| u \|_{H^1(\mathbb{R})}^2 + \| \rho - 1 \|_{L^2(\mathbb{R})}^2 \leq \| u_0 \|_{H^1(\mathbb{R})}^2 + \| \rho_0 - 1 \|_{L^2(\mathbb{R})}^2 \tag{5.1}$$

holds, then $(u, \rho - 1)$ is called an admissible weak solution to the system in (2.1).

Proof of Theorem 1.1. Let $(u, \rho - 1)$ be a pair of functions which we have obtained in Lemma 4.3. Then, we have

$$\begin{cases} u_t + \sigma u u_x = A\partial_x p * u - \bar{P}, & t > 0,\ x \in \mathbb{R}, \\ (\rho - 1)_t + (u(\rho - 1))_x = -u_x, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0,x) - 1 = \rho_0(x) - 1, & x \in \mathbb{R}. \end{cases} \tag{5.2}$$

By Theorem 4.1 and Lemma 4.3, we obtain

$$\begin{aligned} \bar{P} =& p * (\frac{3-\sigma}{2} u^2 + \frac{\sigma}{2} \overline{u_x^2} + \frac{1}{2} \overline{(\rho-1)^2} + (\rho - 1)) \\ =& p * (\frac{3-\sigma}{2} u^2 + \frac{\sigma}{2} u_x^2 + \frac{1}{2} (\rho-1)^2 + (\rho - 1)). \end{aligned} \tag{5.3}$$

Thus $(u, \rho - 1)$ satisfies system (2.1). By (3.4) and (4.39)-(4.40), we get that $(u(t,x), \rho(t,x) - 1) \to (u_0, \rho_0 - 1)$ as $t \to 0^+$ in the sense of distributions. The energy inequality (5.1) is the straight conclusion of (3.11), due to the weak lower-semicontinuity of the norm.

We are now in the position to prove equality (1.6). Firstly, multiplying (4.27) by $\langle u_x \rangle_\varepsilon$ and multiplying (4.28) by $\langle \rho - 1 \rangle_\varepsilon$, we deduce by (4.47) that

$$\begin{aligned} &\partial_t \frac{1}{2} \langle u_x \rangle_\varepsilon^2 \\ =& \langle u_x \rangle_\varepsilon (-\sigma \partial_x (u \langle u_x \rangle_\varepsilon) + \frac{\sigma}{2} \langle u_x^2 \rangle_\varepsilon + \frac{1}{2} \langle (\rho-1)^2 \rangle_\varepsilon + \langle (\rho - 1) \rangle_\varepsilon \\ &+ \frac{3-\sigma}{2} \langle u^2 \rangle_\varepsilon + A \langle \partial_x^2 p * u \rangle_\varepsilon - \langle \bar{P} \rangle_\varepsilon + r_\varepsilon^1) \end{aligned} \tag{5.4}$$

and

$$\partial_t \frac{1}{2}\langle(\rho-1)\rangle_\varepsilon^2 = \langle\rho-1\rangle_\varepsilon(-\partial_x(u\langle\rho-1\rangle_\varepsilon) - \langle u_x\rangle_\varepsilon + r_\varepsilon^2). \tag{5.5}$$

Second, mollifying the first equation in (2.1) and multiplying by $\langle u\rangle_\varepsilon$, we have

$$\partial_t \frac{1}{2}\langle u\rangle_\varepsilon^2 = \langle u\rangle_\varepsilon(-\sigma\langle uu_x\rangle_\varepsilon - A\langle\partial_x p * u\rangle_\varepsilon - \langle\partial_x \bar{P}\rangle_\varepsilon). \tag{5.6}$$

Given any $T > 0$, adding (5.4)-(5.6) and integrating by parts, we obtain that for $0 < t \leq T$,

$$\begin{aligned}
&\int_{\mathbb{R}}(\langle u\rangle_\varepsilon^2 + \langle u_x\rangle_\varepsilon^2 + \langle\rho-1\rangle_\varepsilon^2)(t,x)dx - \int_{\mathbb{R}}(\langle u\rangle_\varepsilon^2 + \langle u_x\rangle_\varepsilon^2 + \langle\rho-1\rangle_\varepsilon^2)(0,x)dx \qquad (5.7)\\
=&2\int_0^t\int_{\mathbb{R}}(-\sigma\langle u_x\rangle_\varepsilon\partial_x(u\langle u_x\rangle_\varepsilon) + \frac{\sigma}{2}\langle u_x\rangle_\varepsilon\langle u_x^2\rangle_\varepsilon + \frac{1}{2}\langle u_x\rangle_\varepsilon\langle(\rho-1)^2\rangle_\varepsilon\\
&+ \frac{3-\sigma}{2}\langle u_x\rangle_\varepsilon\langle u^2\rangle_\varepsilon - \langle\rho-1\rangle_\varepsilon\partial_x(u\langle\rho-1\rangle_\varepsilon) - \sigma\langle u\rangle_\varepsilon\langle uu_x\rangle_\varepsilon\\
&+ \langle u_x\rangle_\varepsilon r_\varepsilon^1 + \langle\rho-1\rangle_\varepsilon r_\varepsilon^2)dx.
\end{aligned}$$

Since $\| u_x \|_{L^\infty(\mathbb{R})} \leq M(T)$ and $\| \rho-1 \|_{L^\infty(\mathbb{R})} \leq M(T)$, we infer that

$$\begin{aligned}
\| \langle u_x\rangle_\varepsilon \|_{L^\infty(\mathbb{R})} &\leq M(T),\\
\| \langle\rho-1\rangle_\varepsilon \|_{L^\infty(\mathbb{R})} &\leq M(T),
\end{aligned}$$

uniformly for ε. Using Lemma 4.2 and taking $\varepsilon \to 0$, and then applying the Lebesgue dominated convergence theorem, we infer that

$$\int_{\mathbb{R}}(u^2 + u_x^2 + (\rho-1)^2)(t,x)dx = \int_{\mathbb{R}}(u_0^2 + u_{0,x}^2 + (\rho_0-1)^2)(x)dx. \tag{5.8}$$

By the arbitrariness of T, we obtain that equality (1.6) holds. Now, we prove the strong continuity of $(u, \rho-1)$. Given any $T > 0$, (4.39)-(4.40) imply that

$$(u, \rho-1) \in C^w([0,T]; H^1(\mathbb{R}) \times L^2(\mathbb{R})). \tag{5.9}$$

Then, (5.8) yields that $\| (u(t), \rho(t)-1) \|_{H^1(\mathbb{R})\times L^2(\mathbb{R})}$ is continuous. The weak continuity and the continuity of norm yields the strong continuity. Thus, for the arbitrary of T, we obtain that

$$(u, \rho-1) \in C(\mathbb{R}^+; H^1(\mathbb{R}) \times L^2(\mathbb{R})). \tag{5.10}$$

This completes the proof of Theorem 1.1. □

Acknowledgments. We thank the referees for valuable comments and suggestions. The work of the second author was partially supported by the NSF grants DMS-0906099 and DMS-1207840, the NSF-China grant-11271192, and the NHARP grant-003599-0001-2009. The work of the first and third authors was partially supported by NNSFC (No. 11271382), RFDP (No. 20120171110014), and the key project of Sun Yat-sen University (No. c1185).

References

[1] R. Beals, D. Sattinger, and J. Szmigielski, Multipeakons and a theorem of Stieltjes, *Inverse Problems,* **15** (1999), 1–4. MR1675325

[2] A. Bressan and A. Constantin, Global conservative solutions of the Camassa-Holm equation, *Arch. Rat. Mech. Anal.,* **183** (2007), 215–239. MR2278406 (2007j:35183)

[3] A. Bressan and A. Constantin, Global dissipative solutions of the Camassa-Holm equation, *Anal. Appl.*, **5** (2007), 1–27. MR2288533 (2007k:35394)

[4] R. Camassa and D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Letters,* **71** (1993), 1661–1664. MR1234453 (94f:35121)

[5] R. Camassa, D. Holm and J. Hyman, A new integrable shallow water equation, *Adv. Appl. Mech.,* **31** (1994), 1–33.

[6] M. Chen, S. Q. Liu and Y. Zhang, A 2-component generalization of the Camassa-Holm equation and its solutions, *Lett. Math. Phys.*, **75** (2006), 1–15. MR2207043 (2006k:37177)

[7] R. M. Chen and Y. Liu, Wave breaking and global existence for a generalized two-component Camassa-Holm system, *Int. Math. Res. Not.,* **6** (2011), 1381–1416. MR2806508

[8] R. M. Chen, Y. Liu and Z. Qiao, Stability of solitary waves and global existence of a generalized two-component Camassa-Holm system, *Comm. Partial Differential Equations,* **36** (2011), 2162–2188. MR2852073

[9] A. Constantin, The Hamiltonian structure of the Camassa-Holm equation, *Exposition. Math.,* **15** (1997), 53–85. MR1438436 (98c:76011)

[10] A. Constantin, Existence of permanent and breaking waves for a shallow water equation: a geometric approach, *Ann. Inst. Fourier, Grenoble,* **50** (2000), 321–362. MR1775353 (2002d:37125)

[11] A. Constantin, On the scattering problem for the Camassa-Holm equation, *Proc. R. Soc. London A,* **457** (2001), 953–970. MR1875310 (2002k:37132)

[12] A. Constantin, The trajectories of particles in Stokes waves, *Invent. Math.,* **166** (2006), 523–535. MR2257390 (2007j:35240)

[13] A. Constantin and J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Mathematica,* **181** (1998), 229–243. MR1668586 (2000b:35206)

[14] A. Constantin and J. Escher, Global existence and blow-up for a shallow water equation, *Annali Sc. Norm. Sup. Pisa,* **26** (1998), 303–328. MR1631589 (99m:35202)

[15] A. Constantin and J. Escher, Well-posedness, global existence and blowup phenomena for a periodic quasi-linear hyperbolic equation, *Comm. Pure Appl. Math.,* **51** (1998), 475–504. MR1604278 (98k:35165)

[16] A. Constantin and J. Escher, On the blow-up rate and the blow-up of breaking waves for a shallow water equation, *Math. Z.,* **233** (2000), 75–91. MR1738352 (2001b:35258)

[17] A. Constantin and J. Escher, Particle trajectories in solitary water waves, *Bull. Amer. Math. Soc.,* **44** (2007), 423–431. MR2318158 (2008m:76019)

[18] A. Constantin and J. Escher, Analyticity of periodic traveling free surface water waves with vorticity, *Ann. of Math.,* **173** (2011), 559-568. MR2753609 (2012a:35367)

[19] G. M. Coclite, H. Holden, and K. H. Karlsen, Global weak solutions to a generalized hyperelastic-rod wave equation, *SIAM J. Math. Anal.,* **37** (2006), 1044–1069. MR2192287 (2007h:35330)

[20] A. Constantin and R. I. Ivanov, On an integrable two-component Camassa-Holm shallow water system, *Phys. Lett. A,* **372** (2008), 7129–7132. MR2474608 (2009m:35418)

[21] A. Constantin and R. S. Johnson, Propagation of very long water waves, with vorticity, over variable depth, with applications to tsunamis, *Fluid Dynam. Res.,* **40** (2008), 175–211. MR2369543 (2009b:76009)

[22] A. Constantin and B. Kolev, Geodesic flow on the diffeomorphism group of the circle, *Comment. Math. Helv.,* **78** (2003), 787–804. MR2016696 (2004k:37163)

[23] A. Constantin and D. Lannes, The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations, *Arch. Rational Mech. Anal.,* **192** (2009), 165–186. MR2481064 (2010f:35334)

[24] A. Constantin and L. Molinet, Global weak solutions for a shallow water equation, *Comm. Math. Phys.,* **211** (2000), 45–61. MR1757005 (2001b:35247)

[25] A. Constantin and W. A. Strauss, Stability of peakons, *Comm. Pure Appl. Math.,* **53** (2000), 603–610. MR1737505 (2001b:35252)

[26] A. Constantin and W. A. Strauss, Stability of a class of solitary waves in compressible elastic rods, *Phys. Lett. A,* **270** (2000), 140–148. MR1763691 (2001e:74048)

[27] H. H. Dai, Model equations for nonlinear dispersive waves in a compressible Mooney-Rivlin rod, *Acta Mech.,* **127** (1998), 193–207. MR1606738 (98m:73037)

[28] R. Danchin, A few remarks on the Camassa-Holm equation, *Differential and Integral Equations* **14** (2001), 953–988. MR1827098 (2002e:35204)

[29] H. R. Dullin, G. A. Gottwald, and D. D. Holm, An integrable shallow water equation with linear and nonlinear dispersion, *Phys. Rev. Letters,* **87** (2001), 4501–4504.

[30] J. Escher, O. Lechtenfeld and Z. Yin, Well-posedness and blow-up phenomena for the 2-component Camassa-Holm equation, *Discrete Contin. Dyn. Syst. Series A,* **19** (2007), 493–513. MR2335761 (2008j:35154)

[31] J. Escher and Z. Yin, Initial boundary value problems of the Camassa-Holm equation, *Commun. Partial Differential Equations,* **33** (2008), 377–395. MR2398234 (2010h:35336)

[32] J. Escher and Z. Yin, Initial boundary value problems for nonlinear dispersive wave equations, *J. Funct. Anal.*, **256** (2009), 479–508. MR2476950 (2009k:35258)

[33] A. Fokas and B. Fuchssteiner, Symplectic structures, their Bäcklund transformation and hereditary symmetries, *Physica D,* **4** (1981), 47–66. MR636470 (84j:58046)

[34] C. Guan and Z. Yin, Global existence and blow-up phenomena for an integrable two-component Camassa-Holm shallow water system, *J. Differential Equations,* **248** (2010), 2003–2014. MR2595712 (2011b:35072)

[35] C. Guan and Z. Yin, Global weak solutions for a two-component Camassa-Holm shallow water system, *J. Funct. Anal.,* **260** (2011), 1132–1154. MR2747018 (2011m:35238)

[36] G. Gui and Y. Liu, On the Cauchy problem for the two-component Camassa-Holm system, *Math. Z.*, **268** (2010), 45–66. MR2805424

[37] G. Gui, and Y. Liu, On the global existence and wave-breaking criteria for the two-component Camassa-Holm system, *J. Funct. Anal.,* **258** (2010), 4527–4578. MR2609545 (2011i:35215)

[38] R. I. Ivanov, Two-component integrable systems modeling shallow water waves: the constant vorticity case, *Wave Motion* **46** (2009), 389–396. MR2598636 (2011a:35415)

[39] R. S. Johnson, Camassa-Holm, Korteweg-de Vries and related models for water waves, *J. Fluid. Mech.,* **457** (2002), 63–82. MR1894796 (2003b:76026)

[40] T. Kato, Quasi-linear equations of evolution, with applications to partial differential equations, in "Spectral Theory and Differential Equations", Lecture Notes in Math., Vol. 448, Springer Verlag, Berlin (1975), 25–70. MR0407477 (53:11252)

[41] B. Kolev, Bi-Hamiltonian systems on the dual of the Lie algebra of vector fields of the circle and periodic shallow water equations, *Philos. Trans. Roy. Soc. London A*, **365** (2007), 2333–2357. MR2329152 (2008e:37071)

[42] P. L. Lions, Mathematical Topics in Fluid Mechanics, Vol. I. *Incompressible models.* Oxford Lecture Series in Mathematics and Applications, 3. Clarendon, Oxford University Press, New York, 1996. MR1422251 (98b:76001)

[43] M. Lakshmanan, Integrable nonlinear wave equations and possible connections to tsunami dynamics, in Tsunami and nonlinear waves, pp. 31–49, Springer, Berlin, 2007. MR2364923 (2008m:35309)

[44] Y. Li and P. Olver, Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation, *J. Differential Equations,* **162** (2000), 27–63. MR1741872 (2002a:35185)

[45] O. Mustafa, On smooth traveling waves of an integrable two-component Camassa-Holm shallow water system, *Wave Motion* **46** (2009), 397–402. MR2598637 (2010m:76037)

[46] Z. Popowicz, A 2-component or $N = 2$ supersymmetric Camassa-Holm equation, *Phys. Lett. A,* **354** (2006), 110–114. MR2220696

[47] P. Olver and P. Rosenau, Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support, *Phys. Rev. E,* **53** (1996), 1900–1906. MR1401317 (97c:35172)

[48] G. Rodriguez-Blanco, On the Cauchy problem for the Camassa-Holm equation, *Nonlinear Anal.,* **46** (2001), 309–327. MR1851854 (2002i:35172)

[49] A. Shabat, and L. Martínez Alonso, On the Prolongation of a Hierarchy of Hydrodynamic Chains, In Proceedings of the NATO Advanced Research Workshop, Cadiz, Spain 2002, 263-280, NATO Science Series, Dordrecht: Kluwer Academic Publishers, 2004. MR2153341 (2006d:37129)

[50] J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pura Appl.,* **146** (1987). MR916688 (89c:46055)

[51] J. F. Toland, Stokes waves, *Topol. Methods Nonlinear Anal.,* **7** (1996), 1–48. MR1422004 (97j:35130)

[52] G. B. Whitham, Linear and nonlinear waves, J. Wiley & Sons, New York, 1980. MR0483954 (58:3905)

[53] Z. Xin and P. Zhang, On the weak solutions to a shallow water equation, *Comm. Pure Appl. Math.,* **53** (2000), 1411–1433. MR1773414 (2001m:35278)

[54] Z. Yin, Well-posedness, blowup, and global existence for an integrable shallow water equation, *Discrete Contin. Dyn. Syst. Series A*, **11** (2004), 393–411. MR2083424 (2005j:35022)

[55] P. Zhang and Y. Liu, Stability of solitary waves and wave-breaking phenomena for the two-component Camassa-Holm system, *International Mathematics Research Notices,* **211** (2009), 41pages. MR2646352 (2011e:35285)