

On the GLR and UMP tests in the family with support dependent on the parameter

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Received: 28 September 2014; Accepted: 27 June 2015

Editor: Rahim Mahmoudvand

Abstract Some general results about the GLR tests, for testing simple hypothesis versus two-sided hypothesis, in the family with support dependent on the parameter, are obtained. In addition, we show that such GLR tests are equivalent to the UMP tests in the same problems. Moreover, we derive the general form of the UMP tests for testing an interval hypothesis versus two-sided alternative.

Keywords Uniformly most powerful test, Generalized likelihood ratio test, Two-sided hypothesis.

AMS 2010 subject classifications 62F03

DOI: 10.19139/soic.v3i3.107

1. Introduction and Background

The problem of finding uniformly most powerful (UMP) test for testing the simple hypothesis $H_0 : \theta = \theta_0$ against two-sided alternative $H_1 : \theta \neq \theta_0$ has been an interesting problem in testing statistical hypotheses. It is well known that, for distributions belong to the exponential family, such UMP tests do not exist (see [7]). On the other hand, there has been some attempts to find UMP tests for other distributions and/or for the problem of testing one-sided hypotheses. Historically, studies on UMP tests started in the 1940's. Lehmann and Stein [6] presented some most powerful tests for the parameter of a normal distribution. Birnbaum [3] and Pratt [12] offered the solutions for UMP tests for composite hypotheses in the uniform distributions with both end points of the support dependent on the parameter. Takeuchi [15] constructed UMP test for the location parameter of an exponential distribution despite the existence of a nuisance parameter. Kabe and Laurent [5] exhibited a UMP test for a two-sided hypothesis in the double exponential distribution with a nuisance parameter. Edelman [4] showed that the usual two-sided test is actually uniformly most powerful within the class of tests with critical regions that are symmetric about the null hypothesis. Nomakuchi [11] studied the UMP tests in the presence of a nuisance parameter for uniform distribution. A new method to determine UMP tests in discrete sample spaces was proposed by Scherb [14]. Wang et al. [16] constructed a UMPU test for comparing the means of two negative binomial distributions. Migliorati [10] demonstrated that a UMP test exists for testing two-sided hypothesis in the family with support dependent on the parameter. Also, McDermott and Wang [9] investigated a construction of uniformly more powerful tests for hypotheses about linear inequalities. Aaberge and Zhang [1] presented a class of exact UMP unbiased tests

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for testing conditional symmetry against various alternative diagonals-parameter. The UMP two-sided test in the family with right and left extreme points of the support dependent on the parameter is obtained by Sayyareh et al. [13].

But, as far as the authors know, there has been neither general result about the existence and formulation of the two-sided generalized likelihood ratio (GLR) test in the family with support dependent to parameter, nor about the formal relationship between such tests and the related UMP tests in this family.

In this paper, we obtain the GLR tests for testing hypothesis $H_0 : \theta = \theta_0$ versus two-sided alternative $H_1 : \theta \neq \theta_0$, in the family of distributions of the following forms

$$f_\theta(x) = a(\theta)b(x) \quad ; \quad c < x < \theta, \quad a(\theta) > 0, \quad (1)$$

and

$$g_\theta(x) = p(\theta)q(x) \quad ; \quad \theta < x < d, \quad p(\theta) > 0. \quad (2)$$

We also demonstrate that UMP tests for testing the above hypotheses are equivalent to the GLR tests. In addition, we derive the general form of UMP tests for testing $H_0 : \theta_1 \leq \theta \leq \theta_2$ versus $H_1 : \theta < \theta_1$ or $\theta > \theta_2$ in the families with the right or the left extreme point of the support dependent on the parameter.

This paper is organized as follows: In Section 2, we obtain the general form of the GLR test for testing simple hypothesis versus two-sided alternative in the family with the right or left extreme point of the support dependent on the parameter. In Section 3, we obtain the UMP test for testing $H_0 : \theta_1 \leq \theta \leq \theta_2$ versus $H_1 : \theta < \theta_1$ or $\theta > \theta_2$ in the families with the right or the left extreme point of the support dependent on the parameter.

In the following, we will use the notations $X_{(1)}$ for $\min(X_1, \dots, X_n)$, and $X_{(n)}$ for $\max(X_1, \dots, X_n)$.

2. GLR two-sided test in the family with right (left) extreme point of the support dependent on the parameter

Let us begin this section with a Lemma.

Lemma 1

i) If

$$f_\theta(x) = a(\theta)b(x) \quad ; \quad c < x < \theta, \quad a(\theta) > 0,$$

be a continuous function, then $a(\cdot)$ is a decreasing function.

ii) If

$$g_\theta(x) = p(\theta)q(x) \quad ; \quad \theta < x < d, \quad p(\theta) > 0,$$

be a continuous function, then $p(\cdot)$ is an increasing function.

Proof

i) $\int_c^\theta a(\theta)b(x) dx = 1$, implies that $\int_c^\theta b(x) dx = 1/a(\theta)$. Now we differentiate from both sides of the second relation to obtain $b(\theta) = -a'(\theta)/(a(\theta))^2$. But $b(\cdot)$ is a nonnegative function, therefore $a(\cdot)$ is a decreasing function.

ii) The proof can be done in a similar way to the case (i). \square

Now, we obtain the general forms of the GLR tests for testing hypothesis $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ in families with density (1) and (2).

Theorem 1

Suppose that $\underline{X} = (X_1, X_2, \dots, X_n)$ is a random sample, with observed value $\underline{x} = (x_1, x_2, \dots, x_n)$, from a distribution with density function (1). Then a GLR test of size α for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ is given by

$$\psi_1^*(\underline{x}) = \begin{cases} 1 & x_{(n)} > \theta_0 \text{ or } x_{(n)} < a^{-1}\left(a(\theta_0)\alpha^{-\frac{1}{n}}\right), \\ 0 & \text{otherwise.} \end{cases}$$

Proof

Let $\Theta = \{\theta : \theta > c\}$, $\Theta_0 = \{\theta : \theta = \theta_0\}$

$$L(\theta; \underline{x}) = (a(\theta))^n \prod_{i=1}^n b(x_i) I_{(c, \theta)}(x_i) = \begin{cases} (a(\theta))^n \prod_{i=1}^n b(x_i) & x_{(n)} \leq \theta, \\ 0 & x_{(n)} > \theta. \end{cases}$$

It is obvious that, $MLE(\theta)_{\theta \in \Theta} = X_{(n)}$ and $MLE(\theta)_{\theta \in \Theta_0} = \theta_0$, and

$$\lambda(\underline{x}) = \frac{\sup_{\Theta_0} L(\theta; \underline{x})}{\sup_{\Theta} L(\theta; \underline{x})} = \frac{(a(\theta_0))^n \prod_{i=1}^n b(x_i)}{(a(x_{(n)}))^n \prod_{i=1}^n b(x_i)} = \begin{cases} \left(\frac{a(\theta_0)}{a(x_{(n)})}\right)^n & x_{(n)} \leq \theta_0, \\ 0 & x_{(n)} > \theta_0. \end{cases}$$

So, the GLR test rejects H_0 , iff $x_{(n)} > \theta_0$ or $\left(a(\theta_0)/a(x_{(n)})\right)^n \leq k_\alpha$ when $x_{(n)} \leq \theta_0$; and iff $x_{(n)} > \theta_0$ or $x_{(n)} < k'_\alpha$ when $x_{(n)} \leq \theta_0$, where k'_α is chosen so that $P_{\theta_0}(X_{(n)} < k'_\alpha) = \alpha$. Using Lemma 1 and distribution of $X_{(n)}$, we have

$$\alpha = \int_c^{k'_\alpha} nb(t) \frac{(a(\theta_0))^n}{(a(t))^{n-1}} dt, \quad k'_\alpha = a^{-1}\left(a(\theta_0)\alpha^{-\frac{1}{n}}\right).$$

Hence the GLR test rejects H_0 iff $X_{(n)} < a^{-1}\left(a(\theta_0)\alpha^{-\frac{1}{n}}\right)$ for $X_{(n)} \leq \theta_0$ or $X_{(n)} > \theta_0$. □

Theorem 2

Suppose that $\underline{X} = (X_1, X_2, \dots, X_n)$ is a random sample, with observed value $\underline{x} = (x_1, x_2, \dots, x_n)$, from a distribution with density function (2). Then a GLR test of size α for the simple hypothesis $H_0 : \theta = \theta_0$ versus two-sided alternative $H_1 : \theta \neq \theta_0$ is given by

$$\psi_2^*(\underline{x}) = \begin{cases} 1 & x_{(1)} < \theta_0 \text{ or } x_{(1)} > p^{-1}\left(p(\theta_0)\alpha^{-\frac{1}{n}}\right), \\ 0 & \text{otherwise.} \end{cases}$$

Proof

The proof is similar to the proof of Theorem 1. □

Example 1

Let $\underline{X} = (X_1, X_2, \dots, X_n)$ is a random sample, with observed value $\underline{x} = (x_1, x_2, \dots, x_n)$, from a distribution with density function

$$f_\theta(x) = \frac{2x}{\theta^2} \quad ; \quad 0 < x < \theta.$$

The GLR test of size α for hypothesis $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ is given by

$$\psi^*(\underline{x}) = \begin{cases} 1 & x_{(n)} > \theta_0 \text{ or } x_{(n)} < \theta_0 \alpha^{\frac{1}{2n}} \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1

It is shown by Birkes [2] that, in the one-parameter exponential family the GLR test for testing one-sided hypotheses coincides with the UMP test. He also shown that in this family under certain conditions on the likelihood function, GLR test for testing a two-sided hypothesis is usually coincide with the UMPU test. But as the authors know, there has been no works on the relationship between UMP tests and GLR tests for testing a simple versus two-sided hypotheses.

Remark 2

The GLR tests obtained in Theorems 1 and 2 are coincided with the UMP tests in the same problems (see Theorems 4.3 and 4.4 in [13]).

3. UMP test for testing composite hypotheses

In this section, we will acquire UMP tests for testing hypothesis $H_0 : \theta_1 \leq \theta \leq \theta_2$ versus $H_1 : \theta < \theta_1$ or $\theta > \theta_2$ in the family with support dependent on the parameter. Note that, such tests don't exist for the exponential family, but it is shown that there exist a level α UMPU test for such hypotheses for one-parameter exponential family (see [7]).

Theorem 3

Suppose that $\underline{X} = (X_1, X_2, \dots, X_n)$ is a random sample, with observed value $\underline{x} = (x_1, x_2, \dots, x_n)$, from a distribution with density function (1). Then a UMP test of size α for a hypothesis $H_0 : \theta_1 \leq \theta \leq \theta_2$ versus $H_1 : \theta < \theta_1$ or $\theta > \theta_2$ is given by

$$\phi^*(\underline{x}) = \begin{cases} 1 & x_{(n)} \leq k_1 \text{ or } x_{(n)} > k_2, \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

where $k_1 = a^{-1} \left(a(\theta_1) \alpha^{-\frac{1}{n}} \right)$ and $k_2 = a^{-1} \left(a(\theta_2) \left\{ 1 - \alpha \left(1 - \left(\frac{a(\theta_2)}{a(\theta_1)} \right)^n \right) \right\}^{-\frac{1}{n}} \right)$ are chosen so that $\alpha = E_{\theta_1}[\phi^*(\underline{X})] = E_{\theta_2}[\phi^*(\underline{X})]$.

Proof

First we divide H_1 into $H'_1 : \theta < \theta_1$ and $H''_1 : \theta > \theta_2$. Then we find a UMP test for H'_1 and H''_1 .

1. Consider the problem of testing hypotheses

$$\begin{cases} H_0 : \theta \geq \theta_1, & (\theta_1 < \theta_2) \\ H'_1 : \theta < \theta_1. \end{cases} \tag{4}$$

Since the probability density function (1) has the monotone likelihood ratio at $X_{(n)}$, so a UMP test for (4) is as follows

$$\phi_1(\underline{x}) = \begin{cases} 1 & x_{(n)} \leq k_1, \\ 0 & x_{(n)} > k_1, \end{cases}$$

which is a UMP test of size α on the condition that $\alpha = E_{\theta_1}[\phi_1(\underline{X})]$. Meanwhile it is easy to show that $k_1 = a^{-1} \left(a(\theta_1) \alpha^{-\frac{1}{n}} \right)$. Therefore, the power function of ϕ_1 is as follows

$$\begin{aligned} \pi_{\phi_1}(\theta) &= P_{\theta}(X_{(n)} < k_1) \\ &= \begin{cases} 1 & \theta \leq k_1, \\ \alpha \left(\frac{a(\theta)}{a(\theta_1)} \right)^n & \theta > k_1. \end{cases} \end{aligned}$$

2. Now, we find a UMP for

$$\begin{cases} H_0 : \theta = \theta_0, \\ H''_1 : \theta > \theta_2 > \theta_0, \end{cases} \tag{5}$$

where $\theta_0 \in [\theta_1, \theta_2]$. For this purpose, first consider the problem of testing the following simple hypotheses

$$\begin{cases} H_0 : \theta = \theta_0, \\ H''_{12} : \theta = \theta'_2 \quad (\theta_0 \leq \theta_2 < \theta'_2). \end{cases} \tag{6}$$

According to Neyman-Pearson Lemma, each test conforming with

$$\phi_2(\underline{x}) = \begin{cases} 1 & r(\underline{x}) > k_2, \\ \gamma & r(\underline{x}) = k_2, \\ 0 & r(\underline{x}) < k_2. \end{cases} \tag{7}$$

is a MP test of size α for the hypotheses (6) on the condition that (see [7])

$$\alpha = E_{\theta_0}[\phi_2(\underline{X})] = P_{\theta_0}(r(\underline{X}) > k_2) + \gamma P_{\theta_0}(r(\underline{X}) = k_2),$$

where

$$r(\underline{x}) = \frac{f_{\theta'_2}(\underline{x})}{f_{\theta_0}(\underline{x})} = \left(\frac{a(\theta'_2)}{a(\theta_0)}\right)^n \frac{I_{(c, \theta'_2)}(x_{(n)})}{I_{(c, \theta_0)}(x_{(n)})}.$$

On the other hand, we know that $\theta_0 \in [\theta_1, \theta_2]$, so we can choose each value in this interval for θ_0 . Now, let $\theta_0 = \theta_2$, thus

$$r(\underline{x}) = \begin{cases} \left(\frac{a(\theta'_2)}{a(\theta_0)}\right)^n & x_{(n)} \leq \theta_0, \\ \infty & x_{(n)} > \theta_0. \end{cases}$$

It is clear that $r(\underline{x})$ is increasing with respect to $x_{(n)}$. So, ϕ_2 consists of

$$\phi_2(\underline{x}) = \begin{cases} 1 & x_{(n)} > k_2, \\ 0 & x_{(n)} \leq k_2, \end{cases}$$

is a MP test of size α for the related hypotheses. But, ϕ_2 depends on θ'_2 only by $\theta_0 \leq \theta_2 < \theta'_2$, thus ϕ_2 is a UMP test of size α for testing hypotheses (5) when

$$\alpha = E_{\theta_2}[\phi_2(\underline{X})].$$

It can be shown that the power function of ϕ_2 is given by

$$\pi_{\phi_2}(\theta) = 1 - (1 - \alpha) \left(\frac{a(\theta)}{a(\theta_2)}\right)^n, \quad \theta \geq k_2.$$

Now, it is seen that $\phi^*(\underline{x})$ is a UMP test of size α for testing the hypothesis $H_0 : \theta_1 \leq \theta \leq \theta_2$ versus $H_1 : \theta < \theta_1$ or $\theta > \theta_2$. Moreover, the power function of ϕ^* is given by

$$\begin{aligned} \pi_{\phi^*}(\theta) &= P_{\theta}(X_{(n)} < k_1) + P_{\theta}(X_{(n)} > k_2) \\ &= \begin{cases} 1 & \theta < k_1, \\ \alpha \left(\frac{a(\theta)}{a(\theta_1)}\right)^n & k_1 \leq \theta < k_2, \\ 1 - (1 - \alpha) \left(\frac{a(\theta)}{a(\theta_2)}\right)^n & \theta \geq k_2. \end{cases} \end{aligned}$$

□

Theorem 4

Suppose that $\underline{X} = (X_1, X_2, \dots, X_n)$ is a random sample, with observed value $\underline{x} = (x_1, x_2, \dots, x_n)$, from a distribution with density function (2). Then a UMP test of size α for a hypothesis $H_0 : \theta_1 \leq \theta \leq \theta_2$ versus $H_1 : \theta < \theta_1$ or $\theta > \theta_2$ is given by

$$\varphi^*(\underline{x}) = \begin{cases} 1 & x_{(1)} < m_1 \text{ or } x_{(1)} \geq m_2, \\ 0 & \text{otherwise,} \end{cases}$$

where m_1 and m_2 are chosen so that $\alpha = E_{\theta_1}[\varphi^*(\underline{X})] = E_{\theta_2}[\varphi^*(\underline{X})]$.

Proof

The proof is similar to the proof of Theorem 3.

□

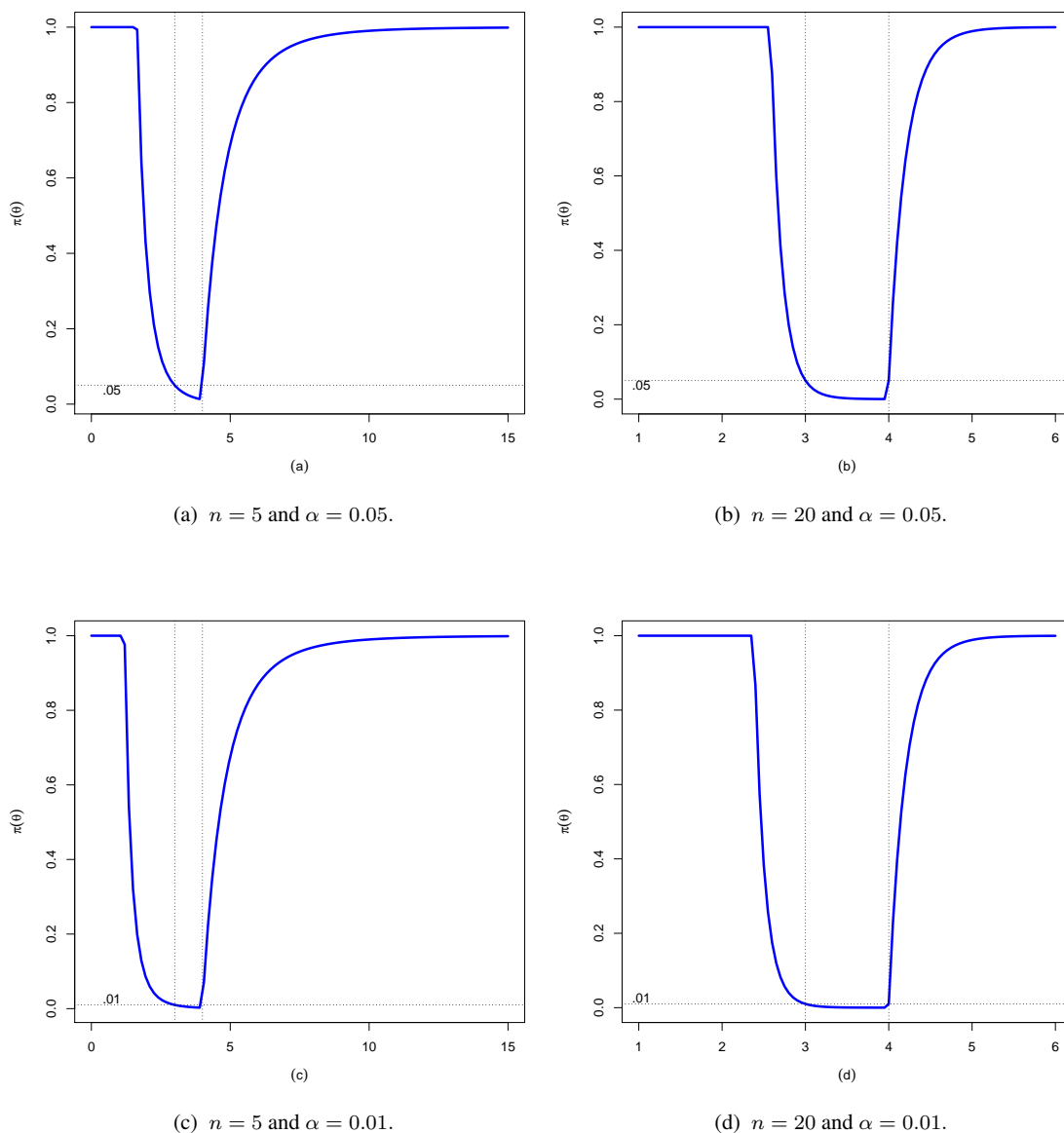


Figure 1. The curves of $\pi_{\phi^*}(\theta)$ in Example 2.

Remark 3

If in Theorems 3 and 4, $\theta_1 = \theta_2$, then, as an especial case, the rejection region of these tests would be more or less similar to the rejection region of Theorems 4.3 and 4.4 in [13].

Example 2

Let X_1, X_2, \dots, X_n be a random sample from a population with the Uniform probability density function

$$f_{\theta}(x) = \frac{1}{\theta} ; 0 < x < \theta.$$

According to Theorem 3, a UMP test of size α for testing the hypothesis $H_0 : \theta_1 \leq \theta \leq \theta_2$ versus $H_1 : \theta < \theta_1$ or $\theta > \theta_2$ is obtained as

$$\phi^*(\underline{x}) = \begin{cases} 1 & x_{(n)} \leq \theta_1 \alpha^{\frac{1}{n}} \text{ or } x_{(n)} > \theta_2 \left\{ 1 - \alpha \left(1 - \left(\frac{\theta_1}{\theta_2} \right)^n \right) \right\}^{\frac{1}{n}}, \\ 0 & \text{otherwise.} \end{cases}$$

It should be mentioned that, if $\theta_1 = \theta_2 = \theta_0$, then ϕ^* is coincided with the UMP test for testing simple hypothesis $H_0 : \theta = \theta_0$ versus two-sided hypothesis $H_1 : \theta \neq \theta_0$ (see [7], pp. 92). The power function of the above test is given by

$$\begin{aligned} \pi_{\phi^*}(\theta) &= P_{\theta}(X_{(n)} < \theta_1 \alpha^{\frac{1}{n}}) + P_{\theta}(X_{(n)} > \theta_2 \left\{ 1 - \alpha \left(1 - \left(\frac{\theta_1}{\theta_2} \right)^n \right) \right\}^{\frac{1}{n}}) \\ &= \begin{cases} 1 & \theta < \theta_1 \alpha^{\frac{1}{n}}, \\ \alpha \left(\frac{\theta_1}{\theta} \right)^n & \theta_1 \alpha^{\frac{1}{n}} \leq \theta < \theta_2 \left\{ 1 - \alpha \left(1 - \left(\frac{\theta_1}{\theta_2} \right)^n \right) \right\}^{\frac{1}{n}}, \\ 1 - (1 - \alpha) \left(\frac{\theta_2}{\theta} \right)^n & \theta \geq \theta_2 \left\{ 1 - \alpha \left(1 - \left(\frac{\theta_1}{\theta_2} \right)^n \right) \right\}^{\frac{1}{n}}. \end{cases} \end{aligned}$$

Figure 1 shows the curves of $\pi_{\phi^*}(\theta)$ for the cases $\theta_1 = 3$, $\theta_2 = 4$, $n = 5$ and $n = 20$, for two usual significance levels $\alpha = 0.05$ and $\alpha = 0.01$.

4. Conclusion

The problem of testing hypothesis in the family with support dependent on the unknown parameter was considered. The GLR test and UMP test were obtained for testing a simple versus two-sided hypotheses in the considered family. The extension of the obtained results to other family of distributions, such as the Pitman family, is a potential work for future research.

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