On the Grassmann modules for the unitary groups — Source link

Bart De Bruyn

Institutions: Ghent University

Published on: 30 Mar 2010 - Linear & Multilinear Algebra (Taylor & Francis Group)

Topics: Unitary group, Group (mathematics), Vector space, Field (mathematics) and Linear algebra

On the Grassmann modules for the unitary groups

Bart De Bruyn

Ghent University, Department of Pure Mathematics and Computer Algebra, Krijgslaan 281 (S22), B-9000 Gent, Belgium, E-mail: bdb@cage.ugent.be

Abstract

Let V be $2n$-dimensional vector space over a field $\mathbb{K}$ equipped with a nondegenerate skew-ψ-Hermitian form f of Witt index $n \geq 1$, let $\mathbb{K}_0 \subseteq \mathbb{K}$ be the fix field of ψ and let G denote the group of isometries of (V, f). For every $k \in \{1, \ldots, 2n\}$, there exist natural representations of the groups $G \cong U(2n, \mathbb{K}/\mathbb{K}_0)$ and $H = G \cap SL(V) \cong SU(2n, \mathbb{K}/\mathbb{K}_0)$ on the k-th exterior power of V. With the aid of linear algebra, we prove some properties of these representations. We also discuss some applications to projective embeddings and hyperplanes of Hermitian dual polar spaces.

Keywords: Grassmann module, unitary group, Hermitian dual polar space, hyperplane
MSC2000: 15A75, 15A63, 20C33, 51A50

1 Introduction

This paper is an essay in which we will use methods based on linear algebra to derive several facts regarding structures which are related to a $2n$-dimensional $\mathbb{K}$-vector space V which is endowed with a nondegenerate skew-Hermitian form f of maximal Witt index n. These methods allow us to give more elegant proofs for some known results, and to state some known results in a language which is more elegant and more suitable for future applications. More precisely, we will do the following:

(1) If ψ denotes the involutary automorphism of $\mathbb{K}$ associated to f and if $\mathbb{K}_0 \subset \mathbb{K}$ denotes the fix field of ψ, then we will prove the irreducibility of certain modules for the groups $U(2n, \mathbb{K}/\mathbb{K}_0)$ and $SU(2n, \mathbb{K}/\mathbb{K}_0)$.

(2) We will give a more elegant description (and a more elegant proof for the existence) of the Baer-$\mathbb{K}_0$-subgeometry $\mathrm{PG}(W^*)$ of $\mathrm{PG}(\bigwedge^n V)$ which affords the Grassmann embedding of the dual polar space $DH(2n-1, \mathbb{K}, \psi)$ associated to (V, f).

(3) Every hyperplane $\mathcal{H}$ of $DH(2n-1, \mathbb{K}, \psi)$ which arises from the Grassmann embedding can be described by a certain vector of W^*, a so-called representative vector of $\mathcal{H}$. De Bruyn and Pralle [9] proved that the finite Hermitian dual polar space $DH(5, q^2)$ has 5 isomorphism classes of hyperplanes arising from the Grassmann embedding. We determine a representative vector for each of these 5 isomorphism classes.

Remark. In [8], we used techniques based on linear algebra to derive several facts regarding structures related to a $2n$-dimensional vector space endowed with a nondegenerate alternating bilinear form.

1.1 Certain representations of unitary groups

Let n be a strictly positive integer and let $\mathbb{K}_0$, $\mathbb{K}$ be two fields such that $\mathbb{K}$ is a quadratic Galois extension of $\mathbb{K}_0$. Put $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$ and $\mathbb{K}_0^* := \mathbb{K}_0 \setminus \{0\}$. Let ψ denote the unique nontrivial element in $Gal(\mathbb{K}/\mathbb{K}_0)$ and let V be a $2n$-dimensional vector space over $\mathbb{K}$ equipped with a nondegenerate skew-ψ-Hermitian form f of Witt index n.

An ordered basis $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V is called a *hyperbolic basis* of V if $f(\bar{e}_i, \bar{e}_j) = f(\bar{f}_i, \bar{f}_j) = 0$ and $f(\bar{e}_i, \bar{f}_j) = \delta_{ij}$ for all $i, j \in \{1, \ldots, n\}$. Let G denote the group of isometries of (V, f), i.e. the set of all $\theta \in GL(V)$ satisfying $f(\theta(\bar{x}), \theta(\bar{y})) = f(\bar{x}, \bar{y})$ for all $\bar{x}, \bar{y} \in V$. Then $G \cong U(2n, \mathbb{K}/\mathbb{K}_0)$ and $H := G \cap SL(V) \cong SU(2n, \mathbb{K}/\mathbb{K}_0)$. The elements of G are precisely those elements of $GL(V)$ which map hyperbolic bases of V to hyperbolic bases of V. It can be proved (see Lemma 2.2) that if $\theta \in G$, then there exists an $\eta \in \mathbb{K}^*$ such that $\det(\theta) = \frac{\eta^\psi}{\eta}$. We denote by η_θ any of the elements of $\mathbb{K}^*$ satisfying this property. The element η_θ is uniquely determined up to a factor of $\mathbb{K}_0^*$. If $\theta_1, \theta_2 \in G$, then $\eta_{\theta_2 \circ \theta_1} \cdot \eta_{\theta_1}^{-1} \cdot \eta_{\theta_2}^{-1} \in \mathbb{K}_0$ since $\det(\theta_2 \circ \theta_1) = \det(\theta_1) \cdot \det(\theta_2)$.

For every $k \in \{0, \ldots, 2n\}$, let $\bigwedge^k V$ be the k-th exterior power of V. Then $\bigwedge^0 V = \mathbb{K}$ and $\bigwedge^1 V = V$. If $k \in \{1, \ldots, 2n\}$, then for every $\theta \in GL(V)$, there exists a unique $\widetilde{\theta}_k \in GL(\bigwedge^k V)$ such that $\widetilde{\theta}_k(\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k) = \theta(\bar{v}_1) \wedge \theta(\bar{v}_2) \wedge \cdots \wedge \theta(\bar{v}_k)$ for all vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k \in V$. The map $\theta \mapsto \widetilde{\theta}_k$ define representations $\mathcal{R}_k$ and $\mathcal{R}'_k$ of the respective groups $G \cong U(2n, \mathbb{K}/\mathbb{K}_0)$ and $H \cong SU(2n, \mathbb{K}/\mathbb{K}_0)$ on the $\binom{2n}{k}$-dimensional vector space $\bigwedge^k V$. We call the corresponding $\mathbb{K}G$-modules (respectively $\mathbb{K}H$-modules) *Grassmann modules* for G (respectively H). We put $\widetilde{G}_k := \{\widetilde{\theta}_k \,|\, \theta \in G\}$ and $\widetilde{H}_k :=$

$\{\widetilde{\theta}_k \,|\, \theta \in H\}$. The following result might be known (during the course of writing this paper, the author observed that a group-theoretical proof of this fact is also contained in the preprint [2]). Anyhow, we will prove it in Section 3 with the aid of elementary linear algebra.

Theorem 1.1 *For every $k \in \{1, \ldots, 2n\}$, the representation $\mathcal{R}'_k$ is irreducible.*

Theorem 1.1 has the following corollary:

Corollary 1.2 (1) *For every $k \in \{1, \ldots, 2n\}$, the representation $\mathcal{R}_k$ is irreducible.*

(2) *For every $k \in \{1, \ldots, n\}$, the subspace of $\bigwedge^k V$ generated by all vectors of the form $\bar{v}_1 \wedge \cdots \wedge \bar{v}_k$ such that $\langle \bar{v}_1, \ldots, \bar{v}_k \rangle$ is totally isotropic with respect to f coincides with $\bigwedge^k V$.*

Proof. Claim (1) follows from the fact that H is a subgroup of G.

Now, let $k \in \{1, \ldots, n\}$. Obviously, the subspace of $\bigwedge^k V$ generated by all vectors of the form $\bar{v}_1 \wedge \cdots \wedge \bar{v}_k$ such that $\langle \bar{v}_1, \ldots, \bar{v}_k \rangle$ is totally isotropic with respect to f is stabilized by $\widetilde{G}_k$. Claim (2) then follows from Claim (1). ■

In Section 4, we prove the following:

Theorem 1.3 *There exists a set W^* of vectors of $\bigwedge^n V$ satisfying the following properties:*

(1) *The set W^* is a $\binom{2n}{n}$-dimensional vector space over $\mathbb{K}_0$ (with addition of vectors and multiplication with scalars inherited from $\bigwedge^n V$).*

(2) *For every $\theta \in G$, $\widetilde{\theta}_n(W^*) = \{\frac{\alpha}{\eta_\theta} \,|\, \alpha \in W^*\}$.*

If $\theta \in H$, then $\eta_\theta \in \mathbb{K}_0^*$ and we have

Corollary 1.4 *If $\theta \in H$, then $\widetilde{\theta}_n(W^*) = W^*$.*

Now, for every map $\theta \in H$, let $\widehat{\theta}$ be the element of $GL(W^*)$ mapping $\alpha \in W^*$ to $\widetilde{\theta}_n(\alpha) \in W^*$. Then the map $\theta \mapsto \widehat{\theta}$ defines a representation $\widehat{\mathcal{R}}$ of the group $H \cong SU(2n, \mathbb{K}/\mathbb{K}_0)$ on the $\binom{2n}{n}$-dimensional $\mathbb{K}_0$-vector space W^*. The corresponding $\mathbb{K}_0 H$-module is also called a *Grassmann module for $SU(2n, \mathbb{K}/\mathbb{K}_0)$*. Put $\widehat{H} := \{\widehat{\theta} \,|\, \theta \in H\}$. As a consequence of Theorem 1.1, we have

Corollary 1.5 *The representation $\widehat{\mathcal{R}}$ is irreducible.*

Proof. Suppose U is a subspace of W^* which is stabilized by $\widehat{H}$. The subspace U is contained in a unique subspace $\overline{U}$ of $\bigwedge^n V$ with the same dimension as U. Obviously, $\overline{U}$ is stabilized by $\widetilde{H}_n$. So by Theorem 1.1, either $\overline{U} = 0$ or $\overline{U} = \bigwedge^n V$. Hence, either $U = 0$ or $U = W^*$. ■

1.2 The Grassmann embedding of the dual polar space $DH(2n-1, \mathbb{K}, \psi)$

A *full (projective) embedding* of a point-line geometry $\mathcal{S}$ is an injective mapping e from the point-set $\mathcal{P}$ of $\mathcal{S}$ to the point-set of a projective space Σ satisfying (i) $\langle e(\mathcal{P}) \rangle_\Sigma = \Sigma$ and (ii) $e(L)$ is a line of Σ for every line L of $\mathcal{S}$.

Let Π be a polar space (Tits [12], Veldkamp [13]) of rank $n \geq 2$. With Π there is associated a point-line geometry Δ which is called a *dual polar space*, see Cameron [3]. The points of Δ are the maximal singular subspaces of Π, the lines of Δ are the next-to-maximal singular subspaces of Π, and incidence is reverse containment. If ω_1 and ω_2 are two maximal singular subspaces of Π, then $\mathrm{d}(\omega_1, \omega_2)$ denotes the distance between ω_1 and ω_2 in the collinearity graph of Δ. We have $\mathrm{d}(\omega_1, \omega_2) = n - 1 - \dim(\omega_1 \cap \omega_2)$. The points ω_1 and ω_2 of Δ are called *opposite* if they lie at maximal distance n from each other. The dual polar space Δ is a *near polygon*, which means that for every point x and every line L there exists a unique point on L nearest to x. If x is a point of Δ, then $x^\perp$ denotes the set of points of Δ equal to or collinear with x. There exists a bijective correspondence between the possibly empty singular subspaces of Π and the nonempty convex subspaces of Δ. If ω is an $(n-1-k)$-dimensional singular subspace of Π, then the set of all maximal singular subspaces of Π containing ω is a convex subspace of Δ of diameter k. These convex subspaces are called *quads* if $k = 2$. Any two points x_1 and x_2 of Δ at distance k from each other are contained in a unique convex subspace $\langle x_1, x_2 \rangle$ of diameter k. If x is a point and S is a convex subspace, then there exists a unique point $\pi_S(x) \in S$ such that $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_S(x)) + \mathrm{d}(\pi_S(x), y)$ for every point $y \in S$. The convex subspaces through a given point x of Δ define an $(n-1)$-dimensional projective space which we will denote by $Res(x)$.

As in Section 1.1, let V be a $2n$-dimensional vector space over $\mathbb{K}$ equipped with a nondegenerate skew-ψ-Hermitian form f of Witt index $n \geq 2$. With the nondegenerate skew-ψ-Hermitian form f, there is associated a Hermitian polar space $H(2n-1, \mathbb{K}, \psi)$ and a Hermitian dual polar space $DH(2n-1, \mathbb{K}, \psi)$. The singular subspaces of $H(2n-1, \mathbb{K}, \psi)$ are the subspaces of $\mathrm{PG}(2n-1, \mathbb{K})$ which are totally isotropic with respect to the Hermitian polarity of $PG(V)$ defined by f. In Section 4, we will prove the following regarding the vector space W^* alluded to in Theorem 1.3.

Theorem 1.6 (1) *For every maximal singular subspace* $\omega = \langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n \rangle$ *of* $H(2n-1, \mathbb{K}, \psi)$*, there exists a unique point* $e_{gr}(\omega) = \langle \beta \rangle$ *in* $\mathrm{PG}(W^*)$ *such that* $\beta \in W^*$ *and* $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n$ *are linearly dependent vectors of* $\bigwedge^n V$*.*

(2) *The map* $\omega \mapsto e_{gr}(\omega)$ *defines a full embedding of* $DH(2n-1, \mathbb{K}, \psi)$ *into the Baer-*$\mathbb{K}_0$*-subgeometry* $\mathrm{PG}(W^*)$ *of* $\mathrm{PG}(\bigwedge^n V)$*.*

The projective embedding e_{gr} mentioned in Theorem 1.6(2) is called the *Grassmann embedding of* $DH(2n-1, \mathbb{K}, \psi)$.

Remark. Another description of the Baer-$\mathbb{K}_0$-subgeometry of $\mathrm{PG}(\bigwedge^n V)$ which affords the Grassmann embedding of $DH(2n-1, \mathbb{K}, \psi)$ was given in [7]. The description and proof which we will give in Section 4 seem more elegant. In [5, Proposition 5.1], there was given a description of a $\mathbb{K}_0$-vector space $W \subseteq \bigwedge^n V$ stabilized by $\widetilde{H}_n$ such that $\mathrm{PG}(W)$ affords the Grassmann embedding of $DH(2n-1, \mathbb{K}, \psi)$. The proof given in [5] is however not correct as was already pointed out in [7]. Also some corrections must be performed in [5] in order to get the right equation for W (e.g., observe the coefficient $(-1)^l$ in the formula at the beginning of Section 4).

A set of points of $DH(2n-1, \mathbb{K}, \psi)$ distinct from the whole point-set is called a *hyperplane* of $DH(2n-1, \mathbb{K}, \psi)$ if it intersects every line in either a singleton or the whole line. If π is a hyperplane of $\mathrm{PG}(W^*)$, then the set of all points p of $DH(2n-1, \mathbb{K}, \psi)$ such that $e_{gr}(p) \in \pi$ is a hyperplane of $DH(2n-1, \mathbb{K}, \psi)$. Any hyperplane of $DH(2n-1, \mathbb{K}, \psi)$ which can be obtained in this way is said to *arise from* e_{gr}.

If $\mathbb{K}$ is the finite field $\mathbb{F}_{q^2}$ with q^2 elements (so, $\mathbb{K}_0 \cong \mathbb{F}_q$ and $\psi : \mathbb{K} \to \mathbb{K} : x \mapsto x^q$), then we will denote $H(2n-1, \mathbb{K}, \psi)$ and $DH(2n-1, \mathbb{K}, \psi)$ also by $H(2n-1, q^2)$ and $DH(2n-1, q^2)$.

Consider now the special case $n = 3$, $\mathbb{K} = \mathbb{F}_{q^2}$ and let $\mathcal{A}$ denote the group of automorphisms of $DH(5, q^2)$. For every $\varphi \in \mathcal{A}$, there exists a unique collineation $\widetilde{\varphi}$ of $\mathrm{PG}(W^*)$ such that $e_{gr}(\varphi(p)) = \widetilde{\varphi}(e_{gr}(p))$ for every point p of $DH(5, q^2)$. By De Bruyn and Pralle [9], the group $\mathcal{A}$ has 5 orbits on the set of hyperplanes of $DH(5, q^2)$ arising from e_{gr}. In Section 5, we will show that this implies that $\widetilde{A} := \{\widetilde{\varphi} \,|\, \varphi \in \mathcal{A}\}$ has 5 orbits on the set of points of $\mathrm{PG}(W^*)$, and we will determine an explicit description of a point of each of these five orbits.

2 Hyperbolic bases of V

In this section, we continue with the notation introduced in Section 1.1. Recall that V is a $2n$-dimensional vector space ($n \geq 1$) over $\mathbb{K}$ which is equipped with a nondegenerate skew-ψ-Hermitian form f of Witt index n, and that $\mathbb{K}_0$ is the fix field of ψ.

If $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ is a hyperbolic basis of V, then

(1) for every permutation σ of $\{1, \ldots, n\}$, also $(\bar{e}_{\sigma(1)}, \bar{f}_{\sigma(1)}, \ldots, \bar{e}_{\sigma(n)}, \bar{f}_{\sigma(n)})$ is a hyperbolic basis of V;

(2) for every $\lambda \in \mathbb{K}^*$, also $(\frac{\bar{e}_1}{\lambda}, \lambda^\psi \bar{f}_1, \bar{e}_2, \bar{f}_2, \ldots, \bar{e}_n, \bar{f}_n)$ is a hyperbolic basis of V;

(3) for every $\lambda \in \mathbb{K}$, also $(\bar{e}_1 + \lambda\bar{e}_2, \bar{f}_1, \bar{e}_2, -\lambda^\psi \bar{f}_1 + \bar{f}_2, \bar{e}_3, \bar{f}_3, \ldots, \bar{e}_n, \bar{f}_n)$ is a hyperbolic basis of V;

(4) for every $\lambda \in \mathbb{K}_0$, also $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_{n-1}, \bar{f}_{n-1}, \bar{e}_n, \bar{f}_n + \lambda\bar{e}_n)$ is a hyperbolic basis of V;

(5) for every $\lambda \in \mathbb{K}_0$, also $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_{n-1}, \bar{f}_{n-1}, \bar{e}_n + \lambda\bar{f}_n, \bar{f}_n)$ is a hyperbolic basis of V.

For every $i \in \{1, 2, 3, 4, 5\}$, let Ω_i denote the set of all ordered pairs (B_1, B_2) of hyperbolic bases of V such that B_2 can be obtained from B_1 as described in (i) above.

Lemma 2.1 *If B and B' are two hyperbolic bases of V, then there exist hyperbolic bases $B_0, B_1, \ldots, B_k$ ($k \geq 0$) of V such that $B_0 = B$, $B_k = B'$ and $(B_{i-1}, B_i) \in \Omega_1 \cup \cdots \cup \Omega_5$ for every $i \in \{1, \ldots, k\}$.*

Proof. Put $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ and $B' = (\bar{e}'_1, \bar{f}'_1, \ldots, \bar{e}'_n, \bar{f}'_n)$. Put $E = \langle \bar{e}_1, \ldots, \bar{e}_n \rangle$, $E' = \langle \bar{e}'_1, \ldots, \bar{e}'_n \rangle$, $F = \langle \bar{f}_1, \ldots, \bar{f}_n \rangle$ and $F' = \langle \bar{f}'_1, \ldots, \bar{f}'_n \rangle$. The proof of the lemma will occur in 3 steps.

(1) Suppose $E = E'$ and $F = F'$. Since the maps $(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n) \mapsto (\bar{g}_{\sigma(1)}, \bar{g}_{\sigma(2)}, \ldots, \bar{g}_{\sigma(n)})$, $(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n) \mapsto (\frac{\bar{g}_1}{\lambda}, \bar{g}_2, \ldots, \bar{g}_n)$ and $(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n) \mapsto (\bar{g}_1 + \lambda\bar{g}_2, \bar{g}_2, \ldots, \bar{g}_n)$ allow us to transform any basis of E to any other basis of E, there exist hyperbolic bases $B_0, B_1, \ldots, B_k$ ($k \geq 0$) of V such that (i) $B_0 = B$, (ii) $(B_{i-1}, B_i) \in \Omega_1 \cup \Omega_2 \cup \Omega_3$ for every $i \in \{1, \ldots, k\}$, and (iii) B_k is of the form $(\bar{e}'_1, \bar{f}''_1, \ldots, \bar{e}'_n, \bar{f}''_n)$ with $F = \langle \bar{f}''_1, \ldots, \bar{f}''_n \rangle$. The vector $\bar{f}''_i$, $i \in \{1, \ldots, n\}$, is uniquely determined by the vectors $\bar{e}'_1, \ldots, \bar{e}'_n$: it is the unique vector of F which is f-orthogonal with every $\bar{e}'_j$, $j \neq i$, and which satisfies $(\bar{e}'_i, \bar{f}''_i) = 1$. Hence, $\bar{f}''_i = \bar{f}'_i$ for every $i \in \{1, \ldots, k\}$, i.e. $B_k = B'$.

(2) Suppose ($E = E'$ and $\dim(F \cap F') = n - 1$) or ($F = F'$ and $\dim(E \cap E') = n - 1$). We will only treat the case $E = E'$ and $\dim(F \cap F') = n - 1$, since the other case is completely similar. By (1), the lemma will hold for

the pair (B, B') as soon as it holds for at least one pair (B, B') giving rise to the same subspaces $E = E'$, F and F'. So, without loss of generality, we may suppose that B and B' are in such a way that $\{\bar{f}_1, \dots, \bar{f}_{n-1}\}$ is a basis of $F \cap F'$ and $\bar{e}'_i = \bar{e}_i$ for every $i \in \{1, \dots, n\}$. Then $\bar{f}'_i = \bar{f}_i$ for every $i \in \{1, \dots, n-1\}$ and there exists a $\lambda \in \mathbb{K}_0^*$ such that $\bar{f}'_n = \bar{f}_n + \lambda \bar{e}_n$. So, $(B, B') \in \Omega_4$.

(3) Consider the following graph Γ. The vertices of Γ are the pairs (X, Y) where X and Y are two complementary totally isotropic n-dimensional subspaces of V. Two vertices (X, Y) and (X', Y') of Γ are adjacent if either ($X = X'$ and $\dim(Y \cap Y') = n - 1$) or ($Y = Y'$ and $\dim(X \cap X') = n - 1$). We will now prove that the graph Γ is connected. This fact, combined with (1) and (2), then finishes the proof of the lemma. Notice that the vertices of Γ are the pairs (x, y) of opposite points of $DH(2n - 1, \mathbb{K}, \psi)$. We will now prove by induction on $\mathrm{d}(x_1, x_2)$ that any two vertices (x_1, y_1) and (x_2, y_2) of Γ are connected by a path.

Suppose first that $\mathrm{d}(x_1, x_2) = 0$, i.e. $x_1 = x_2$. Then the claim follows from the fact that the subgraph of Γ induced on the set of points opposite to a given vertex is connected, see e.g. [6, Theorem 2.7].

Suppose $\mathrm{d}(x_1, x_2) \geq 1$. Let x_3 be a point collinear with x_2 at distance $\mathrm{d}(x_1, x_2) - 1$ from x_1. By the induction hypothesis, it suffices to show that there exists a point y_3 opposite to x_3 such that (x_2, y_2) and (x_3, y_3) are contained in the same connected component of Γ. This clearly holds if $\mathrm{d}(x_3, y_2) = n$. (Take $y_3 = y_2$.) Suppose therefore that $\mathrm{d}(x_3, y_2) = n - 1$. Let L denote a line through y_2 which is not contained in the convex subspace $\langle x_3, y_2 \rangle$, and let y_3 be a point of $L \setminus \{y_2\}$ distinct from $\pi_L(x_2)$. Then $\mathrm{d}(x_2, y_3) = \mathrm{d}(x_3, y_3) = n$. So, $(x_2, y_2) \sim_\Gamma (x_2, y_3) \sim_\Gamma (x_3, y_3)$. This is precisely what we needed to show. ■

Lemma 2.2 *If $\theta \in G$, then there exists an element $\eta \in \mathbb{K}^*$ such that $\det(\theta) = \frac{\eta^\psi}{\eta}$. The element η is determined up to a factor of $\mathbb{K}_0^*$.*

Proof. Let $B = (\bar{e}_1, \bar{f}_1, \dots, \bar{e}_n, \bar{f}_n)$ be an arbitrary hyperbolic basis of V.

(i) Let σ be a permutation of $\{1, \dots, n\}$ and let θ be the element of G mapping B to $B' = (\bar{e}_{\sigma(1)}, \bar{f}_{\sigma(1)}, \dots, \bar{e}_{\sigma(n)}, \bar{f}_{\sigma(n)})$. Then $\det(\theta) = 1 = \frac{1^\psi}{1}$.

(ii) Let $\lambda \in \mathbb{K}^*$ and let θ be the element of G mapping B to $B' = (\frac{\bar{e}_1}{\lambda}, \lambda^\psi \bar{f}_1, \bar{e}_2, \bar{f}_2, \dots, \bar{e}_n, \bar{f}_n)$. Then $\det(\theta) = \frac{\lambda^\psi}{\lambda}$.

(iii) Let $\lambda \in \mathbb{K}$ and let θ be the element of G mapping B to $B' = (\bar{e}_1 + \lambda \bar{e}_2, \bar{f}_1, \bar{e}_2, -\lambda^\psi \bar{f}_1 + \bar{f}_2, \bar{e}_3, \bar{f}_3, \dots, \bar{e}_n, \bar{f}_n)$. Then $\det(\theta) = 1 = \frac{1^\psi}{1}$.

(iv) Let $\lambda \in \mathbb{K}_0$ and let θ be the element of G mapping B to $B' = (\bar{e}_1, \bar{f}_1, \dots, \bar{e}_{n-1}, \bar{f}_{n-1}, \bar{e}_n, \bar{f}_n + \lambda \bar{e}_n)$. Then $\det(\theta) = 1 = \frac{1^\psi}{1}$.

(v) Let $\lambda \in \mathbb{K}_0$ and let θ be the element of G mapping B to $B' = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_{n-1}, \bar{f}_{n-1}, \bar{e}_n + \lambda \bar{f}_n, \bar{f}_n)$. Then $\det(\theta) = 1 = \frac{1^\psi}{1}$.

(vi) If $\theta_1, \theta_2 \in G$ such that $\det(\theta_i) = \frac{\eta_i^\psi}{\eta_i}$, $i \in \{1,2\}$, then $\det(\theta_2 \circ \theta_1) = \det(\theta_1) \cdot \det(\theta_2) = \frac{(\eta_1\eta_2)^\psi}{\eta_1\eta_2}$.

The first claim of the lemma now follows from Lemma 2.1 and (i)–(vi) above. Notice also that if $\eta_1, \eta_2 \in \mathbb{K}^*$ such that $\frac{\eta_1^\psi}{\eta_1} = \frac{\eta_2^\psi}{\eta_2}$, then $(\frac{\eta_1}{\eta_2})^\psi = \frac{\eta_1}{\eta_2}$ and hence $\frac{\eta_1}{\eta_2} \in \mathbb{K}_0^*$. This also proves the second claim of the lemma. ■

3 Proof of Theorem 1.1

3.1 A useful lemma

Suppose that $2 \leq k \leq 2n-1$ and that $\bar{e}_1$ and $\bar{f}_1$ are two vectors of V such that $f(\bar{e}_1, \bar{f}_1) = 1$. Let V' denote the set of vectors of V which are f-orthogonal with $\bar{e}_1$ and $\bar{f}_1$ and let f' denote the skew-ψ-Hermitian form of V' induced by f. Let G' denote the group of isometries of (V', f'), $H' := G' \cap SL(V')$ and let $\widetilde{G}'_{k-1}$ and $\widetilde{H}'_{k-1}$ denote the subgroups of $GL(\bigwedge^{k-1} V')$ corresponding to G' and H' (see Section 1.1). For every vector α of $\bigwedge^{k-1} V'$, let $\mu_k(\alpha)$ be the vector $\bar{e}_1 \wedge \alpha$ of $\bigwedge^k V$. Then μ_k defines a linear isomorphism between $\bigwedge^{k-1} V'$ and the subspace $\mu_k(\bigwedge^{k-1} V')$ of $\bigwedge^k V$.

Lemma 3.1 *Suppose U is a subspace of $\bigwedge^k V$ which is stabilized by $\widetilde{H}_k$. Then $\mu_k^{-1}(U \cap \mu_k(\bigwedge^{k-1} V'))$ is a subspace of $\bigwedge^{k-1} V'$ which is stabilized by $\widetilde{H}'_{k-1}$.*

Proof. Let α be an arbitrary vector of $\mu_k^{-1}(U \cap \mu_k(\bigwedge^{k-1} V'))$ and let $\widetilde{\theta}$ be an arbitrary element of $\widetilde{H}'_{k-1}$ corresponding to an element $\theta \in H'$. We need to show that $\widetilde{\theta}(\alpha) \in \mu_k^{-1}(U \cap \mu_k(\bigwedge^{k-1} V'))$.

We extend θ to an element $\overline{\theta}$ of H by defining $\overline{\theta}(\bar{e}_1) = \bar{e}_1$ and $\overline{\theta}(\bar{f}_1) = \bar{f}_1$. Let $\widetilde{\overline{\theta}}$ be the element of $\widetilde{H}_k$ corresponding to $\overline{\theta}$. Then for every vector α' of $\bigwedge^{k-1} V'$, $\mu_k \circ \widetilde{\theta}(\alpha') = \widetilde{\overline{\theta}} \circ \mu_k(\alpha')$. Hence, $\widetilde{\overline{\theta}}$ stabilizes $\mu_k(\bigwedge^{k-1} V')$.

Now, since $\mu_k(\alpha) \in U \cap \mu_k(\bigwedge^{k-1} V')$, also $\widetilde{\overline{\theta}} \circ \mu_k(\alpha) \in U \cap \mu_k(\bigwedge^{k-1} V')$. Hence, $\widetilde{\theta}(\alpha) = \mu_k^{-1} \circ \widetilde{\overline{\theta}} \circ \mu_k(\alpha) \in \mu_k^{-1}(U \cap \mu_k(\bigwedge^{k-1} V'))$. ■

3.2 Proof of Theorem 1.1

The following proposition is precisely Theorem 1.1.

Proposition 3.2 *Let $k \in \{1, \ldots, 2n\}$. If U is a proper subspace of $\bigwedge^k V$ which is stabilized by $\widetilde{H}_k$, then $U = 0$.*

Proof.

If $k = 2n$, then $U = 0$ since 0 is the only proper subspace of $\bigwedge^{2n} V$.

Suppose $k = 1$ and $U \neq 0$. Then U contains a nonzero vector $\chi = \lambda_1 \bar{e}_1 + \lambda'_1 \bar{f}_1 + \cdots + \lambda_n \bar{e}_n + \lambda'_n \bar{f}_n$, where $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ is some given hyperbolic basis of V. Without loss of generality, we may suppose that $\lambda'_1 \neq 0$. If θ is the element of H mapping the hyperbolic basis $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ to the hyperbolic basis $(\bar{e}_1, \bar{e}_1 + \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$, then since $\chi \in U$, also $\frac{1}{\lambda'_1}\left(\widetilde{\theta}(\chi) - \chi\right) = \bar{e}_1 \in U$. Since for any $\bar{g} \in \{\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n\}$, there exists an element of H mapping $\bar{e}_1$ to $\bar{g}$, we have $U = \langle \bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n \rangle = V$, a contradiction.

We will now prove the lemma by induction on n. By the previous two paragraphs, we may suppose that $n \geq 2$, $k \in \{2, \ldots, 2n-1\}$ and that the lemma holds for smaller values of n. Let $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ be a given hyperbolic basis of V and let V', μ_k and $\widetilde{H}'_{k-1}$ as in Section 3.1.

Let χ be an arbitrary vector of U. Then χ can be written in a unique way as

$$\chi = \bar{e}_1 \wedge \bar{f}_1 \wedge \alpha(\chi) + \bar{e}_1 \wedge \beta(\chi) + \bar{f}_1 \wedge \gamma(\chi) + \delta(\chi),$$

where $\alpha(\chi) \in \bigwedge^{k-2} V'$, $\beta(\chi), \gamma(\chi) \in \bigwedge^{k-1} V'$ and $\delta(\chi) \in \bigwedge^k V'$. Let θ be the unique element of H mapping the hyperbolic basis $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V to the hyperbolic basis $(\bar{e}_1, \bar{e}_1 + \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V. Then $\widetilde{\theta}_k(\chi) = \chi + \bar{e}_1 \wedge \gamma(\chi)$. Since $\chi \in U$, also $\widetilde{\theta}_k(\chi) \in U$ and hence also $\bar{e}_1 \wedge \gamma(\chi) \in U$. We show that $\gamma(\chi) = 0$.

Suppose $\gamma(\chi) \neq 0$. Then since $\gamma(\chi) \in \mu_k^{-1}(U \cap \mu_k(\bigwedge^{k-1} V'))$ and $\mu_k^{-1}(U \cap \mu_k(\bigwedge^{k-1} V'))$ is stabilized by $\widetilde{H}'_{k-1}$ (Lemma 3.1), $\mu_k^{-1}(U \cap \mu_k(\bigwedge^{k-1} V')) = \bigwedge^{k-1} V'$ by the induction hypothesis. So, $\mu_k(\bigwedge^{k-1} V') \subseteq U$. Hence, U contains a vector of the form $\bar{e}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k$ where $\langle \bar{e}_1, \bar{v}_2, \ldots, \bar{v}_k \rangle$ is a k-dimensional subspace of V which is totally isotropic with respect to f. Since H acts transitively on the set of all k-dimensional subspaces of V which are totally isotropic with respect to f, we would have that $U = \bigwedge^k V$, which is impossible.

Hence, $\gamma(\chi) = 0$. In a similar way, one can prove that $\beta(\chi) = 0$. What we have just done, we can also do for any pair $(\bar{e}_i, \bar{f}_i)$, $i \in \{1, \ldots, n\}$. We can conclude:

(P1) For every $i \in \{1, \ldots, n\}$ and every $\chi \in U$, χ can be written in the form $\bar{e}_i \wedge \bar{f}_i \wedge \alpha_i(\chi) + \delta_i(\chi)$ where $\alpha_i(\chi) \in \bigwedge^{k-2} \langle \bar{e}_1, \bar{f}_1, \ldots, \widehat{\bar{e}_i}, \widehat{\bar{f}_i}, \ldots, \bar{e}_n, \bar{f}_n \rangle$ and $\delta_i(\chi) \in \bigwedge^k \langle \bar{e}_1, \bar{f}_1, \ldots, \widehat{\bar{e}_i}, \widehat{\bar{f}_i}, \ldots, \bar{e}_n, \bar{f}_n \rangle$.

If k is odd, then (P1) implies that $U = 0$. Suppose therefore that $k = 2m$ is even. By (P1), every element χ of U is of the form $\sum \lambda_I \cdot \bar{e}_{i_1} \wedge \bar{f}_{i_1} \wedge \cdots \wedge \bar{e}_{i_m} \wedge \bar{f}_{i_m}$, with the summation ranging over all subsets $I = \{i_1, \ldots, i_m\}$ of size m of $\{1, \ldots, n\}$ satisfying $i_1 < i_2 < \cdots < i_m$. We will now show that all the coefficients λ_I are equal to each other.

Suppose first that I_1 and I_2 are two subsets of size m of $\{1, 2, \ldots, n\}$ such that $|I_1 \cap I_2| = m - 1$. Without loss of generality, we may suppose that $I_1 \setminus I_2 = \{1\}$ and $I_2 \setminus I_1 = \{2\}$. Write $\chi = \sum \lambda_I \cdot \bar{e}_{i_1} \wedge \bar{f}_{i_1} \wedge \cdots \wedge \bar{e}_{i_m} \wedge \bar{f}_{i_m}$ in the form

$$\bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 \wedge \alpha + \bar{e}_1 \wedge \bar{f}_1 \wedge \beta + \bar{e}_2 \wedge \bar{f}_2 \wedge \gamma + \delta,$$

where $\alpha \in \bigwedge^{k-4} \langle \bar{e}_3, \bar{f}_3, \ldots, \bar{e}_n, \bar{f}_n \rangle$, $\beta, \gamma \in \bigwedge^{k-2} \langle \bar{e}_3, \bar{f}_3, \ldots, \bar{e}_n, \bar{f}_n \rangle$ and $\delta \in \bigwedge^{k} \langle \bar{e}_3, \bar{f}_3, \ldots, \bar{e}_n, \bar{f}_n \rangle$. [If $k = 2$, then we omit the term $\bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 \wedge \alpha$.] Let θ denote the element of H mapping the hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \ldots, \bar{e}_n, \bar{f}_n)$ of V to the hyperbolic basis $(\bar{e}_1 + \bar{e}_2, \bar{f}_1, \bar{e}_2, -\bar{f}_1 + \bar{f}_2, \bar{e}_3, \bar{f}_3, \ldots, \bar{e}_n, \bar{f}_n)$ of V. Then $\widetilde{\theta}(\chi) = \chi + \bar{e}_2 \wedge \bar{f}_1 \wedge (\beta - \gamma)$. Since $\chi \in U$, also $\widetilde{\theta}(\chi) \in U$ and hence $\bar{e}_2 \wedge \bar{f}_1 \wedge (\beta - \gamma) \in U$. By (P1), $\beta = \gamma$. Hence, $\lambda_{I_1} = \lambda_{I_2}$.

Consider now the most general case and let I_1 and I_2 be two arbitrary subsets of size m of $\{1, \ldots, n\}$. Put $|I_1 \cap I_2| = m - l$. Then there exist $l + 1$ subsets $J_0, \ldots, J_l$ of size m of $\{1, \ldots, n\}$ such that $J_0 = I_1$, $J_l = I_2$ and $|J_{i-1} \cap J_i| = m - 1$ for every $i \in \{1, \ldots, l\}$. By the previous paragraph, we know that $\lambda_{I_1} = \lambda_{J_0} = \lambda_{J_1} = \cdots = \lambda_{J_l} = \lambda_{I_2}$.

So, we can conclude

(P2) Every element χ of U is of the form $\lambda \cdot \sum \bar{e}_{i_1} \wedge \bar{f}_{i_1} \wedge \cdots \wedge \bar{e}_{i_m} \wedge \bar{f}_{i_m}$, with the summation ranging over all subsets $I = \{i_1, \ldots, i_m\}$ of size m of $\{1, \ldots, n\}$ satisfying $i_1 < i_2 < \cdots < i_m$.

Now, consider an arbitrary element $\eta \in \mathbb{K} \setminus \mathbb{K}_0$ satisfying $\eta^\psi \notin \{-\eta, \eta\}$ (if ϵ is an arbitrary element of $\mathbb{K} \setminus \mathbb{K}_0$, then at least one of $\epsilon, \epsilon + 1$ satisfies this condition) and let θ' be the unique element of H mapping the hyperbolic basis $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V to the hyperbolic basis $(\frac{\bar{e}_1}{\eta}, \eta^\psi \cdot \bar{f}_1, \eta \cdot \bar{e}_2, \frac{\bar{f}_2}{\eta^\psi}, \ldots, \bar{e}_n, \bar{f}_n)$ of V. Then the fact that $\widetilde{\theta'}(\chi) \in U$ implies that the λ mentioned in (P2) must be equal to 0. So, $U = 0$. ■

4 The $\mathbb{K}_0$-vector space W^*

For every hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V and every $\lambda \in \mathbb{K} \setminus \mathbb{K}_0$, we will now define a basis $\mathcal{B}_\lambda(B)$ of $\bigwedge^n V$. The basis $\mathcal{B}_\lambda(B)$ consists of all the vectors

$$\left(\bar{g}_{\sigma(1)} \wedge \cdots \wedge \bar{g}_{\sigma(k)}\right) \wedge \Big(\epsilon \cdot \bar{e}_{\sigma(k+1)} \wedge \bar{f}_{\sigma(k+1)} \wedge \cdots \wedge \bar{e}_{\sigma(k+l)} \wedge \bar{f}_{\sigma(k+l)}$$

$$+(-1)^l \epsilon^\psi \cdot \bar{e}_{\sigma(k+l+1)} \wedge \bar{f}_{\sigma(k+l+1)} \wedge \cdots \wedge \bar{e}_{\sigma(n)} \wedge \bar{f}_{\sigma(n)}\Big),$$

where (1) $k, l \in \{0, \ldots, n\}$ such that $k+2l = n$, (2) $\epsilon \in \{1, \lambda\}$, (3) $\bar{g}_i \in \{\bar{e}_i, \bar{f}_i\}$ for every $i \in \{\sigma(1), \ldots, \sigma(k)\}$, (4) σ is a permutation of $\{1, \ldots, n\}$ satisfying (i) $\sigma(1) < \sigma(2) < \cdots < \sigma(k)$, (ii) $\sigma(k+1) < \sigma(k+2) < \cdots < \sigma(k+l)$, (iii) $\sigma(k+l+1) < \sigma(k+l+2) < \cdots < \sigma(n)$, (iv) $\sigma(k+1) < \sigma(k+l+1)$.

Let $W_\lambda(B)$ denote the set of all $\mathbb{K}_0$-linear combinations of the elements of $\mathcal{B}_\lambda(B)$. Now, for all $\lambda_1, \lambda_2 \in \mathbb{K} \setminus \mathbb{K}_0$, there exist $a_1, a_2 \in \mathbb{K}_0$ such that $(\lambda_2, \lambda_2^\psi) = a_1(1,1) + a_2(\lambda_1, \lambda_1^\psi)$. From this we readily observe that $W_{\lambda_1}(B) = W_{\lambda_2}(B)$ for any two $\lambda_1, \lambda_2 \in \mathbb{K} \setminus \mathbb{K}_0$. We define $W(B) := W_\lambda(B)$ where λ is an arbitrary element of $\mathbb{K} \setminus \mathbb{K}_0$.

Lemma 4.1 *If B_1 and B_2 are two hyperbolic bases of V such that $(B_1, B_2) \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_5$ and if θ is the unique element of G mapping B_1 to B_2, then $W(B_2) = \{\frac{\alpha}{\eta_\theta} \,|\, \alpha \in W(B_1)\}$. In particular, if $(B_1, B_2) \in \Omega_1 \cup \Omega_3 \cup \Omega_4 \cup \Omega_5$, then $W(B_2) = W(B_1)$.*

Proof. Put $B_1 = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ and $B_2 = (\bar{e}'_1, \bar{f}'_1, \ldots, \bar{e}'_n, \bar{f}'_n)$. Notice that $\eta_\theta \cdot \eta_{\theta^{-1}} \in \mathbb{K}_0^*$ and if $(B_1, B_2) \in \Omega_i$, $i \in \{1, \ldots, 5\}$, then also $(B_2, B_1) \in \Omega_i$. So, it suffices to prove that $W(B_2) \subseteq \{\frac{\alpha}{\eta_\theta} \,|\, \alpha \in W(B_1)\}$, or equivalently, that $\mathcal{B}_\lambda(B_2) \subseteq \{\frac{\alpha}{\eta_\theta} \,|\, \alpha \in W(B_1)\}$, where λ is a given element of $\mathbb{K} \setminus \mathbb{K}_0$. The latter statement is easily seen to be true if $(B_1, B_2) \in \Omega_1 \cup \Omega_2 \cup \Omega_4 \cup \Omega_5$. We will now treat the harder case $(B_1, B_2) \in \Omega_3$. Then there exists a $\eta \in \mathbb{K}$ such that $\bar{e}'_1 = \bar{e}_1 + \eta \bar{e}_2$, $\bar{f}'_1 = \bar{f}_1$, $\bar{e}'_2 = \bar{e}_2$, $\bar{f}'_2 = -\eta^\psi \bar{f}_1 + \bar{f}_2$, $\bar{e}'_3 = \bar{e}_3$, $\bar{f}'_3 = \bar{f}_3$, ..., $\bar{e}'_n = \bar{e}_n$ and $\bar{f}'_n = \bar{f}_n$. Let

$$\chi = \left(\bar{g}'_{\sigma(1)} \wedge \cdots \wedge \bar{g}'_{\sigma(k)}\right) \wedge \Big(\epsilon \cdot \bar{e}'_{\sigma(k+1)} \wedge \bar{f}'_{\sigma(k+1)} \wedge \cdots \wedge \bar{e}'_{\sigma(k+l)} \wedge \bar{f}'_{\sigma(k+l)}$$

$$+(-1)^l \epsilon^\psi \cdot \bar{e}'_{\sigma(k+l+1)} \wedge \bar{f}'_{\sigma(k+l+1)} \wedge \cdots \wedge \bar{e}'_{\sigma(n)} \wedge \bar{f}'_{\sigma(n)}\Big),$$

be an arbitrary element of $\mathcal{B}_\lambda(B_2)$, where (1) $k, l \in \{0, \ldots, n\}$ such that $k + 2l = n$, (2) $\epsilon \in \{1, \lambda\}$, (3) $\bar{g}'_i \in \{\bar{e}'_i, \bar{f}'_i\}$ for every $i \in \{\sigma(1), \ldots, \sigma(k)\}$, (4) σ is a permutation of $\{1, \ldots, n\}$ satisfying (i) $\sigma(1) < \sigma(2) < \cdots < \sigma(k)$, (ii) $\sigma(k+1) < \sigma(k+2) < \cdots < \sigma(k+l)$, (iii) $\sigma(k+l+1) < \sigma(k+l+2) < \cdots < \sigma(n)$, (iv) $\sigma(k+1) < \sigma(k+l+1)$. There are 10 possibilities:

(1) Suppose $\sigma(1) = 1$, $\sigma(2) = 2$, $\bar{g}'_1 = \bar{e}'_1$ and $\bar{g}'_2 = \bar{e}'_2$. Then χ can be written in a natural way as $\chi = \bar{e}'_1 \wedge \bar{e}'_2 \wedge \alpha \wedge \beta$, where $\alpha = \bar{g}'_{\sigma(3)} \wedge \cdots \wedge \bar{g}'_{\sigma(k)}$

and β does not involve indices which are equal to either 1 or 2. We have $\chi = (\bar{e}_1 + \eta\bar{e}_2) \wedge \bar{e}_2 \wedge \alpha \wedge \beta = \bar{e}_1 \wedge \bar{e}_2 \wedge \alpha \wedge \beta \in \mathcal{B}_\lambda(B_1)$.

(2) Suppose $\sigma(1) = 1$, $\sigma(2) = 2$, $\bar{g}'_1 = \bar{e}'_1$ and $\bar{g}'_2 = \bar{f}'_2$. Then χ can be written in a natural way as $\chi = \bar{e}'_1 \wedge \bar{f}'_2 \wedge \alpha \wedge (\epsilon \cdot \beta + (-1)^l \epsilon^\psi \cdot \gamma)$, where α, β and γ do not involve indices which are equal to 1 or 2. We have $\chi = (\bar{e}_1 + \eta\bar{e}_2) \wedge (-\eta^\psi \bar{f}_1 + \bar{f}_2) \wedge \alpha \wedge (\epsilon \cdot \beta + (-1)^l \epsilon^\psi \cdot \gamma) = \bar{e}_1 \wedge \bar{f}_2 \wedge \alpha \wedge (\epsilon \cdot \beta + (-1)^l \epsilon^\psi \cdot \gamma) - \eta^{\psi+1} \bar{e}_2 \wedge \bar{f}_1 \wedge \alpha \wedge (\epsilon \cdot \beta + (-1)^l \epsilon^\psi \cdot \gamma) + \alpha \wedge (-\eta^\psi \epsilon \cdot \bar{e}_1 \wedge \bar{f}_1 \wedge \beta - (-1)^{l+1} (\eta^\psi \epsilon)^\psi \cdot \bar{e}_2 \wedge \bar{f}_2 \wedge \gamma) + \alpha \wedge (\epsilon\eta \cdot \bar{e}_2 \wedge \bar{f}_2 \wedge \beta + (-1)^{l+1} (\epsilon\eta)^\psi \cdot \bar{e}_1 \wedge \bar{f}_1 \wedge \gamma)$ and this is clearly a $\mathbb{K}_0$-linear combination of elements of $\mathcal{B}_\lambda(B_1)$.

(3) Suppose $\sigma(1) = 1$, $\sigma(2) = 2$, $\bar{g}'_1 = \bar{f}'_1$ and $\bar{g}'_2 = \bar{e}'_2$. Then χ can be written in a natural way as $\chi = \bar{f}'_1 \wedge \bar{e}'_2 \wedge \alpha \wedge \beta$, where $\alpha = \bar{g}'_{\sigma(3)} \wedge \cdots \wedge \bar{g}'_{\sigma(k)}$ and β does not involve indices which are equal to either 1 or 2. We have $\chi = \bar{f}_1 \wedge \bar{e}_2 \wedge \alpha \wedge \beta \in \mathcal{B}_\lambda(B_1)$.

(4) Suppose $\sigma(1) = 1$, $\sigma(2) = 2$, $\bar{g}'_1 = \bar{f}'_1$ and $\bar{g}'_2 = \bar{f}'_2$. Then χ can be written in a natural way as $\chi = \bar{f}'_1 \wedge \bar{f}'_2 \wedge \alpha \wedge \beta$, where $\alpha = \bar{g}'_{\sigma(3)} \wedge \cdots \wedge \bar{g}'_{\sigma(k)}$ and β does not involve indices which are equal to either 1 or 2. We have $\chi = \bar{f}_1 \wedge (-\eta^\psi \bar{f}_1 + \bar{f}_2) \wedge \alpha \wedge \beta = \bar{f}_1 \wedge \bar{f}_2 \wedge \alpha \wedge \beta \in \mathcal{B}_\lambda(B_1)$.

(5) Suppose $\sigma(1) = 1$, $\sigma(k+1) = 2$ and $\bar{g}'_1 = \bar{e}'_1$. Then χ can be written in a natural way as $\chi = \bar{e}'_1 \wedge \alpha \wedge (\epsilon \cdot \bar{e}'_2 \wedge \bar{f}'_2 \wedge \beta + (-1)^l \epsilon^\psi \cdot \gamma)$ where α, β and γ does not involve indices which are equal to 1 or 2. We have $\chi = (\bar{e}_1 + \eta\bar{e}_2) \wedge \alpha \wedge (\epsilon \cdot \bar{e}_2 \wedge (-\eta^\psi \bar{f}_1 + \bar{f}_2) \wedge \beta + (-1)^l \epsilon^\psi \cdot \gamma) = \bar{e}_1 \wedge \alpha \wedge (\epsilon \cdot \bar{e}_2 \wedge \bar{f}_2 \wedge \beta + (-1)^l \epsilon^\psi \cdot \gamma) + \bar{e}_2 \wedge \alpha \wedge (\epsilon\eta^\psi \cdot \bar{e}_1 \wedge \bar{f}_1 \wedge \beta + (-1)^l (\epsilon\eta^\psi)^\psi \cdot \gamma)$ and this is clearly a $\mathbb{K}_0$-linear combination of the elements of $\mathcal{B}_\lambda(B_1)$.

(6) Suppose $\sigma(1) = 1$, $\sigma(k+1) = 2$ and $\bar{g}'_1 = \bar{f}'_1$. Then χ can be written in a natural way as $\chi = \bar{f}'_1 \wedge \alpha \wedge (\epsilon \cdot \bar{e}'_2 \wedge \bar{f}'_2 \wedge \beta + (-1)^l \epsilon^\psi \cdot \gamma)$ where α, β and γ does not involve induces which are equal to 1 or 2. We have $\chi = \bar{f}_1 \wedge \alpha \wedge (\epsilon \cdot \bar{e}_2 \wedge (-\eta^\psi \bar{f}_1 + \bar{f}_2) \wedge \beta + (-1)^l \epsilon^\psi \cdot \gamma) = \bar{f}_1 \wedge \alpha \wedge (\epsilon \cdot \bar{e}_2 \wedge \bar{f}_2 \wedge \beta + (-1)^l \epsilon^\psi \cdot \gamma) \in \mathcal{B}_\lambda(B_1)$.

(7) Suppose $\sigma(1) = 2$, $\sigma(k+1) = 1$ and $\bar{g}'_2 = \bar{e}'_2$. Then χ can be written in natural way as $\chi = \bar{e}'_2 \wedge \alpha \wedge (\epsilon \cdot \bar{e}'_1 \wedge \bar{f}'_1 \wedge \beta + (-1)^l \epsilon^\psi \cdot \gamma)$, where α, β and γ do not involve indices which are equal to 1 or 2. We have $\chi = \bar{e}_2 \wedge \alpha \wedge (\epsilon \cdot (\bar{e}_1 + \eta\bar{e}_2) \wedge \bar{f}_1 \wedge \beta + (-1)^l \epsilon^\psi \cdot \gamma) = \bar{e}_2 \wedge \alpha \wedge (\epsilon \cdot \bar{e}_1 \wedge \bar{f}_1 \wedge \beta + (-1)^l \epsilon^\psi \cdot \gamma) \in \mathcal{B}_\lambda(B_1)$.

(8) Suppose $\sigma(1) = 2$, $\sigma(k+1) = 1$ and $\bar{g}'_2 = \bar{f}'_2$. Then χ can be written in a natural way as $\bar{f}'_2 \wedge \alpha \wedge (\epsilon \cdot \bar{e}'_1 \wedge \bar{f}'_1 \wedge \beta + (-1)^l \epsilon^\psi \cdot \gamma)$, where α, β and γ do not involve indices which are equal to 1 of 2. We have $\chi = (-\eta^\psi \bar{f}_1 + \bar{f}_2) \wedge \alpha \wedge (\epsilon \cdot (\bar{e}_1 + \eta\bar{e}_2) \wedge \bar{f}_1 \wedge \beta + (-1)^l \epsilon^\psi \cdot \gamma) = \bar{f}_2 \wedge \alpha \wedge (\epsilon \cdot \bar{e}_1 \wedge \bar{f}_1 \wedge \beta + (-1)^l \epsilon^\psi \cdot \gamma) - \bar{f}_1 \wedge \alpha \wedge (\epsilon\eta \cdot \bar{e}_2 \wedge \bar{f}_2 \wedge \beta + (-1)^l (\epsilon\eta)^\psi \cdot \gamma)$ and this is clearly a $\mathbb{K}_0$-linear combination of elements of $\mathcal{B}_\lambda(B_1)$.

(9) Suppose $\sigma(k+1) = 1$ and $\sigma(k+2) = 2$. Then χ can be written in a natural way as $\chi = \alpha \wedge (\epsilon \cdot \bar{e}'_1 \wedge \bar{f}'_1 \wedge \bar{e}'_2 \wedge \bar{f}'_2 \wedge \beta + (-1)^l \epsilon^\psi \cdot \gamma)$ where α, β and γ do not involve indices which are equal to 1 or 2. We have $\chi = \alpha \wedge (\epsilon \cdot (\bar{e}_1 + \eta\bar{e}_2) \wedge \bar{f}_1 \wedge \bar{e}_2 \wedge (-\eta^\psi \bar{f}_1 + f_2) \wedge \beta + (-1)^l \epsilon^\psi \cdot \gamma) = \alpha \wedge (\epsilon \cdot$

$\bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 \wedge \beta + (-1)^l \epsilon^\psi \cdot \gamma) \in \mathcal{B}_\lambda(B_1)$.

(10) Suppose $\sigma(k+1) = 1$ and $\sigma(k+l+1) = 2$. Then χ can be written in a natural way as $\chi = \alpha \wedge (\epsilon \cdot \bar{e}_1 \wedge \bar{f}_1 \wedge \beta + (-1)^l \epsilon^\psi \cdot \bar{e}_2 \wedge \bar{f}_2 \wedge \gamma)$, where α, β and γ do not involve indices which are equal to 1 or 2. We have $\chi = \alpha \wedge (\epsilon \cdot (\bar{e}_1 + \eta \bar{e}_2) \wedge \bar{f}_1 \wedge \beta + (-1)^l \epsilon^\psi \cdot \bar{e}_2 \wedge (-\eta^\psi \bar{f}_1 + \bar{f}_2) \wedge \gamma) = \alpha \wedge (\epsilon \cdot \bar{e}_1 \wedge \bar{f}_1 \wedge \beta + (-1)^l \epsilon^\psi \cdot \bar{e}_2 \wedge \bar{f}_2 \wedge \gamma) - \bar{f}_1 \wedge \bar{e}_2 \wedge \alpha \wedge (\epsilon\eta \cdot \beta + (-1)^{l+1} (\epsilon\eta)^\psi \cdot \gamma)$ and this is clearly a $\mathbb{K}_0$-linear combination of elements of $\mathcal{B}_\lambda(B_1)$. ■

Lemma 4.2 *If B_1 and B_2 are two hyperbolic bases of V and if θ is the unique element of G mapping B_1 to B_2, then $W(B_2) = \{\frac{\alpha}{\eta_\theta} \,|\, \alpha \in W(B_1)\}$.*

Proof. Let B_1, B_2 and B_3 be three hyperbolic bases of V and let θ_i, $i \in \{1, 2\}$, be the unique element of G mapping B_i to B_{i+1}. Then $\theta_3 := \theta_2 \circ \theta_1$ is the unique element of G mapping B_1 to B_3. In view of Lemmas 2.1 and 4.1, it suffices to show that if the lemma holds for the pairs (B_1, B_2) and (B_2, B_3), then it also holds for the pair (B_1, B_3). As remarked in Section 1.1, $\eta_{\theta_3} \cdot \eta_{\theta_2}^{-1} \cdot \eta_{\theta_1}^{-1} \in \mathbb{K}_0$. Now, since $W_{B_3} = \{\frac{\alpha}{\eta_{\theta_2}} \,|\, \alpha \in W(B_2)\}$ and $W_{B_2} = \{\frac{\alpha}{\eta_{\theta_1}} \,|\, \alpha \in W(B_1)\}$, we have $W_{B_3} = \{\frac{\alpha}{\eta_{\theta_1} \cdot \eta_{\theta_2}} \,|\, \alpha \in W(B_1)\} = \{\frac{\alpha}{\eta_{\theta_3}} \,|\, \alpha \in W(B_1)\}$. ■

Now, let B^* be a fixed hyperbolic basis of V and put $W^* := W(B^*)$. Then Theorem 1.3 is an immediate consequence of Lemma 4.2. (Notice that since $\widetilde{\theta}_n$ maps every element of $\mathcal{B}_\lambda(B^*)$ to an element of $\mathcal{B}_\lambda(\theta(B^*))$, we have $\widetilde{\theta}_n(W^*) = W(\theta(B^*))$.) We will now also give a proof of Theorem 1.6.

Proof. Let $\langle \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n \rangle$ be an n-dimensional subspace of V which is totally isotropic with respect to f. Extend $(\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n)$ to a hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V. Then $\bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_n \in W(B)$. Claim (1) of Theorem 1.6 now follows from Lemma 4.2.

Let $\langle \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{n-1} \rangle$ be an $(n-1)$-dimensional subspace of V which is totally isotropic with respect to f. Let $\bar{e}_n$ and $\bar{f}_n$ be two vectors of V which are f-orthogonal with $\langle \bar{e}_1, \ldots, \bar{e}_{n-1} \rangle$ and which satisfy $f(\bar{e}_n, \bar{f}_n) = 1$. Then the n-dimensional subspaces of V through $\langle \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{n-1} \rangle$ which are totally isotropic with respect to f are precisely the subspaces $\langle \bar{e}_1, \ldots, \bar{e}_n \rangle$, $\langle \bar{e}_1, \ldots, \bar{e}_{n-1}, \bar{f}_n + \lambda \bar{e}_n \rangle$, $\lambda \in \mathbb{K}_0$. Now, extend $(\bar{e}_1, \ldots, \bar{e}_n, \bar{f}_n)$ to a hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V. Then for every $\lambda \in \mathbb{K}_0$, also $B_\lambda := (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_{n-1}, \bar{f}_{n-1}, \bar{e}_n, \bar{f}_n + \lambda \bar{e}_n)$ is a hyperbolic basis of V. Now, by Lemma 4.1, $W(B_\lambda) = W(B)$ for every $\lambda \in \mathbb{K}_0$. Now, $\bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_n \in W(B)$ and $\bar{e}_1 \wedge \cdots \wedge \bar{e}_{n-1} \wedge (\bar{f}_n + \lambda \bar{e}_n) = \bar{e}_1 \wedge \cdots \wedge \bar{e}_{n-1} \wedge \bar{f}_n + \lambda \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_n \in W(B_\lambda) = W(B)$. It now follows that the line of $DH(2n-1, \mathbb{K}, \psi)$ corresponding to the subspace $\langle \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{n-1} \rangle$ is mapped by e to a line of PG(W^*). ■

Now, let $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ be a hyperbolic basis of V. For every two vectors α_1 and α_2 of $\bigwedge^n V$, we define $\widetilde{f}_B(\alpha_1, \alpha_2) \in \mathbb{K}$ in such a way that

$$\alpha_1 \wedge \alpha_2 \;=\; \widetilde{f}_B(\alpha_1, \alpha_2) \cdot (\bar{e}_1 \wedge \bar{f}_1) \wedge \cdots \wedge (\bar{e}_n \wedge \bar{f}_n).$$

Clearly, $\widetilde{f}_B$ is a nondegenerate form which is symmetric if n is even and alternating if n is odd.

Lemma 4.3 (1) *If $\theta \in G$ and B is a hyperbolic basis, then $\widetilde{f}_B = \det(\theta) \cdot \widetilde{f}_{\theta(B)}$. In particular, if $\theta \in H$, then $\widetilde{f}_B = \widetilde{f}_{\theta(B)}$.*

(2) *If B is a hyperbolic basis and $\alpha_1, \alpha_2 \in W(B)$, then $\widetilde{f}_B(\alpha_1, \alpha_2) \in \mathbb{K}_0$.*

Proof. (1) If $\alpha_1, \alpha_2 \in \bigwedge^n V$, then $\widetilde{f}_B(\alpha_1, \alpha_2) \cdot (\bar{e}_1 \wedge \bar{f}_1) \wedge \cdots \wedge (\bar{e}_n \wedge \bar{f}_n) = \alpha_1 \wedge \alpha_2 = \widetilde{f}_{\theta(B)}(\alpha_1, \alpha_2) \cdot \theta(\bar{e}_1) \wedge \theta(\bar{f}_1) \wedge \cdots \wedge \theta(\bar{e}_n) \wedge \theta(\bar{f}_n) = \det(\theta) \cdot \widetilde{f}_{\theta(B)}(\alpha_1, \alpha_2) \cdot \bar{e}_1 \wedge \bar{f}_1 \wedge \cdots \wedge \bar{e}_n \wedge \bar{f}_n$. Hence, $\widetilde{f}_B(\alpha_1, \alpha_2) = \det(\theta) \cdot \widetilde{f}_{\theta(B)}(\alpha_1, \alpha_2)$.

(2) Let λ be an arbitrary element of $\mathbb{K} \setminus \mathbb{K}_0$. It suffices to prove that $\widetilde{f}(\alpha_1, \alpha_2) \in \mathbb{K}_0$ for every two vectors $\alpha_1, \alpha_2 \in \mathcal{B}_\lambda(B)$. We readily observe that $\widetilde{f}(\alpha_1, \alpha_2)$ is always equal to 0 if $\alpha_1, \alpha_2 \in \mathcal{B}_\lambda(B)$, except in the following cases:

(a) $\alpha_1 = \bar{g}_1 \wedge \bar{g}_2 \wedge \cdots \wedge \bar{g}_n$ and $\alpha_2 = \bar{g}'_1 \wedge \bar{g}'_2 \wedge \cdots \wedge \bar{g}'_n$ where $\{\bar{g}_i, \bar{g}'_i\} = \{\bar{e}_i, \bar{f}_i\}$ for every $i \in \{1, \ldots, n\}$. One readily verifies that $\widetilde{f}_B(\alpha_1, \alpha_2) \in \{-1, 1\} \subseteq \mathbb{K}_0$.

(b) $\alpha_1 = \Big(\bar{g}_{\sigma(1)} \wedge \cdots \wedge \bar{g}_{\sigma(k)}\Big) \wedge \Big(\epsilon \cdot \bar{e}_{\sigma(k+1)} \wedge \bar{f}_{\sigma(k+1)} \wedge \cdots \wedge \bar{e}_{\sigma(k+l)} \wedge \bar{f}_{\sigma(k+l)} + (-1)^l \epsilon^\psi \cdot \bar{e}_{\sigma(k+l+1)} \wedge \bar{f}_{\sigma(k+l+1)} \wedge \cdots \wedge \bar{e}_{\sigma(n)} \wedge \bar{f}_{\sigma(n)}\Big)$ and $\alpha_2 = \Big(\bar{g}'_{\sigma(1)} \wedge \cdots \wedge \bar{g}'_{\sigma(k)}\Big) \wedge \Big(\epsilon' \cdot \bar{e}_{\sigma(k+1)} \wedge \bar{f}_{\sigma(k+1)} \wedge \cdots \wedge \bar{e}_{\sigma(k+l)} \wedge \bar{f}_{\sigma(k+l)} + (-1)^l \epsilon'^\psi \cdot \bar{e}_{\sigma(k+l+1)} \wedge \bar{f}_{\sigma(k+l+1)} \wedge \cdots \wedge \bar{e}_{\sigma(n)} \wedge \bar{f}_{\sigma(n)}\Big)$, where (1) $k, l \in \{0, \ldots, n\}$ such that $l > 0$ and $k + 2l = n$, (2) $\epsilon, \epsilon' \in \{1, \lambda\}$, (3) $\{\bar{g}_i, \bar{g}'_i\} = \{\bar{e}_i, \bar{f}_i\}$ for every $i \in \{\sigma(1), \ldots, \sigma(k)\}$, (4) σ is a permutation of $\{1, \ldots, n\}$ satisfying (i) $\sigma(1) < \sigma(2) < \cdots < \sigma(k)$, (ii) $\sigma(k+1) < \sigma(k+2) < \cdots < \sigma(k+l)$, (iii) $\sigma(k+l+1) < \sigma(k+l+2) < \cdots < \sigma(n)$, (iv) $\sigma(k+1) < \sigma(k+l+1)$. One readily verifies that $\widetilde{f}_B(\alpha_1, \alpha_2) \in \{(\epsilon \cdot \epsilon'^\psi + \epsilon^\psi \epsilon'), -(\epsilon \cdot \epsilon'^\psi + \epsilon^\psi \epsilon')\} \subseteq \mathbb{K}_0$. ∎

Again consider a fixed hyperbolic basis B^* of V and let $\widetilde{f}^*$ be the restriction of $\widetilde{f}_{B^*}$ to the $\mathbb{K}_0$-vector space $W^* = W(B^*)$. Then $\widetilde{f}^*$ is a nondegenerate bilinear form on the vector space W^*. This form defines a polarity ζ^* of $\mathrm{PG}(W^*)$. If n is odd or $char(\mathbb{K}) = 2$, then ζ^* is a symplectic polarity. Otherwise, ζ^* is an orthogonal polarity. If U is a subspace of W^*, then we define $U^\perp := \{x \in W^* \mid \widetilde{f}^*(x, u) = 0, \forall u \in U\}$.

Remark. If $e : \Delta \to \Sigma$ is the so-called minimal full polarized embedding (see [4] for the definition) of a thick dual polar space Δ in a finite-dimensional projective space Σ, then there exists a unique polarity ζ of Σ such that two points p_1 and p_2 of Δ are not opposite if and only if $e(p_2) \in e(p_1)^\zeta$. The polarity ζ^* defined above is a special case of this (take $e = e_{gr}$, $\Delta = DH(2n-1, \mathbb{K}, \psi)$ and $\Sigma = \mathrm{PG}(W^*)$). We refer to Cardinali, De Bruyn and Pasini [4] for more information on minimal full polarized embeddings. The existence of the polarity ζ is an immediate consequence of the isomorphism between the embedding e and its so-called dual embedding e^*.

5 Hyperplanes of $DH(2n-1, \mathbb{K}, \psi)$

5.1 Representative vectors

By Shult [11, Lemma 6.1], every hyperplane of a thick dual polar space (in particular, of $DH(2n-1, \mathbb{K}, \psi)$) is a maximal subspace. So, if H is a hyperplane of $DH(2n-1, \mathbb{K}, \psi)$ arising from e_{gr}, then $\langle e_{gr}(H) \rangle$ necessarily is a hyperplane of $\mathrm{PG}(W^*)$ and there exists a unique 1-space U of W^* such that $\langle e_{gr}(H) \rangle = \mathrm{PG}(U^\perp)$. Any nonzero vector of U is called a *representative vector* of H.

Consider now the special case $n = 3$. Recall that $B^* = (\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ is a given hyperbolic basis of V and that $W^* = W(B^*)$. Let λ be an arbitrary point of $\mathbb{K} \setminus \mathbb{K}_0$. By Section 4, a basis of the $\mathbb{K}_0$-vector space W^* is given by the following 20 vectors:

$$\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3,\ \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3,\ \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{e}_3,\ \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{f}_3,$$

$$\bar{f}_1 \wedge \bar{e}_2 \wedge \bar{e}_3,\ \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_3,\ \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{e}_3,\ \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{f}_3,$$

$$\bar{e}_1 \wedge (\bar{e}_2 \wedge \bar{f}_2 - \bar{e}_3 \wedge \bar{f}_3),\ \bar{e}_1 \wedge (\lambda \cdot \bar{e}_2 \wedge \bar{f}_2 - \lambda^\psi \cdot \bar{e}_3 \wedge \bar{f}_3),$$

$$\bar{f}_1 \wedge (\bar{e}_2 \wedge \bar{f}_2 - \bar{e}_3 \wedge \bar{f}_3),\ \bar{f}_1 \wedge (\lambda \cdot \bar{e}_2 \wedge \bar{f}_2 - \lambda^\psi \cdot \bar{e}_3 \wedge \bar{f}_3),$$

$$\bar{e}_2 \wedge (\bar{e}_1 \wedge \bar{f}_1 - \bar{e}_3 \wedge \bar{f}_3),\ \bar{e}_2 \wedge (\lambda \cdot \bar{e}_1 \wedge \bar{f}_1 - \lambda^\psi \cdot \bar{e}_3 \wedge \bar{f}_3),$$

$$\bar{f}_2 \wedge (\bar{e}_1 \wedge \bar{f}_1 - \bar{e}_3 \wedge \bar{f}_3),\ \bar{f}_2 \wedge (\lambda \cdot \bar{e}_1 \wedge \bar{f}_1 - \lambda^\psi \cdot \bar{e}_3 \wedge \bar{f}_3),$$

$$\bar{e}_3 \wedge (\bar{e}_1 \wedge \bar{f}_1 - \bar{e}_2 \wedge \bar{f}_2),\ \bar{e}_3 \wedge (\lambda \cdot \bar{e}_1 \wedge \bar{f}_1 - \lambda^\psi \cdot \bar{e}_2 \wedge \bar{f}_2),$$

$$\bar{f}_3 \wedge (\bar{e}_1 \wedge \bar{f}_1 - \bar{e}_2 \wedge \bar{f}_2),\ \bar{f}_3 \wedge (\lambda \cdot \bar{e}_1 \wedge \bar{f}_1 - \lambda^\psi \cdot \bar{e}_2 \wedge \bar{f}_2).$$

We now discuss two classes of hyperplanes of $DH(5, \mathbb{K}, \psi)$.

(I) Let $\mathcal{H}$ be the hyperplane of $DH(5, \mathbb{K}, \psi)$ with representative vector $\alpha = \eta_1 \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \eta_2 \cdot \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{f}_3 + \eta_3 \cdot \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_3$. Let p be the point $\langle \bar{e}_1, \bar{e}_2, \bar{f}_3 \rangle$ of

$DH(5,\mathbb{K},\psi)$. Since $\alpha\wedge\bar{e}_1\wedge\bar{e}_2\wedge\bar{f}_3=0$, the point p belongs to $\mathcal{H}$. An arbitrary line of $DH(5,\mathbb{K},\psi)$ through p corresponds to a line $\langle\bar{v}_1,\bar{v}_2\rangle\subseteq\langle\bar{e}_1,\bar{e}_2,\bar{f}_3\rangle$ of $H(5,\mathbb{K},\psi)$. Since $\langle\bar{v}_1,\bar{v}_2\rangle$ meets each of $\langle\bar{e}_1,\bar{e}_2\rangle$, $\langle\bar{e}_1,\bar{f}_3\rangle$ and $\langle\bar{e}_2,\bar{f}_3\rangle$, we necessarily have $\alpha\wedge\bar{v}_1\wedge\bar{v}_2=0$. So, every line of $DH(5,\mathbb{K},\psi)$ through p is contained in $\mathcal{H}$. This implies that every quad Q through p is either deep (i.e. $Q\subseteq\mathcal{H}$) or singular with deep point p (i.e. $Q\cap\mathcal{H}=p^\perp\cap\mathcal{H}$).

Lemma 5.1 *Let $a_1,a_2,a_3\in\mathbb{K}$ with $(a_1,a_2,a_3)\neq(0,0,0)$. The quad Q through p corresponding to the point $\langle a_1\bar{e}_1+a_2\bar{e}_2+a_3\bar{f}_3\rangle$ of $H(5,\mathbb{K},\psi)$ is contained in $\mathcal{H}$ if and only if $\eta_3a_1^{\psi+1}+\eta_2a_2^{\psi+1}-\eta_1a_3^{\psi+1}=0$.*

Proof. Suppose $a_3\neq 0$. Then $p'=\langle a_1\bar{e}_1+a_2\bar{e}_2+a_3\bar{f}_3, a_3^\psi\bar{f}_2+a_2^\psi\bar{e}_3, a_3^\psi\bar{f}_1+a_1^\psi\bar{e}_3\rangle$ is a point of Q at distance 2 from p. Clearly, Q is deep if and only if $p'\in\mathcal{H}$, i.e. if and only if

$$(\eta_1\cdot\bar{e}_1\wedge\bar{e}_2\wedge\bar{e}_3+\eta_2\cdot\bar{e}_1\wedge\bar{f}_2\wedge\bar{f}_3+\eta_3\cdot\bar{f}_1\wedge\bar{e}_2\wedge\bar{f}_3)$$

$$\wedge(a_1\bar{e}_1+a_2\bar{e}_2+a_3\bar{f}_3)\wedge(a_3^\psi\bar{f}_2+a_2^\psi\bar{e}_3)\wedge(a_3^\psi\bar{f}_1+a_1^\psi\bar{e}_3)=0.$$

One readily verifies that this is the case if and only if $\eta_3a_1^{\psi+1}+\eta_2a_2^{\psi+1}-\eta_1a_3^{\psi+1}=0$ holds in this case.

Similar calculations as above show that if $a_1\neq 0$ or $a_2\neq 0$, then $Q\subseteq\mathcal{H}$ if and only if $\eta_3a_1^{\psi+1}+\eta_2a_2^{\psi+1}-\eta_1a_3^{\psi+1}=0$. ■

So, the deep quads through p determine a possibly degenerate Hermitian variety in the dual projective plane of *Res*(p). If the Hermitian variety $\eta_3X_1^{\psi+1}+\eta_2X_2^{\psi+1}-\eta_1X_3^{\psi+1}=0$ is empty (which is impossible in the finite case but possible in the infinite case, for instance when ψ is the complex conjugation of $\mathbb{K}=\mathbb{C}$), then $\mathcal{H}$ is a so-called *semi-singular hyperplane with deepest point* p, i.e. $\mathcal{H}$ is of the form $p^\perp\cup O$, where O is a set of points of $DH(5,\mathbb{K},\psi)$ at distance 3 from p such that every line at distance 2 from p meets O in a unique point.

(II) Recall that $H(5,\mathbb{K},\psi)$ is the Hermitian variety of $\mathrm{PG}(5,\mathbb{K})=\mathrm{PG}(V)$ associated to (V,f). With respect to the reference system B^*, $H(5,\mathbb{K},\psi)$ has equation

$$(X_1X_2^\psi-X_2X_1^\psi)+(X_3X_4^\psi-X_4X_3^\psi)+(X_5X_6^\psi-X_6X_5^\psi)=0.$$

Now, let $a_1,b_1,a_2,b_2,a_3,b_3\in\mathbb{K}_0$ and let ω be the plane of $\mathrm{PG}(5,\mathbb{K})$ with equation $X_1=(a_1+b_1\lambda)\cdot X_2$, $X_3=(a_2+b_2\lambda)\cdot X_4$, $X_5=(a_3+b_3\lambda)\cdot X_6$. Then $\omega\cap H(5,\mathbb{K},\psi)$ is the Hermitian variety of ω with equation $b_1\cdot X_2^{\psi+1}+$

$b_2 \cdot X_4^{\psi+1} + b_3 \cdot X_6^{\psi+1} = 0$. Let X be the subspace of W^* consisting of all vectors $\chi \in W^*$ satisfying

$$(\bar{f}_1 + (a_1 + b_1\lambda)\bar{e}_1) \wedge (\bar{f}_2 + (a_2 + b_2\lambda)\bar{e}_2) \wedge (\bar{f}_3 + (a_3 + b_3\lambda)\bar{e}_3) \wedge \chi = 0.$$

If $(b_1, b_2, b_3) = (0, 0, 0)$, then X is 19-dimensional and given by the equation

$$(\bar{f}_1 + a_1\bar{e}_1) \wedge (\bar{f}_2 + a_2\bar{e}_2) \wedge (\bar{f}_3 + a_3\bar{e}_3) \wedge \chi = 0.$$

If $(b_1, b_2, b_3) \neq (0, 0, 0)$, then using the explicit description of the vector space W^* given above, we see that any $\chi \in X$ also satisfies the equation

$$(\bar{f}_1 + (a_1 + b_1\lambda^\psi)\bar{e}_1) \wedge (\bar{f}_2 + (a_2 + b_2\lambda^\psi)\bar{e}_2) \wedge (\bar{f}_3 + (a_3 + b_3\lambda^\psi)\bar{e}_3) \wedge \chi = 0.$$

Now, for every $\eta \in \mathbb{K}^*$, the vector

$$\chi_\eta := \eta \cdot (\bar{f}_1 + (a_1 + b_1\lambda)\bar{e}_1) \wedge (\bar{f}_2 + (a_2 + b_2\lambda)\bar{e}_2) \wedge (\bar{f}_3 + (a_3 + b_3\lambda)\bar{e}_3)$$

$$+\eta^\psi \cdot (\bar{f}_1 + (a_1 + b_1\lambda^\psi)\bar{e}_1) \wedge (\bar{f}_2 + (a_2 + b_2\lambda^\psi)\bar{e}_2) \wedge (\bar{f}_3 + (a_3 + b_3\lambda^\psi)\bar{e}_3).$$

belongs to W^*. One readily verifies that $\dim(X) = 18$ and that the hyperplanes of W^* with equations $\chi_\eta \wedge \chi = 0$, $\eta \in \mathbb{K}^*$, are the $|\mathbb{K}_0| + 1$ hyperplanes of W^* containing X.

From the definition of X, the following is also clear: a maximal singular subspace p of $H(5, \mathbb{K}, \psi)$ meets ω if and only if $e_{gr}(p) \in \mathrm{PG}(X)$. If (b_1, b_2, b_3) can be chosen in such a way that the Hermitian variety $b_1X_2^{\psi+1} + b_2X_4^{\psi+1} + b_3X_6^{\psi+1} = 0$ of ω is empty, then $\mathrm{PG}(X) \cap e_{gr}(\mathcal{P}) = \emptyset$, where $\mathcal{P}$ denotes the point-set of $DH(5, \mathbb{K}, \psi)$. (Again, this is impossible in the finite case, but possible when ψ is the complex conjugation of $\mathbb{K} = \mathbb{C}$.) This implies that every hyperplane of $DH(5, \mathbb{K}, \psi)$ arising from a hyperplane of $\mathrm{PG}(W^*)$ through $\mathrm{PG}(X)$ cannot contain lines. Each such hyperplane is a so-called *ovoid* of $DH(5, \mathbb{K}, \psi)$, i.e. a set of points of $DH(5, \mathbb{K}, \psi)$ meeting each line in a unique point.

5.2 The hyperplanes of $DH(5, q^2)$ arising from the Grassmann embedding

In this section, we suppose that $n = 3$, $\mathbb{K} = \mathbb{F}_{q^2}$ and $\mathbb{K}_0 = \mathbb{F}_q$. Then $x^\psi = x^q$ for every $x \in \mathbb{F}_{q^2}$. Let $\mathcal{P}$ denote the point set of $DH(5, q^2)$, let $B^* = (\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ be a given hyperbolic basis of V, let $W^* = W(B^*)$ and let e_{gr} denote the Grassmann embedding of $DH(5, q^2)$ in $\mathrm{PG}(W^*)$. Every quad of $DH(5, q^2)$ is isomorphic to $Q^-(5, q)$. The generalized quadrangle $Q^-(5, q)$ admits subquadrangles isomorphic to $Q(4, q)$, see Payne and Thas [10]. For any two hyperplanes $\mathcal{H}_1$ and $\mathcal{H}_2$ of $DH(5,q^2)$ arising from e_{gr}, let $[[\mathcal{H}_1,\mathcal{H}_2]]$ denote the set of all hyperplanes of $DH(5,q^2)$ of the form $e_{gr}^{-1}(e_{gr}(\mathcal{P})\cap\pi)$, where π is one of the $q+1$ hyperplanes of $\mathrm{PG}(W^*)$ containing $\langle e_{gr}(\mathcal{H}_1)\rangle \cap \langle e_{gr}(\mathcal{H}_2)\rangle$. If $\alpha_i \in W^*$, $i \in \{1,2\}$, is a representative vector of $\mathcal{H}_i$, then the representative vectors of the hyperplanes of $[[\mathcal{H}_1,\mathcal{H}_2]]$ are precisely the vectors $\lambda_1\alpha_1+\lambda_2\alpha_2$, where $\lambda_1,\lambda_2 \in \mathbb{F}_q$ with $(\lambda_1,\lambda_2)\neq(0,0)$.

By De Bruyn and Pralle [9], $DH(5,q^2)$ has 5 isomorphism classes of hyperplanes which arise from e_{gr}. We now give a description and a representative vector of a hyperplane of each of these classes.

(I) The hyperplanes of *Type I* of $DH(5,q^2)$ are the so-called singular hyperplanes. If x is a point of $DH(5,q^2)$, then the set $\mathcal{H}_x$ of points of $DH(5,q^2)$ at distance at most 2 from x is a hyperplane of $DH(5,q^2)$, the so-called *singular hyperplane of $DH(5,q^2)$ with deepest point x*. If x coincides with the point $\langle \bar{e}_1,\bar{e}_2,\bar{e}_3\rangle$, then $\bar{e}_1\wedge\bar{e}_2\wedge\bar{e}_3$ is a representative vector of $\mathcal{H}_x$.

(II) The hyperplanes of *Type II* of $DH(5,q^2)$ are the so-called extensions of the $Q(4,q)$-subquadrangles of the quads. If ρ is a $Q(4,q)$-subquadrangle of a quad Q, then the set $\mathcal{H}_\rho$ of points at distance at most 1 from ρ is a hyperplane of $DH(5,q^2)$, the so-called *extension of ρ*. By De Bruyn and Pralle [9], if x_1 and x_2 are two points of $DH(5,q^2)$ at distance 2 from each other, then every hyperplane of $[[\mathcal{H}_{x_1},\mathcal{H}_{x_2}]]\setminus\{\mathcal{H}_{x_1},\mathcal{H}_{x_2}\}$ is the extension of a $Q(4,q)$-subquadrangle of the quad $\langle x_1,x_2\rangle$. If $x_1=\langle\bar{e}_1,\bar{e}_2,\bar{e}_3\rangle$ and $x_2=\langle\bar{e}_1,\bar{f}_2,\bar{f}_3\rangle$, then $\bar{e}_1\wedge\bar{e}_2\wedge\bar{e}_3+\bar{e}_1\wedge\bar{f}_2\wedge\bar{f}_3$ is a representative vector of a hyperplane of the set $[[\mathcal{H}_{x_1},\mathcal{H}_{x_2}]]\setminus\{\mathcal{H}_{x_1},\mathcal{H}_{x_2}\}$.

(III) A hyperplane of $DH(5,q^2)$ is said to be of *Type III* if it belongs to some set $[[\mathcal{H}_{x_1},\mathcal{H}_{x_2}]]\setminus\{\mathcal{H}_{x_1},\mathcal{H}_{x_2}\}$ where x_1 and x_2 are two points of $DH(5,q^2)$ at distance 3 from each other. The vector $\bar{e}_1\wedge\bar{e}_2\wedge\bar{e}_3+\bar{f}_1\wedge\bar{f}_2\wedge\bar{f}_3$ is a representative vector of such a hyperplane.

(IV) A hyperplane of $DH(5,q^2)$ is said to be of *Type IV* if it belongs to some set $[[\mathcal{H}_{\rho_1},\mathcal{H}_{\rho_2}]]\setminus\{\mathcal{H}_{\rho_1},\mathcal{H}_{\rho_2}\}$ where (i) ρ_i, $i\in\{1,2\}$, is a $Q(4,q)$-subquadrangle of a quad Q_i of $DH(5,q^2)$, (ii) $Q_1\cap Q_2$ is a line L, (iii) $L\subseteq\rho_1$ and $|\rho_2\cap L|=1$ (see [9, Section 4.5]). By De Bruyn and Pralle [9], a hyperplane $\mathcal{H}$ of $DH(5,q^2)$ is of type IV if and only if there exists a (necessarily unique) point x such that (i) $x^\perp\subseteq\mathcal{H}$ and (ii) the set of deep quads through x is a nondegenerate Hermitian curve in the dual projective plane of $Res(x)$. By Lemma 5.1, the vector $\bar{e}_1\wedge\bar{e}_2\wedge\bar{e}_3+\bar{e}_1\wedge\bar{f}_2\wedge\bar{f}_3+\bar{f}_1\wedge\bar{e}_2\wedge\bar{f}_3$ is a representative vector of a hyperplane of Type IV.

(V) With respect to the reference system B^*, $H(5,q^2)$ has the equation $(X_1X_2^q-X_2X_1^q)+(X_3X_4^q-X_4X_3^q)+(X_5X_6^q-X_6X_5^q)=0$. Let ω be a plane of $\mathrm{PG}(5,q^2)$ which intersects $H(5,q^2)$ in a unital of ω and let S_ω be the set of planes of $H(5,q^2)$ meeting ω. By De Bruyn and Pralle [9, Corollary 4.29], $\langle e_{gr}(S_\omega)\rangle$ is a 17-dimensional subspace of $\mathrm{PG}(W^*)$. A hyperplane of $DH(5,q^2)$ is said to be of *Type V* if it is isomorphic to some hyperplane of the form $e_{gr}^{-1}(e_{gr}(\mathcal{P})\cap\pi)$, where π is one of the $q+1$ hyperplanes of $\mathrm{PG}(W^*)$ containing $\langle e_{gr}(S_\omega)\rangle$.

Now, for every $\lambda\in\mathbb{F}_{q^2}\setminus\mathbb{F}_q$, let ω_λ be the plane of $\mathrm{PG}(5,q^2)$ with equation $X_1=\lambda\cdot X_2$, $X_3=\lambda\cdot X_4$, $X_5=\lambda\cdot X_6$. Then $\omega_\lambda\cap H(5,q^2)$ is a unital of ω_λ. The 17-dimensional subspace $\langle e_{gr}(S_{\omega_\lambda})\rangle$ of $\mathrm{PG}(W^*)$ consists of all points $\langle\chi\rangle$ of $\mathrm{PG}(W^*)$, where χ is a nonzero vector of W^* satisfying

$$(\bar{f}_1+\lambda\bar{e}_1)\wedge(\bar{f}_2+\lambda\bar{e}_2)\wedge(\bar{f}_3+\lambda\bar{e}_3)\wedge\chi \;=\; 0.$$

Now, for every $\eta\in\mathbb{F}_{q^2}^*$, $\chi_{\lambda,\eta}:=\eta\cdot(\bar{f}_1+\lambda\bar{e}_1)\wedge(\bar{f}_2+\lambda\bar{e}_2)\wedge(\bar{f}_3+\lambda\bar{e}_3)+\eta^q\cdot(\bar{f}_1+\lambda^q\bar{e}_1)\wedge(\bar{f}_2+\lambda^q\bar{e}_2)\wedge(\bar{f}_3+\lambda^q\bar{e}_3)$ is a vector of W^* and the equations $\chi_{\lambda,\eta}\wedge\chi=0$, $\eta\in\mathbb{F}_{q^2}^*$, determine the $q+1$ hyperplanes of $\mathrm{PG}(W^*)$ containing $\langle e_{gr}(S_{\omega_\lambda})\rangle$, see Section 5.1 (II). So, for any $\lambda\in\mathbb{F}_{q^2}\setminus\mathbb{F}_q$ and any $\eta\in\mathbb{F}_{q^2}^*$, $\chi_{\lambda,\eta}$ is a representative vector of a hyperplane of Type V. Our aim is now to give a representative vector of a nicer form.

Let $\eta_1,\eta_2\in\mathbb{F}_q^*$ such that the polynomial $\eta_2X^2+(\eta_1\eta_2+\eta_1+\eta_2)X+\eta_1\in\mathbb{F}_q[X]$ is irreducible. Such a polynomial exists by the following lemma.

Lemma 5.2 *For every irreducible monic quadratic polynomial $X^2+aX+b\in\mathbb{F}_q[X]$, there exist unique elements $\eta_1,\eta_2\in\mathbb{F}_q\setminus\{0\}$ such that $\eta_2(X^2+aX+b)=\eta_2X^2+(\eta_1\eta_2+\eta_1+\eta_2)X+\eta_1$.*

Proof. Since X^2+aX+b is irreducible, its values at the points -1 and 0 are nonzero. Hence, $b\neq 0$ and $a-1-b\neq 0$. After an easy and straightforward computation, we find that there is only one solution for η_1 and η_2, namely $\eta_1=a-1-b$ and $\eta_2=\frac{a-1-b}{b}$. ■

Suppose now that $\lambda\in\mathbb{F}_{q^2}\setminus\mathbb{F}_q$ is a root of the polynomial $\eta_2X^2-(\eta_1\eta_2+\eta_1+\eta_2)X+\eta_1$ and $\eta=\frac{\lambda^q-1}{\lambda(\lambda^q-\lambda)}$. Then $\lambda+\lambda^q=\frac{\eta_1+\eta_2+\eta_1\eta_2}{\eta_2}$ and $\lambda^{q+1}=\frac{\eta_1}{\eta_2}$. One calculates that $\chi_{\lambda,\eta}=\eta_1\cdot\bar{e}_1\wedge\bar{e}_2\wedge\bar{e}_3+\eta_2\cdot\bar{f}_1\wedge\bar{f}_2\wedge\bar{f}_3+(\bar{e}_1+\bar{f}_1)\wedge(\bar{e}_2+\bar{f}_2)\wedge(\bar{e}_3+\bar{f}_3)=0$.

Let $\mathcal{A}$ denote the group of automorphisms of $DH(5,q^2)$. By De Bruyn and Pralle [9], $\mathcal{A}$ has 5 orbits on the set of hyperplanes of $DH(5,q^2)$ arising from e_{gr}. [A more careful inspection of the proof of [9] would reveal that there are still five orbits if we restrict to those automorphisms which arise from projectivities of $\mathrm{PG}(5,q^2)=\mathrm{PG}(V)$.] For every $\varphi\in\mathcal{A}$, there exists a unique projectivity $\widetilde{\varphi}$ of $\mathrm{PG}(W^*)$ such that $e_{gr}(\varphi(p))=\widetilde{\varphi}(e_{gr}(p))$ for every point

p of $DH(5, q^2)$. In view of the bijective correspondence between the set of hyperplanes of $DH(5, q^2)$, the set of hyperplanes of $\mathrm{PG}(W^*)$ and the set of points of $\mathrm{PG}(W^*)$ (use the polarity ζ^* defined in Section 4), the group $\widetilde{A} := \{\widetilde{\varphi} \,|\, \varphi \in \mathcal{A}\}$ has 5 orbits on the set of hyperplanes of $\mathrm{PG}(W^*)$ and also 5 orbits on the set of points of $\mathrm{PG}(W^*)$. Representatives of these 5 orbits are the points $\langle \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 \rangle$, $\langle \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{f}_3 \rangle$, $\langle \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{f}_3 \rangle$, $\langle \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{f}_3 + \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_3 \rangle$ and $\langle \eta_1 \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \eta_2 \cdot \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{f}_3 + (\bar{e}_1 + \bar{f}_1) \wedge (\bar{e}_2 + \bar{f}_2) \wedge (\bar{e}_3 + \bar{f}_3) \rangle$ (with η_1 and η_2 as above).

References

[1] R. J. Blok, I. Cardinali, B. De Bruyn and A. Pasini. Polarized and homogeneous embeddings of dual polar spaces. *J. Algebraic Combin.*, to appear.

[2] R. J. Blok and B. N. Cooperstein. The generating rank of the unitary and symplectic grassmannians. preprint.

[3] P. J. Cameron. Dual polar spaces. *Geom. Dedicata* 12 (1982), 75–85.

[4] I. Cardinali, B. De Bruyn and A. Pasini. Minimal full polarized embeddings of dual polar spaces. *J. Algebraic Combin.* 25 (2007), 7–23.

[5] B. N. Cooperstein. On the generation of dual polar spaces of unitary type over finite fields. *European J. Combin.* 18 (1997), 849–856.

[6] B. De Bruyn. *Near polygons.* Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.

[7] B. De Bruyn. On the Grassmann-embeddings of the hermitian dual polar spaces. *Linear Multilinear Algebra* 56 (2008), 665–677.

[8] B. De Bruyn. On the Grassmann modules for the symplectic groups. Preprint.

[9] B. De Bruyn and H. Pralle. The hyperplanes of $DH(5, q^2)$. *Forum Math.* 20 (2008), 239–264.

[10] S. E. Payne and J. A. Thas. *Finite Generalized Quadrangles.* Research Notes in Mathematics 110. Pitman, Boston, 1984.

[11] E. E. Shult. On Veldkamp lines. *Bull. Belg. Math. Soc. Simon Stevin* 4 (1997), 299–316.

[12] J. Tits. *Buildings of spherical type and finite BN-pairs*. Lecture Notes in Mathematics 386. Springer, Berlin, 1974.

[13] F. D. Veldkamp. Polar Geometry. I-IV, V. *Indag. Math.* 21 and 22 (1959), 512–551 and 207–212.