

## ON THE GREATEST PRIME FACTORS OF DECOMPOSABLE FORMS AT INTEGER POINTS

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### 1. Introduction

Let  $f \in \mathbf{Z}[x, y]$  be a binary form and assume that among the linear factors in the factorization of  $f$  at least three are distinct. Mahler [12] proved that  $P(f(x, y)) \rightarrow \infty$  if  $X = \max(|x|, |y|) \rightarrow \infty$  with  $x, y \in \mathbf{Z}$ ,  $(x, y) = 1$ , where  $P(n)$  denotes the greatest prime factor of  $n$ . Mahler's work was generalized by Parry [14]. For irreducible forms  $f$  Coates [4] improved Mahler's result by showing that if  $\alpha = 1/4$ , then for any coprime integers  $x, y$

$$(1) \quad P(f(x, y)) > c_1(\log \log X)^\alpha, \quad X \cong X_1,$$

where  $c_1 > 0$  and  $X_1 > 0$  depend only on  $f$  and can be given explicitly. Sprindžuk [21], [22] established (1) with  $\alpha = 1$  for all such forms of degree at least 5 and for so-called non-exceptional forms of degree 4. Kotov [11] generalized Sprindžuk's result to binary forms with algebraic integer coefficients. Shorey, van der Poorten, Tijdeman and Schinzel [20] proved that if  $f \in \mathbf{Z}[x, y]$  has at least three distinct linear factors in its factorization and  $\alpha = 1$ , then (1) holds for any  $x, y \in \mathbf{Z}$  with  $(x, y) = d$ , where  $d$  is a fixed positive integer.

Schlickewei [17], [18] proved that for a large class of norm forms  $F \in \mathbf{Z}[x_1, \dots, x_m]$  in  $m \geq 2$  variables and for  $\mathbf{x} = (x_1, \dots, x_m) \in \mathbf{Z}^m$  with relatively prime components,  $P(F(\mathbf{x})) \rightarrow \infty$  as  $|\mathbf{x}| = \max(|x_1|, \dots, |x_m|) \rightarrow \infty$ . For index forms  $F \in \mathbf{Z}[x_1, \dots, x_m]$  Trelina [24] showed that

$$P(F(\mathbf{x})) > c_2(\log \log |\mathbf{x}| \log \log \log |\mathbf{x}|)^{1/2}, \quad |\mathbf{x}| \cong X_2.$$

Independently, for discriminant forms and index forms  $F \in \mathbf{Z}[x_1, \dots, x_m]$

$$(2) \quad P(F(\mathbf{x})) > c_3 \log \log |\mathbf{x}|, \quad |\mathbf{x}| \cong X_3,$$

have been established by Papp and the author [8]. Here  $\mathbf{x} \in \mathbf{Z}^m$  with  $(x_1, \dots, x_m) = 1$  and  $c_2, c_3, X_2, X_3$  are effectively computable positive numbers depending only on  $F$ . Recently the author [10] proved (2) for a wide class of irreducible norm forms  $F(\mathbf{x})$  in  $m \geq 2$  variables (including all binary forms). In [8] and [10] our estimates are established for forms  $F(\mathbf{x}) \in \mathbf{Z}_L[x_1, \dots, x_m]$  at integer points  $\mathbf{x} \in \mathbf{Z}_L^m$ , where  $\mathbf{Z}_L$  denotes the ring of integers of an arbitrary but fixed algebraic number field  $L$ .

In this paper we give a common generalization of our results mentioned above and compute an explicit value of the constant corresponding to  $c_3$ . Our main result implies the above-quoted theorems of Sprindžuk [21], [22], Kotov [11], Shorey, van der Poorten, Tijdeman and Schinzel [20], Trelina [24], Győry and Papp [8] and Győry [10].

## 2. Results

Before we state our theorem, we establish our notation and introduce some definitions.

A system  $\mathcal{L}$  of  $n \geq 2$  linear forms  $L_1(\mathbf{x}), \dots, L_n(\mathbf{x})$  in  $\mathbf{x} = (x_1, \dots, x_m)$  with algebraic coefficients will be called triangularly connected or, more briefly,  $\Delta$ -connected (cf. [7]) if for any distinct  $i, j$  with  $1 \leq i, j \leq n$  there is a sequence  $L_i = L_{i_1}, \dots, L_{i_v} = L_j$  in  $\mathcal{L}$  such that for each  $u$  with  $1 \leq u \leq v-1$ ,  $L_{i_u}, L_{i_{u+1}}$  have a linear combination with non-zero algebraic coefficients which belongs to  $\mathcal{L}$ . If in particular  $m=2$ , then every system  $\mathcal{L}$  which contains at least three pairwise non-proportional linear forms is  $\Delta$ -connected.

Throughout the paper,  $L$  will denote a fixed algebraic number field of degree  $l \geq 1$  with ring of integers  $\mathbf{Z}_L$ , and  $U_L$  will be the group of units in  $L$ . We denote by  $\omega(\alpha)$  the number of distinct prime ideal divisors  $\mathfrak{p}$  of a non-zero integer  $\alpha$  in  $L$  and by  $\mathcal{P}(\alpha)$  the greatest of the norms  $N(\mathfrak{p})$  of these prime ideals. For  $\alpha \in U_L$  we take  $\mathcal{P}(\alpha) = 1$  and  $\omega(\alpha) = 0$ .

If  $F(x_1, \dots, x_m) \in \mathbf{Z}_L[x_1, \dots, x_m]$  is a form in  $m \geq 2$  variables, then  $F(x_1, \dots, x_m)$  and  $F(\varepsilon x_1, \dots, \varepsilon x_m)$  have the same prime ideal decomposition for any  $\mathbf{x} = (x_1, \dots, x_m) \in \mathbf{Z}_L^m$  and  $\varepsilon \in U_L$ . It will be useful to introduce the notation  $\overline{|\mathbf{x}|}$  defined by<sup>1)</sup>

$$\overline{|\mathbf{x}|} = \min_{\varepsilon \in U_L} \max(|\varepsilon x_1|, \dots, |\varepsilon x_m|), \quad m \geq 2,$$

where  $\mathbf{x} = (x_1, \dots, x_m) \in \mathbf{Z}_L^m$ .  $s_0 \overline{|\mathbf{x}|}$  can be effectively determined and clearly

$$(3) \quad N^{1/l} \overline{|\mathbf{x}|} \leq \max(|x_1|, \dots, |x_m|)$$

for any  $\mathbf{x} \in \mathbf{Z}_L^m$ , where  $N = \max_{1 \leq i \leq m} (|N_{L/Q}(x_i)|)$ . Further, it is clear that in the special case  $L = \mathbf{Q}$   $\overline{|\mathbf{x}|}$  coincides with  $|\mathbf{x}|$ .

Our main result is the following

**Theorem.** *Let  $F(\mathbf{x}) = F(x_1, \dots, x_m) \in \mathbf{Z}_L[x_1, \dots, x_m]$  be a decomposable form of degree  $n \geq 3$  in  $m \geq 2$  variables with splitting field  $G$  over  $L$ , and let  $[G : \mathbf{Q}] = g$ ,  $[G : L] = f$ . Suppose that the linear factors  $L_1(\mathbf{x}), \dots, L_n(\mathbf{x})$  in the factorization of*

<sup>1)</sup>  $|\gamma|$  denotes the maximum absolute value of the conjugates of an algebraic number  $\gamma$ .

$F$  form a  $\Delta$ -connected system and that there is no  $0 \neq \mathbf{x} \in L^m$  for which  $L_j(\mathbf{x}) = 0$ ,  $j = 1, \dots, n$ . Let  $d$  be a positive integer. Then there exists an effectively computable number  $X_4$  depending only on  $F$ ,  $d$  and  $L$ , such that

$$(4) \quad (13f+1)s \log(s+1) + (g+1) \log \mathcal{P} > \log \log |\overline{\mathbf{x}}|$$

and

$$(5) \quad \mathcal{P} > ((13f+1)l)^{-\alpha} (\log \log |\overline{\mathbf{x}}|)^\alpha$$

for any  $\mathbf{x} \in \mathbf{Z}_L^m$  with  $N((x_1, \dots, x_m)) \leq d$  and  $|\overline{\mathbf{x}}| \geq X_4$ , where  $\mathcal{P} = \mathcal{P}(F(\mathbf{x}))$ ,  $s = \omega(F(\mathbf{x}))$ ,  $\mathcal{P} = P^\alpha$  and  $P$  is the maximal rational prime for which  $(F(\mathbf{x}), P) \neq 1$ .

It is easily seen that under the conditions and notations of the theorem we have  $1 \leq \alpha \leq l$ ,

$$(4') \quad (13f+1)s \log(s+1) + (g+1) \log \mathcal{P} > \log \log N$$

and

$$(5') \quad \mathcal{P} > ((13f+1)l)^{-\alpha} (\log \log N)^\alpha$$

for any  $\mathbf{x} \in \mathbf{Z}_L^m$  with  $N((x_1, \dots, x_m)) \leq d$  and  $N = \max_{1 \leq i \leq m} (|N_{L/Q}(x_i)|) \geq N_1$ . For small values of  $s$  the estimates (4) and (4') are obviously much better than (5) and (5').

Our theorem has several consequences. We first mention an application to diophantine equations. Let  $F(\mathbf{x})$  and  $d$  be as in the theorem and let  $\beta, \pi_1, \dots, \pi_t$  be fixed non-zero algebraic integers in  $L$ . Consider the equation

$$(6) \quad F(\mathbf{x}) = \beta \pi_1^{z_1} \dots \pi_t^{z_t}$$

in  $\mathbf{x} \in \mathbf{Z}_L^m, z_1, \dots, z_t \in \mathbf{Z}$  with  $N((x_1, \dots, x_m)) \leq d$  and  $z_1, \dots, z_t \geq 0$ . Then (4) gives

$$\max(|\overline{\mathbf{x}}|, e^{\max_k(z_k)}) < C$$

for all solutions  $\mathbf{x}, z_1, \dots, z_t$  of (6), where  $C$  is an effectively computable number<sup>2)</sup> depending only on  $F, d, \mathcal{P}(\beta \pi_1 \dots \pi_t), \omega(\beta \pi_1 \dots \pi_t)$  and  $L$ . This result can be regarded as a  $p$ -adic analogue of our Theorem 1 in [7]. (In [7] it is not assumed  $F \in \mathbf{Z}_L[\mathbf{x}]$ ; however, in the applications of Theorem 1 of [7]  $F \in \mathbf{Z}_L[\mathbf{x}]$  is always supposed. Thus this is not an essential restriction.)

The following corollary enables us to obtain some information about the arithmetical structure of those algebraic integers of  $L$  which can be represented by a decomposable form of the above type.

*Corollary 1. Suppose  $F(x_1, \dots, x_m)$  and  $d$  are as in the Theorem. Let  $F$  be any algebraic integer in  $L$  represented by  $F(x_1, \dots, x_m)$ , where  $x_1, \dots, x_m \in \mathbf{Z}_L$  with  $N((x_1, \dots, x_m)) \leq d$ . Then*

$$(7) \quad (13f+1)\omega(F) \log(\omega(F)+1) + (g+1) \log \mathcal{P}(F) > \log \log N$$

<sup>2)</sup> We could easily obtain an explicit expression for  $C$  by computing each constant in the proof of our theorem. *Added in proof:* In my paper „Explicit upper bounds for the solutions of some diophantine equations” (to appear) I explicitly evaluated  $C$  in terms of each constant, (generalizing many earlier effective results on norm form, discriminant form and index form equations).

and

$$(8) \quad \mathcal{P}(F) > ((13f+1)l)^{-1} \log \log N$$

if  $N = |N_{L/Q}(F)| \geq N_2$ , where  $N_2$  is an effectively computable positive number depending only on  $d, L$  and the form  $F(x_1, \dots, x_m)$ .

Our Corollary 1 generalizes and improves Sprindžuk's theorems [22], [23] concerning rational integers represented by a binary form  $f \in \mathbf{Z}[x, y]$ .

**Corollary 2.** *Let  $F(\mathbf{x}) \in \mathbf{Z}_L[x_1, \dots, x_m]$  be a decomposable form with the properties specified in the Theorem. Let  $d$  and  $A$  be positive numbers with  $d \geq 1$  and  $A < 1/(g+1)$ . Then there exists an effectively computable number  $X_5$  depending only on  $F, d, L$  and  $A$  such that if*

$$\mathcal{P}(F(\mathbf{x})) \leq (\log \overline{|\mathbf{x}|})^A, \quad \mathbf{x} \in \mathbf{Z}_L^m, \quad \overline{|\mathbf{x}|} \geq X_5$$

and  $N((x_1, \dots, x_m)) \leq d$ , then

$$(9) \quad \omega(F(\mathbf{x})) > c_4 \frac{\log \log \overline{|\mathbf{x}|}}{\log \log \log \overline{|\mathbf{x}|}},$$

where  $c_4 = (1 - A(g+1))/(13f+1)$ .

Let  $f \in \mathbf{Z}_L[x]$  be a polynomial with at least three distinct roots. Since  $|\overline{x}|^{1/l} \leq \max(|\overline{\varepsilon x}|, |\overline{\varepsilon}|)$  for any  $x \in \mathbf{Z}_L$  and  $\varepsilon \in U_L$ , our estimates (4), (5), (7), (8) and (9) remain obviously valid for  $\mathcal{P}(f(x))$  and  $\omega(f(x))$  with  $|\overline{x}|$  instead of  $|\overline{\mathbf{x}}|$ , where  $x \in \mathbf{Z}_L$  and  $|\overline{x}| > X_6$ . We remark that for polynomials  $f(x)$  with rational integer coefficients Shorey and Tijdeman [19] obtained a much better result than our Corollary 2; they proved  $\omega(f(x)) \gg (\log \log |x|)/(\log \log \log |x|)$  under the condition  $P(f(x)) \leq \exp((\log \log |x|)^A)$ , where  $A$  is any positive number. As an immediate consequence of this result they derived a good lower bound for  $\max_{1 \leq i \leq y} P(f(x+i))$ .

As a consequence of our theorem we obtain the following generalization and improvement, respectively, of the theorems of Coates [4], Sprindžuk [21], [22], Kotov [11] and Shorey, van der Poorten, Tijdeman and Schinzel [20] on the maximal prime factors of binary forms.

**Corollary 3.** *Let  $f(x, y) \in \mathbf{Z}_L[x, y]$  be a binary form with splitting field  $G$  over  $L$  and suppose that among the linear factors in the factorization of  $f$  at least three are distinct<sup>3)</sup>. Let  $[G: \mathbf{Q}] = g$ ,  $[G: L] = f$  and  $d \geq 1$ . Then there exists an effectively computable positive number  $X_7$  depending only on  $d, L$  and the form  $f(x, y)$  such that for all pairs  $x, y \in \mathbf{Z}_L$  with  $N((x, y)) \leq d$  and  $|\overline{\mathbf{x}}| = \min_{\varepsilon \in U_L} \max(|\overline{\varepsilon x}|, |\overline{\varepsilon y}|) > X_7$ , (4) and (5) hold, where  $\mathcal{P} = \mathcal{P}(f(x, y))$ ,  $s = \omega(f(x, y))$ ,  $\mathcal{P} = P^z$  and  $P$  is the maximal rational prime with  $(f(x, y), P) \neq 1$ .*

<sup>3)</sup> In other words  $f$  has at least three pairwise nonproportional linear factors in its factorization.

It follows from (5') that

$$(10) \quad \mathcal{P}(f(x, y)) > c_5(\log \log N)^\alpha$$

for all  $x, y \in \mathbf{Z}_L$  with  $(x, y) = 1$  and  $N = \max(|N_{L/Q}(x)|, |N_{L/Q}(y)|) \geq N_3$ , where  $c_5 = ((13f+1)l)^{-\alpha}$ . For irreducible forms  $f \in \mathbf{Z}_L[x, y]$  of degree  $\geq 5$  (10) was earlier proved by Kotov [11].

An important special case of Corollary 3 is when  $f(x, y) = (x - \alpha_1 y) \dots (x - \alpha_n y)$ , where  $\alpha_1, \dots, \alpha_n \in \mathbf{Z}_L$  and at least three of them are distinct. This special case of Corollary 3 can be used to obtain an effective result on the diophantine equation  $az^d = f(x, y)$  (cf. [20], pp. 63–65).

Corollary 4. *Let  $K$  be an extension of degree  $n \geq 3$  of  $L$  and let  $F(\mathbf{x}) = \alpha_0 N_{K/L}(x_1 + \alpha_2 x_2 + \dots + \alpha_m x_m) \in \mathbf{Z}_L[x_1, \dots, x_m]$  be a norm form in  $m \geq 2$  variables such that  $[L(\alpha_i): L] = n_i \geq 3, i = 2, \dots, m$ , and  $n_2 \dots n_m = n$ . Then with the notations of the Theorem we have (4) and (5).*

Corollary 4 implies Corollary 2 of [10] and Theorem 3 of Kotov [11].

Corollary 5. *Let  $K$  be as in Corollary 4. Let  $\alpha_1, \dots, \alpha_m$  be  $m \geq 2$  algebraic integers in  $K$  with  $K = L(\alpha_1, \dots, \alpha_m)$  and suppose that  $1, \alpha_1, \dots, \alpha_m$  are linearly independent over  $L$ . Let  $F(\mathbf{x})$  denote the discriminant form  $\text{Discr}_{K/L}(\alpha_1 x_1 + \dots + \alpha_m x_m)$ . Under the notations of the Theorem, for  $F(\mathbf{x})$  (4) and (5) hold.*

Corollary 5 improves Corollary 1 of our paper [8].

Let again  $K$  be an extension of degree  $n \geq 3$  of  $L$  and let  $G$  be the smallest normal extension of  $L$  containing  $K$ . Write  $[G: Q] = g$  and  $[G: L] = f$ . Consider an order  $O$  of the field extension  $K/L$  (i.e. a subring of  $\mathbf{Z}_K$  containing  $\mathbf{Z}_L$  that has the full dimension  $n$  as a  $\mathbf{Z}_L$ -module) and suppose that  $O$  has a relative integral basis  $1, \alpha_1, \dots, \alpha_{n-1}$  over  $L$ . (Such an integral basis exists for a number of orders of  $K/L$ ; see e.g. [2], [13] and [8].) Then we have (cf. [8])

$$(11)$$

$$\text{Discr}_{K/L}(\alpha_1 x_1 + \dots + \alpha_{n-1} x_{n-1}) = [\text{Ind}_{K/L}(\alpha_1 x_1 + \dots + \alpha_{n-1} x_{n-1})]^2 D_{K/L}(1, \alpha_1, \dots, \alpha_{n-1}),$$

where  $I(x) = \text{Ind}_{K/L}(\alpha_1 x_1 + \dots + \alpha_{n-1} x_{n-1}) \in \mathbf{Z}_L[x_1, \dots, x_{n-1}]$  is a decomposable form of degree  $n(n-1)/2$ . It is called the index form of the basis  $1, \alpha_1, \dots, \alpha_{n-1}$  of  $O$  over  $L$ .

In the special case  $L = Q$  Trelina [24] obtained lower bounds for  $P(I(\mathbf{x}))$ . Corollary 1 and Theorem 3 in our paper [8], established independently of Trelina, give lower bounds for  $\mathcal{P}(I(\mathbf{x}))$  in the above general case. As a consequence of Corollary 5 we obtain the following generalization and improvement of the estimates of Trelina [24] and Györy and Papp [8].

Corollary 6. Let  $L, K, d$  and  $I(\mathbf{x})$  be defined as above. Then there exists an effectively computable positive number  $X_8$  depending only on  $I(\mathbf{x}), d, L$  and  $D_{K/L}(1, \alpha_1, \dots, \alpha_{n-1})$  such that (4) and (5) hold for any  $\mathbf{x} \in \mathbf{Z}_L^{n-1}$  with  $N((x_1, \dots, x_{n-1})) \leq d$  and  $|\bar{\mathbf{x}}| \leq X_8$ , where  $\mathcal{P} = \mathcal{P}(I(\mathbf{x})), s = \omega(I(\mathbf{x})D_{K/L}(1, \alpha_2, \dots, \alpha_{n-1})), \mathcal{P} = P^2$  and  $P$  is the maximal rational prime with  $(I(\mathbf{x}), P) \neq 1$ .

The proof of our theorem depends on two deep theorems, due to van der Poorten and Loxton [16] and van der Poorten [15], which are essentially sharp inequalities on linear forms in the complex and in the  $p$ -adic case.

### 3. Proof of the Theorem

We first show that we can make certain assumptions without loss of generality. By using a well-known argument we can easily see that there exist algebraic integers  $a_2, \dots, a_m$  in  $L$  such that  $F(1, a_2, \dots, a_m) \neq 0$  (see e.g. [3], p. 77). It suffices to prove the theorem for  $F(x_1, a_2x_1 + x_2, \dots, a_mx_1 + x_m)$ , where the coefficient of  $x_1^n$  is non-zero. Hence we may suppose that

$$F(\mathbf{x}) = a_0 L_1(\mathbf{x}) \dots L_n(\mathbf{x})$$

with  $0 \neq a_0 \in \mathbf{Z}_L$  and

$$L_j(\mathbf{x}) = x_1 + \alpha_{2j}x_2 + \dots + \alpha_{mj}x_m, \quad j = 1, \dots, n,$$

where  $\alpha_{ij} \in G, 2 \leq i \leq m, 1 \leq j \leq n$ . Writing  $\alpha'_{ij} = a_0 \alpha_{ij}$  for  $i \geq 2$  and  $\alpha'_{ij} = a_0$  for  $i = 1$ , we have  $\alpha'_{ij} \in \mathbf{Z}_G$  for each  $i$  and  $j$ . We shall prove our theorem for

$$f(\mathbf{x}) = a_0^{n-1} F(\mathbf{x}) = \prod_{j=1}^n L'_j(\mathbf{x}),$$

where  $L'_j(\mathbf{x}) = \alpha'_{1j}x_1 + \dots + \alpha'_{mj}x_m$ . This will imply at once the assertion of the theorem for  $F(\mathbf{x})$ .

We suppose that there are  $r_1$  real and  $2r_2$  complex conjugate fields to  $G$  and that they are chosen in the usual manner: if  $\theta$  is in  $G$ , then  $\theta^{(i)}$  is real for  $1 \leq i \leq r_1$  and  $\theta^{(i+r_2)} = \overline{\theta^{(i)}}$  for  $r_1 + 1 \leq i \leq r_1 + r_2$ . Put  $r = r_1 + r_2 - 1$ . It is well-known that there exist fundamental units  $\eta_1, \dots, \eta_r$  in  $G$  and constants  $c_6, c_7$  such that  $|\log |\eta_h^{(i)}|| \leq c_6$  for  $1 \leq h \leq r, 1 \leq i \leq g$  and  $R_G > c_7$ , where  $R_G$  denotes the regulator of  $G$ . Here, and below,  $c_6, c_7, \dots$  will denote effectively computable positive numbers which depend only on  $F(\mathbf{x}), L$  and (some of them) on  $d$ .

Let  $x_1, \dots, x_m$  be any  $m$ -tuple of algebraic integers in  $L$  with  $N((x_1, \dots, x_m)) \leq d$ . Put

$$(12) \quad \beta_j = \alpha'_{1j}x_1 + \dots + \alpha'_{mj}x_m, \quad j = 1, \dots, n,$$

and

$$(13) \quad (f(\mathbf{x})) = (\beta_1 \dots \beta_n) = \mathfrak{p}_1^{v_1} \dots \mathfrak{p}_s^{v_s},$$

where  $\mathfrak{p}_1, \dots, \mathfrak{p}_s$  are distinct prime ideals in  $L$ . If  $X_4$  is sufficiently large and  $\overline{|\mathbf{x}|} \geq X_4$ , then Theorem 1 of [7] implies  $s > 0$  and  $P > 1$ . Let  $\mathfrak{P}_1, \dots, \mathfrak{P}_t$  be all distinct prime ideals in  $G$  lying above  $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ . Clearly  $t \leq sf$ . Applying now the unique factorization theorem to (13) we get in  $\mathbf{Z}_G$

$$(14) \quad (\beta_j) = \mathfrak{P}_1^{u_{1j}} \dots \mathfrak{P}_t^{u_{tj}}, \quad j = 1, \dots, n,$$

where the  $U_{kj}$  are non-negative rational integers. Denote by  $h_G$  the class number of  $G$  and write  $U_{kj} = h_G u_{kj} + r_{kj}$  with  $0 \leq r_{kj} < h_G$ . We have  $\mathfrak{P}_k^{h_G} = (\mu_k)$  with some  $\mu_k \in \mathbf{Z}_G$ . Then from (14) we see that

$$(15) \quad (\beta_j) = (\chi_j) (\mu_1)^{u_{1j}} \dots (\mu_t)^{u_{tj}},$$

where  $(\chi_j) = \mathfrak{P}_1^{r_{1j}} \dots \mathfrak{P}_t^{r_{tj}}$  and

$$|N_{G/Q}(\mu_k)| \leq P^{gh_G}, \quad |N_{G/Q}(\chi_j)| \leq P^{gh_G t}.$$

So, following a well-known argument (see e.g. [1], p. 188), we may choose  $\mu_k$  and  $\chi_j$  such that

$$(16) \quad |\log |\mu_k^{(i)}|| \leq c_8 \log P, \quad |\log |\chi_j^{(i)}|| \leq c_8 s \log P, \quad i = 1, \dots, g,$$

and, by (15), we have

$$\beta_j = \varepsilon_j \chi_j \mu_1^{u_{1j}} \dots \mu_t^{u_{tj}}, \quad j = 1, \dots, n,$$

for some unit  $\varepsilon_j$  of  $G$ .

Put  $\mathcal{L} = \{L'_1, \dots, L'_n\}$ . By hypothesis there are two forms in  $\mathcal{L}$ , say  $L'_1$  and  $L'_2$ , such that  $\lambda_1 L'_1(\mathbf{x}) + \lambda_2 L'_2(\mathbf{x}) \in \mathcal{L}$  with non-zero algebraic numbers  $\lambda_1, \lambda_2$ . Suppose, for convenience, that

$$\lambda_1 L'_1(\mathbf{x}) + \lambda_2 L'_2(\mathbf{x}) + \lambda_3 L'_3(\mathbf{x}) = 0$$

with  $\lambda_1 \lambda_2 \lambda_3 \neq 0$ . Further, we may assume that  $\lambda_1, \lambda_2, \lambda_3 \in \mathbf{Z}_G$  and  $\max(\overline{|\lambda_1|}, \overline{|\lambda_2|}, \overline{|\lambda_3|}) \leq c_9$ . We obtain now

$$(17) \quad \lambda_1 \beta_1 + \lambda_2 \beta_2 + \lambda_3 \beta_3 = 0.$$

Put  $a_k = \min_q u_{kq}$  and  $u'_{kq} = u_{kq} - a_k$  for  $q = 1, 2, 3$  and  $k = 1, \dots, t$ . We may suppose without loss of generality that  $U = \max_{k,q} u'_{kq} = u'_{11}$  and  $u'_{13} = 0$ . Since  $\eta_1, \dots, \eta_r$  are fundamental units, we can write

$$\varepsilon_1/\varepsilon_3 = \varrho_1 \eta_1^{w_{11}} \dots \eta_r^{w_{r1}}, \quad \varepsilon_2/\varepsilon_3 = \varrho_2 \eta_1^{w_{12}} \dots \eta_r^{w_{r2}},$$

where  $\varrho_1, \varrho_2$  are roots of unity in  $G$  and  $w_{11}, \dots, w_{r1}, w_{12}, \dots, w_{r2}$  are rational integers. With the notation

$$(18) \quad \beta_q = \sigma \delta_q, \quad \sigma = \varepsilon_3 \mu_1^{u_1} \dots \mu_t^{u_t}, \quad \delta_q = \chi_q \varrho_q \eta_1^{w_{1q}} \dots \eta_r^{w_{rq}} \mu_1^{u_{1q}} \dots \mu_t^{u_{tq}}$$

and  $w_{13} = \dots = w_{r3} = 0, \varrho_3 = 1$  we get from (17)

$$(19) \quad A = -\frac{\lambda_2 \delta_2}{\lambda_3 \delta_3} - 1 = \frac{\lambda_1 \delta_1}{\lambda_3 \delta_3} \neq 0.$$

We are now going to derive an upper bound for  $H = \max(U, W)$ , where  $W = \max_{j,q} |w_{jq}|$ . First suppose that  $c_{10}s \log P \cdot U > H$  with a sufficiently large  $c_{10}$ . We may assume that  $U \cong c_{11}s \log P$  with a sufficiently large  $c_{11}$ , for otherwise (21) immediately follows. We see from (19) that

$$\infty > \text{ord}_{\mathfrak{p}_1} A \cong U - c_{12}s \log P \cong c_{13}U \cong \frac{c_{14}}{s \log P} H.$$

Further, by (19) we have

$$(20) \quad A = -\frac{\lambda_2 \chi_2 \varrho_2}{\lambda_3 \chi_3} \eta_1^{w_{12}} \dots \eta_r^{w_{r2}} \mu_1^{u'_{12} - u_{13}} \dots \mu_t^{u'_{t2} - u_{t3}} - 1.$$

Applying now Theorem 4 of van der Poorten [15] to  $A$ , we obtain by (16)

$$(21) \quad H < c_{15}(c_{16}s)^{12(r+sf)+28} P^g (\log P)^{sf+4}.$$

Suppose now that  $c_{10}s \log P \cdot U \cong H$ . Assume, for convenience, that  $W = |w_{11}|$ . From (18) we conclude

$$w_{11} \log |\eta_1^{(i)}| + \dots + w_{r1} \log |\eta_r^{(i)}| = \log |\delta_1^{(i)}| - \log |\chi_1^{(i)}| - \sum_k u'_{k1} \log |\mu_k^{(i)}|$$

for each conjugate with  $i=1, \dots, r$ . So for some  $h$  we must have

$$W \cong c_{17} \left( |\log |\delta_1^{(h)}|| + |\log |\chi_1^{(h)}|| + \sum_k u'_{k1} |\log |\mu_k^{(h)}|| \right).$$

Thus, by (16) we obtain

$$|\log |\delta_1^{(h)}|| \cong c_{18}W - c_{19}s \log P - c_{20}Us \log P \cong c_{21}H,$$

provided that  $c_{10}$  is sufficiently large. Further, by (16) and (18) we have

$$\log |N_{G/Q}(\delta_1)| \cong \log |N_{G/Q}(\chi_1)| + U \cdot \sum_k \log |N_{G/Q}(\mu_k)| \cong c_{22}Us \log P.$$

Hence we get for some  $m$

$$(22) \quad \log |\delta_1^{(m)}| \cong -c_{23}H.$$

Formulae (16) and (18) imply

$$(23) \quad \log \left| \frac{\lambda_1^{(m)}}{\lambda_3^{(m)} \delta_3^{(m)}} \right| \cong c_{24} + (g-1) \log |\overline{\delta_3}| \cong c_{25}Us \log P < \frac{c_{23}}{2} H.$$

We now omit the superscript  $(m)$ . It then follows from (22) and (23) that

$$\log |A| < -\frac{c_{23}}{2} H.$$

Write  $\eta_0 = -1$ . By taking the principal values of the logarithms we obtain from (19) and (18)

$$(24) \quad 0 < \left| \log \left( -\frac{\lambda_2 \delta_2}{\lambda_3 \delta_3} \right) \right| \\ = \left| \sum_{j=0}^r w_{j2} \log \eta_j + \sum_{k=1}^t (u'_{k2} - u_{k3}) \log \mu_k - \log \left( -\frac{\lambda_3 \chi_3}{\lambda_2 \chi_2 \varrho_2} \right) \right| < e^{-\delta^*(r+t+1)H},$$



where  $\delta^* = (c_{26}(r+t+1))^{-1}$  and  $w_{02}$  is a rational integer satisfying

$$|w_{02}| \leq (r+t+1)H.$$

We can now apply Theorem 3 of van der Poorten and Loxton [16] to (24) and obtain

$$(25) \quad H < c_{27}(c_{28}s)^{10(r+sf)+33}(\log P)^{sf+3}.$$

So (21) and (25) imply

$$(26) \quad H < c_{29}(c_{30}s)^{12(r+sf)+31}P^g(\log P)^{sf+4}$$

and, by (16), (18) and (26), we have

$$(27) \quad \begin{aligned} |\overline{\delta}_q| &< \exp \{c_{31}s \log P + c_{32}H + c_{33}Hs \log P\} < \\ &< \exp \{c_{34}(c_{30}s)^{12(r+sf)+32}P^g(\log P)^{sf+5}\} = T_1, \quad q = 1, 2, 3. \end{aligned}$$

Consider now any  $\beta_j$  with  $3 \leq j \leq n$ . By the assumption made on  $L'_1, \dots, L'_n$  there is a sequence  $\beta_2 = \beta_{i_1}, \dots, \beta_{i_v} = \beta_j$  such that for each  $u$  with  $1 \leq u \leq v-1$

$$\lambda_{i_u}\beta_{i_u} + \lambda_{i_{u+1}}\beta_{i_{u+1}} + \lambda_{i_u, u+1}\beta_{i_u, u+1} = 0$$

holds with some non-zero  $\lambda_{i_u}, \lambda_{i_{u+1}}, \lambda_{i_u, u+1} \in \mathbf{Z}_G$  satisfying  $\max(|\overline{\lambda_{i_u}}|, |\overline{\lambda_{i_{u+1}}}|, |\overline{\lambda_{i_u, u+1}}|) \leq c_{35}$ . Further, we may assume  $v \leq n$ . We can see in the same way as above that

$$(28) \quad \beta_1 = \sigma\delta_1, \quad \beta_2 = \sigma\delta_2$$

and

$$(29) \quad \beta_{i_u} = \sigma_u\delta_{u, i_u}, \quad \beta_{i_{u+1}} = \sigma_u\delta_{u, i_{u+1}}$$

for  $u=1, \dots, v-1$ , where  $\delta_{u, i_u}, \delta_{u, i_{u+1}} \in \mathbf{Z}_G$  with

$$(30) \quad \max_{1 \leq u \leq v-1} (|\overline{\delta_{u, i_u}}|, |\overline{\delta_{u, i_{u+1}}}|) < T_1$$

and  $\sigma_u = \vartheta_u \mu_1^{a_{1u}} \dots \mu_t^{a_{tu}}$  with units  $\vartheta_u \in G$  and non-negative rational integers  $a_{1u}, \dots, a_{tu}$ . It follows from (28) and (29) that

$$(31) \quad \beta_j = \beta_{i_v} = \sigma\varphi_j/\psi_j$$

with

$$\varphi_j = \delta_2 \prod_{u=1}^{v-1} \delta_{u, i_{u+1}} \quad \text{and} \quad \psi_j = \prod_{u=1}^{v-1} \delta_{u, i_u}.$$

Write  $\psi_1 = \psi_2 = 1$  and  $\varphi_j = \delta_j$  for  $j=1, 2$ . It is clear that

$$(32) \quad \max(|\overline{\varphi_j}|, |\overline{\psi_j}|) < T_1^n, \quad j = 1, \dots, n.$$

We recall that  $\sigma = \varepsilon_3 \mu_1^{a_1} \dots \mu_t^{a_t}$ . Denote by  $\mu_k^{b_k}$  the highest power of  $\mu_k$  with  $b_k \leq a_k$  that divides at least one of the  $\psi_1, \dots, \psi_n$ . By taking norms we see that

$$b_k \leq c_{36} \log T_1, \quad k = 1, \dots, t.$$

Putting

$$b_k^* = \min(a_k, b_k + 1), \quad d_k = a_k - b_k^*, \quad k = 1, \dots, t,$$

and

$$\tau_j = \mu_1^{b_1^*} \dots \mu_t^{b_t^*} \varphi_j / \psi_j,$$

we get

$$(33) \quad \beta_j = \vartheta \mu_1^{d_1} \dots \mu_t^{d_t} \tau_j, \quad j = 1, \dots, n,$$

where  $\vartheta = \varepsilon_3$  is a unit and  $\tau_j$  are algebraic integers in  $G$  satisfying

$$(34) \quad |\overline{\tau_j}| < \exp \{c_{37} s \log P \log T_1\} = T_2.$$

Further, by (13) we have

$$(35) \quad \mathfrak{p}_1^{v_1} \dots \mathfrak{p}_s^{v_s} = (\beta_1 \dots \beta_n) = ((\vartheta \mu_1^{d_1} \dots \mu_t^{d_t})^n \tau_1 \dots \tau_n).$$

Let  $k$ ,  $1 \leq k \leq s$ , be an arbitrary but fixed subscript, and let  $\mathfrak{P}$  denote an arbitrary prime ideal in  $G$  lying above  $\mathfrak{p}_k$ . If  $\mathfrak{P}^{e_k} | \mathfrak{p}_k$ ,  $e_k$  does not depend on the choice of  $\mathfrak{P}$ . Moreover,  $\mathfrak{P}$  divides only one of the  $\mu_1, \dots, \mu_t$ . We shall now follow an argument used in the proof of Theorem 1 of [5] (cf. the deduction (36)  $\Rightarrow$  (41) of [5]). Let  $y_k$  be the greatest rational integer for which

$$(36) \quad \min \left( v_k e_k - \text{ord}_{\mathfrak{P}} \left( \prod_{j=1}^n \tau_j \right), v_k e_k \right) \cong n h_L y_k e_k$$

holds for each  $\mathfrak{P}$  with  $\mathfrak{P} | \mathfrak{p}_k$ , where  $h_L$  denotes the class number of  $L$ . From (35) it follows that  $y_k \cong 0$ . By the definition of the  $y_k$  there is a  $\mathfrak{P}$ , lying above  $\mathfrak{p}_k$ , such that

$$(37) \quad n h_L (y_k + 1) e_k > \min \left( v_k e_k - \text{ord}_{\mathfrak{P}} \left( \prod_{j=1}^n \tau_j \right), v_k e_k \right).$$

Since (34) implies

$$\text{ord}_{\mathfrak{P}} \left( \prod_{j=1}^n \tau_j \right) \leq c_{38} \log T_2,$$

we get from (36) and (37)

$$(38) \quad 0 \leq v_k e_k - n h_L y_k e_k \leq c_{39} \log T_2.$$

If now  $\mathfrak{P}$  is an arbitrary prime ideal in  $G$  lying above  $\mathfrak{p}_k$  and  $\mathfrak{P} | (\mu_p)$ , then (35), (36) and (38) give

$$(39) \quad 0 \leq d_p \text{ord}_{\mathfrak{P}} \mu_p - h_L y_k e_k \leq c_{40} \log T_2.$$

Let now  $\mathfrak{p}_1^{h_L y_1} \dots \mathfrak{p}_s^{h_L y_s} = (\kappa)$ , where  $\kappa \in \mathbf{Z}_L$ , and choose  $\zeta$  in such a way that

$$(40) \quad \mu_1^{d_1} \dots \mu_t^{d_t} = \kappa \zeta.$$

In view of (39)  $\zeta$  is an algebraic integer in  $G$  and

$$(41) \quad |N_{G/Q}(\zeta)| \leq \exp \{c_{41} s \log P \log T_2\}.$$

It follows from (33) and (40) that

$$\omega = \vartheta^n \xi^n \tau_1 \dots \tau_n \in \mathbf{Z}_L.$$

Further, Lemma 3 of [6] together with (34) and (41) imply that there is a unit  $\theta_1 \in L$  and an  $\omega' \in \mathbf{Z}_L$  such that

$$\omega = \theta_1^n \omega'$$

and

$$(42) \quad |\overline{\omega'}| < \exp \{c_{42} s \log P \log T_2\}.$$

Thus by (34) and (42) we have

$$(43) \quad |\overline{\theta_1^{-1} \vartheta \xi}| < \exp \{c_{43} s \log P \log T_2\}.$$

Finally, writing  $\xi_j = \theta_1^{-1} \vartheta \xi \tau_j$  we get

$$(44) \quad \beta_j = \theta_1 \varkappa \xi_j, \quad j = 1, \dots, n,$$

and, by (34) and (43),

$$(45) \quad |\overline{\xi_j}| < \exp \{c_{44} s \log P \log T_2\} = T_3.$$

By hypothesis there is no  $0 \neq \mathbf{x} \in L^m$  for which  $L'_j(\mathbf{x}) = 0, j = 1, \dots, n$ . Consequently, the only solution in  $L$  of the system of equations

$$(46) \quad L'_j(\mathbf{x}) = \beta_j, \quad j = 1, \dots, n,$$

is the  $\mathbf{x} = (x_1, \dots, x_m)$  considered above. Since  $f(\mathbf{x})/a_0^n$  is a product of irreducible norm forms over  $L$ , (46) contains all conjugates of each equation over  $L$ . Following now an argument of the proof of Lemma 2 of [7], we can easily see that (46) has no other solutions in the complex field. So  $m \leq nf$ , and by Cramer's rule we have

$$(47) \quad x_i = \theta_1 \varkappa v_i / v, \quad i = 1, \dots, m,$$

where  $v, v_i \in \mathbf{Z}_G, v_1, \dots, v_m$  are not all zero,

$$(48) \quad |\overline{v}| \leq c_{45}$$

and, by (45),

$$(49) \quad |\overline{v_i}| \leq c_{46} T_3, \quad i = 1, \dots, m.$$

In view of (47) we obtain in  $\mathbf{Z}_G$

$$|N_{G/Q}(\varkappa)| N((v_1, \dots, v_m)) = |N_{G/Q}(v)| N((x_1, \dots, x_m)).$$

Hence, by (48),

$$(50) \quad |N_{L/Q}(\varkappa)| \leq |N_{G/Q}(v)|^{1/f} d \leq c_{47}.$$

Thus we can write  $\theta_1 \varkappa = \theta_2^{-1} \varkappa'$  with a unit  $\theta_2 \in L$  and an algebraic integer  $\varkappa' \in L$  satisfying

$$(51) \quad |\overline{\varkappa'}| \leq c_{48}.$$

It follows now from (47) that

$$x'_i = \theta_2 x_i = \kappa' v_i/v, \quad i = 1, \dots, m,$$

and this implies

$$x'_i{}^f = N_{G/L}(x'_i) = N_{G/L}(\kappa' v_i)/N_{G/L}(v), \quad i = 1, \dots, m.$$

By the inequality (24) of [7] we have

$$|\overline{x'_i}|^f \leq |\overline{N_{G/L}(\kappa' v_i)}| |\overline{N_{G/L}(v)}|^{t-1} \leq |\kappa' v_i|^f |v|^{(t-1)f},$$

whence, by (48), (49), (51), (45), (34) and (27) we obtain

$$(52) \quad \max_{1 \leq i \leq m} |\overline{x'_i}| < c_{49} T_3 \leq \exp \{c_{50} (c_{51} s)^{12(r+sf)+34} P^g (\log P)^{sf+7}\}.$$

From (52) we deduce

$$(53) \quad \log \log |\overline{\mathbf{x}}| < \log c_{50} + (12(r+sf) + 34) \log (c_{51} s) + g \log P + (sf+7) \log \log P.$$

If  $X_4$  is sufficiently large, then  $P$  is also sufficiently large and  $s > (\log P)^{3f/(3f+1)}$  implies

$$\begin{aligned} \log c_{50} + (12(r+sf) + 34) \log c_{51} + (12r+34) \log (s+1) + (sf+7) \log \log P \\ \leq \left(f + \frac{1}{2}\right) s \log (s+1). \end{aligned}$$

On the other hand, for  $s \leq (\log P)^{3f/(3f+1)}$  we have

$$\log c_{50} + (12(r+sf) + 34) \log c_{51} + (12r+34) \log (s+1) + (sf+7) \log \log P \leq \log P.$$

Hence (53) gives

$$(54) \quad \log \log |\overline{\mathbf{x}}| < \left(13f + \frac{1}{2}\right) s \log (s+1) + (g+1) \log P,$$

whence (4) follows.

By prime number theory we can choose  $X_4$  such that even  $\pi(P) \geq (1+\delta)P/\log P$  holds with  $\delta = 1/(2(26f+1))$ . Then  $s \leq l\pi(P) \leq (1+\delta)lP/\log P$  and thus

$$(55) \quad \left(13f + \frac{1}{2}\right) s \log (s+1) + (g+1) \log P \leq (13f+1)lP.$$

Finally, in consequence of (54), (55) and  $\mathcal{P} = P^\alpha$  we obtain (5).

In order to prove (4') and (5') it suffices to observe that (53) and (3) imply

$$\log \log N < \log (lc_{50}) + (12(r+sf) + 34) \log (c_{51} s) + g \log P + (sf+7) \log \log P.$$

If  $N$  is sufficiently large, we get (4') and (5') in the same way as we deduced (4) and (5) from (53).

### 4. Proofs of the Corollaries

*Proof of Corollary 1.* Let  $\varepsilon$  be a unit in  $L$  such that  $|\overline{\mathbf{x}}| = \max(|\overline{\varepsilon x_1}|, \dots, |\overline{\varepsilon x_m}|)$ . Then

$$(56) \quad N = |N_{L/Q}(F)| = |N_{L/Q}(F(\varepsilon \mathbf{x}))| \cong c_{52} |\overline{\mathbf{x}}|^{nt}.$$

Therefore, for sufficiently large  $N$ , (4) implies (7), but only with  $\log \log N - \log(2ln)$  in place of  $\log \log N$ . Following the argument applied at the end of the above proof, we obtain (7) and (8) from (53) and (56).

*Proof of Corollary 2.* Suppose

$$\omega(F(\mathbf{x})) \cong c_4 \frac{\log \log |\overline{\mathbf{x}}|}{\log \log \log |\overline{\mathbf{x}}|}$$

for some  $\mathbf{x} \in \mathbf{Z}_L^n$  with  $|\overline{\mathbf{x}}| \cong X_5$  and  $N((x_1, \dots, x_m)) \cong d$ . Then by our theorem we have

$$\begin{aligned} \log \log |\overline{\mathbf{x}}| &< (13f+1)\omega(F(\mathbf{x})) \log(\omega(F(\mathbf{x}))+1) + (g+1) \log \mathcal{P}(F(\mathbf{x})) \\ &\cong (13f+1)c_4 \log \log |\overline{\mathbf{x}}| + A(g+1) \log \log |\overline{\mathbf{x}}|, \end{aligned}$$

provided that  $X_5$  is sufficiently large. Since  $(13f+1)c_4 + A(g+1) = 1$ , we have arrived at a contradiction and thus (9) is proved.

*Proof of Corollary 3.* By assumption there are at least three pairwise non-proportional linear factors in the factorization

$$f(x, y) = \prod_{i=1}^n (\alpha_{i1}x_1 + \alpha_{i2}y).$$

Consequently, the linear factors  $\alpha_{i1}x + \alpha_{i2}y$ ,  $i = 1, \dots, n$ , form a  $\Delta$ -connected system and the system of equations

$$\alpha_{i1}x + \alpha_{i2}y = 0, \quad i = 1, \dots, n,$$

has no non-trivial solution  $x, y$  in  $L$ . So the assertion of Corollary 3 follows at once from our theorem.

*Proof of Corollary 4.*  $F(\mathbf{x})$  can be written in the form

$$F(\mathbf{x}) = a_0 \prod_{i=1}^n (x_1 + \alpha_2^{(i)}x_2 + \dots + \alpha_m^{(i)}x_m),$$

where  $\alpha_j^{(1)}, \dots, \alpha_j^{(n)}$  denote the conjugates of  $\alpha_j$  over  $L$ . As we showed in [7] (see also [9]), the conjugates  $x_1 + \alpha_2^{(i)}x_2 + \dots + \alpha_m^{(i)}x_m$  of  $x_1 + \alpha_2x_2 + \dots + \alpha_mx_m$  over  $L$  form a  $\Delta$ -connected system. Further, by virtue of the assumption  $[L(\alpha_1): L] \dots [L(\alpha_m): L] = n$ , the only solution of the system of equations

$$x_1 + \alpha_2^{(i)}x_2 + \dots + \alpha_m^{(i)}x_m = 0, \quad i = 1, \dots, n,$$

in  $L$  is  $x_1 = \dots = x_m = 0$ . So our theorem implies the required assertion.

*Proof of Corollary 5.* Let  $L(\mathbf{x}) = \alpha_1 x_1 + \dots + \alpha_m x_m$  and let  $L^{(1)}(\mathbf{x}), \dots, L^{(n)}(\mathbf{x})$  be the conjugates of  $L(\mathbf{x})$  over  $L$ . Put

$$l_{ij}(\mathbf{x}) = L^{(i)}(\mathbf{x}) - L^{(j)}(\mathbf{x}).$$

In proving Theorem 4 in [7] we showed that

$$F(\mathbf{x}) = \text{Discr}_{K/L}(\alpha_1 x_1 + \dots + \alpha_m x_m) = (-1)^{n(n-1)/2} \prod_{\substack{i, j=1 \\ i \neq j}}^n l_{ij}(\mathbf{x})$$

satisfies all conditions made in our theorem. Thus (4) and (5) clearly follow.

*Proof of Corollary 6.* If  $X_8$  is sufficiently large and  $|\overline{\mathbf{x}}| \cong X_8$ , by Corollary 5 and (11) we have  $\mathcal{P}(D(\mathbf{x})) = \mathcal{P}(I(\mathbf{x}))$ , where  $D(\mathbf{x}) = \text{Discr}_{K/L}(\alpha_1 x_1 + \dots + \alpha_{n-1} x_{n-1})$ . Thus Corollary 5 proves the assertion of Corollary 6.

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*Added in proof.* The results of this paper were presented with detailed proofs in my course given at the University of Paris VI, March—June 1979.

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