ON THE GROUP OF DIFFEOMORPHISMS PRESERVING A LOCALLY CONFORMAL SYMPLECTIC STRUCTURE

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Abstract. The automorphism group of a locally conformal symplectic structure is studied. It is shown that this group possesses essential features of the symplectomorphism group. By using a special type of cohomology the flux and Calabi homomorphisms are introduced. The main theorem states that the kernels of these homomorphisms are simple groups, for the precise statement see chapter 7. Some of the methods used, may also be interesting in the symplectic case.

Introduction

The concept of locally conformal symplectic (l.c.s. for short) structure is an interesting generalization of the symplectic geometry. Its importance comes from the fact that the l.c.s. manifolds can serve as natural phase spaces of Hamiltonian dynamical systems. One geometric motivation for a study of l.c.s. structures consists in their relations with the contact structures, cf. [19]. Even more important is the fact that each even dimensional leaf of a Jacobi manifold possesses a l.c.s. structure, cf. [7].

In this paper we study the automorphism group of a l.c.s. structure. Our investigations are motivated by a well known paper of A. Banyaga [1] on the symplectomorphism group. We show that surprisingly many properties of the symplectomorphism group have still their analogues on the ground of the l.c.s. structures. Let us add that some basic facts concerning automorphism groups of almost symplectic and l.c.s. structures were stated by J. Lefebvre in [11, 12].

First section is devoted to the d^ω-cohomology, the concept which has been initiated by A. Lichnerowicz. As it was observed in [7] the d^ω-cohomology rather than the ordinary de Rham cohomology is a proper tool in the l.c.s. geometry. We prove the finite dimensionality of the d^ω-cohomology groups and other results probably not explicitly mentioned in the literature.

In the second section we recall some facts on diffeomorphism groups and fix the notation. Basic properties of a l.c.s. manifold and its automorphism group are presented in the third section. We consider the k-transivity of this group and indicate only that its algebraic structure determines the geometric structure itself.

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In the next three sections we study invariants of the automorphism group of a l.c.s. manifold, namely the Lee extended homomorphism and its integral counterpart, the flux homomorphism and the Calabi homomorphism. By making use of the d^ω-cohomology we establish basic properties of these invariants. We observe also, by appealing to a difficult theory of ILH-Lie groups of H. Omori [15], the countability property of the first homotopy groups of the kernels of these homomorphisms. This enables us to give a characterization of Hamiltonian isotopies on l.c.s. manifolds.

In the seventh section we prove a simplicity theorem which is a consequence of that for the symplectomorphism group. In the final section we introduce some invariants of the equivalence class of a l.c.s. structure. These invariants are defined in the cohomology of the automorphism group and its Lie algebra.

Throughout we will assume that all objects as manifolds, tensors, diffeomorphisms etc. are of class C^∞. We do not know whether possible analogues of some of the presented results are known in C^r or C^ω categories even for the symplectomorphism group.

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1. d^ω-Cohomology

Let ω be a closed 1-form on a manifold M and define

$$d^\omega : \Omega^*(M) \to \Omega^{*+1}(M) \qquad d^\omega(\alpha) := d\alpha + \omega \wedge \alpha$$

Obviously we have $d^\omega \circ d^\omega = 0$ and we may define the d^ω-cohomology $H^*_{d^\omega}(M)$. Similarly we define d^ω_c-cohomology with compact supports $H^*_{d^\omega_c}(M)$. Suppose $[\omega'] = [\omega] \in H^1(M)$ and choose $a \in \Omega^0(M)$ with $\omega' = \omega + \frac{da}{a} = \omega + d(\ln |a|)$. Then there are isomorphisms $\frac{1}{a} : H^*_{d^\omega} \cong H^*_{d^{\omega'}}(M)$ and $\frac{1}{a} : H^*_{d^\omega_c} \cong H^*_{d^{\omega'}_c}(M)$ given by multiplication with $\frac{1}{a}$. So for an exact ω the d^ω-cohomology is isomorphic to the ordinary de Rham cohomology.

For closed 1-forms ω_1, ω_2 an easy calculation shows

$$d^{\omega_1+\omega_2}(\sigma \wedge \tau) = d^{\omega_1}\sigma \wedge \tau + (-1)^{|\sigma|}\sigma \wedge d^{\omega_2}\tau.$$

Hence the wedge product induces a bilinear mapping

$$\wedge : H^k_{d^{\omega_1}}(M) \times H^l_{d^{\omega_2}}(M) \to H^{k+l}_{d^{\omega_1+\omega_2}}(M)$$

and likewise for the cohomology with compact supports.

For a smooth $g : M \to N$ we have an induced mapping $g^* : H^*_{d^\omega}(N) \to H^*_{d^{g^*\omega}}(M)$. If g is proper then we also have an induced mapping $g^* : H^*_{d^\omega_c}(N) \to H^*_{d^{g^*\omega}_c}(M)$.

1.1. Lemma. *Let ω be a closed 1-form on N and let $g : M \times I \to N$ be a smooth homotopy. Define $a \in C^\infty(M \times I, \mathbb{R})$ by $a_t := \exp\left(\int_0^t \mathrm{inc}_s^* i_{\partial_t} g^*\omega ds\right)$ where $\mathrm{inc}_s : M \to M \times I$, $\mathrm{inc}_s(x) := (x, s)$. Then*

$$a_1 g_1^* = a_0 g_0^* : H^*_{d^\omega}(N) \to H^*_{d^{g_0^*\omega}}(M)$$

If g is proper the same holds with compact supports.

Proof. Notice that the definition of a is such that $g_t^*\omega = g_0^*\omega + d(\ln|a_t|)$. One defines a mapping $H : \Omega^*(N) \to \Omega^{*-1}(M)$ by $H(\sigma) := \int_0^1 a_t \operatorname{inc}_t^* i_{\partial_t} g^*\sigma dt$ and checks that it is a chain homotopy, i.e.

$$d^{g_0^*\omega} H(\sigma) + H(d^\omega\sigma) = a_1 g_1^*\sigma - a_0 g_0^*\sigma$$

The calculation is straightforward and uses the fact $\frac{\partial}{\partial_t} a_t = a_t \operatorname{inc}_t^* i_{\partial_t} g^*\omega$. $\square$

Suppose M is the union of two open subsets U, V. Then the following is a short exact sequence of cochain complexes

$$0 \to \big(\Omega^*(M), d^\omega\big) \xrightarrow{\alpha} \big(\Omega^*(U) \oplus \Omega^*(V), d^{\omega|_U} \oplus d^{\omega|_V}\big) \xrightarrow{\beta} \big(\Omega^*(U \cap V), d^{\omega|_{U\cap V}}\big) \to 0$$

where $\alpha(\sigma) = (\sigma|_U, \sigma|_V)$ and $\beta(\sigma,\tau) = \sigma|_{U\cap V} - \tau|_{U\cap V}$. So we obtain the following Mayer-Vietoris sequence:

1.2. Lemma. *Let M be the union of two open subsets U and V. Then there exists a long exact sequence*

$$\cdots \to H^k_{d^\omega}(M) \xrightarrow{\alpha_*} H^k_{d^{\omega|_U}}(U) \oplus H^k_{d^{\omega|_V}}(V) \xrightarrow{\beta_*} H^k_{d^{\omega|_{U\cap V}}}(U\cap V) \xrightarrow{\delta} H^{k+1}_{d^\omega}(M) \to \cdots$$

and $\delta([\sigma]) = [d\lambda_V \wedge \sigma] = -[d\lambda_U \wedge \sigma]$, where $\{\lambda_U, \lambda_V\}$ is a partition of unity subordinate to $\{U,V\}$ and the forms under consideration are assumed to be extended by 0 to the whole M.

Similarly there is an exact sequence of cochain complexes

$$0 \to \big(\Omega_c^*(U\cap V), d_c^{\omega|_{U\cap V}}\big) \xrightarrow{\beta} \big(\Omega_c^*(U) \oplus \Omega_c^*(V), d_c^{\omega|_U} \oplus d_c^{\omega|_V}\big) \xrightarrow{\alpha} \big(\Omega_c^*(M), d_c^\omega\big) \to 0$$

where $\beta(\sigma) = (\sigma, -\sigma)$ and $\alpha(\sigma,\tau) = \sigma + \tau$ and everything is assumed to be extended by 0. So we also get a Mayer-Vietoris sequence with compact supports:

1.3. Lemma. *If M is the union of two open subsets U and V then there exists a long exact sequence*

$$\cdots \to H^{k-1}_{d_c^\omega}(M) \xrightarrow{\delta} H^k_{d_c^{\omega|_{U\cap V}}}(U\cap V) \xrightarrow{\beta_*} H^k_{d_c^{\omega|_U}}(U) \oplus H^k_{d_c^{\omega|_V}}(V) \xrightarrow{\alpha_*} H^k_{d_c^\omega}(M) \to \cdots$$

where $\delta[\sigma] = [d\lambda_U \wedge \sigma|_{U\cap V}] = -[d\lambda_V \wedge \sigma|_{U\cap V}]$ and $\{\lambda_U, \lambda_V\}$ is a partition of unity subordinate to $\{U,V\}$.

A covering $\mathcal{U}$ of a manifold M is called good if for all $m \in \mathbb{N}$ and $U_1, \dots, U_m \in \mathcal{U}$ the intersection $U_1 \cap \cdots \cap U_m$ is either empty or contractible. Using a Riemannian metric and geodesically convex open sets one easily sees that every manifold admits a good covering and these coverings are cofinal in the set of all coverings.

Using the Mayer-Vietoris sequence inductively and the fact that for contractible sets the d^ω-cohomology is isomorphic to the de Rham cohomology, and hence finite dimensional, we immediately obtain

1.4. Corollary. *Suppose M admits a finite good covering. Then $H^*_{d^\omega}(M)$ and $H^*_{d^\omega_c}(M)$ are finite dimensional. Especially this is true for compact manifolds.*

For an oriented manifold of dimension n we may define a pairing by

$$\langle\cdot,\cdot\rangle_\omega : H^*_{d-\omega}(M)\times H^{n-*}_{d^\omega_c}(M)\xrightarrow{\wedge} H^n_c(M)\xrightarrow{\int_M}\mathbb{R}$$

If $\omega' = \omega + \frac{da}{a} = \omega + d(\ln|a|)$ then $-\omega' = -\omega + d(\ln|\frac{1}{a}|)$ so $\frac{1}{a} : H^*_{d^\omega_c}(M) \cong H^*_{d^{\omega'}_c}(M)$, $a : H^*_{d-\omega}(M) \cong H^*_{d-\omega'}(M)$ and $\langle a[\sigma], \frac{1}{a}[\tau]\rangle_{\omega'} = \langle[\sigma],[\tau]\rangle_\omega$. Hence if ω is exact this pairing is non-degenerate by the Poincaré duality.

1.5. Proposition. *On an oriented manifold of dimension n the mappings defined by*

$$D^k_\omega : H^k_{d-\omega}(M)\to H^{n-k}_{d^\omega_c}(M)^* \qquad D^k_\omega([\sigma])([\tau]) := \langle[\sigma],[\tau]\rangle_\omega$$

are isomorphisms.

Proof. If M is a disjoint union of open balls then we have

$$H^k_{d^\omega_c}(\bigsqcup U_i)^* \cong \big(\bigoplus H^k_{d^\omega_c}(U_i)\big)^* \cong \prod H^k_{d^\omega_c}(U_i)^*$$

and via this isomorphism D^k_ω corresponds to $\prod D^k_{\omega|_{U_i}}$ and is therefore an isomorphism. Using the explicit description of the connecting homomorphisms δ in Lemma 1.2 and Lemma 1.3 one easily checks that the following diagram commutes up to sign:

$$\begin{array}{ccccccc}
H^k_{d-\omega}(M) & \xrightarrow{\alpha_*} & H^k_{d-\omega}(U)\oplus H^k_{d-\omega}(V) & \xrightarrow{\beta_*} & H^k_{d-\omega}(U\cap V) & \xrightarrow{\delta} & H^{k+1}_{d-\omega}(M)\\
\downarrow D^k_\omega & & \downarrow D^k_\omega\oplus D^k_\omega & & \downarrow D^k_\omega & & \downarrow D^{k+1}_\omega\\
H^{n-k}_{d^\omega_c}(M)^* & \xrightarrow{(\alpha_*)^*} & H^{n-k}_{d^\omega_c}(U)^*\oplus H^{n-k}_{d^\omega_c}(V)^* & \xrightarrow{(\beta_*)^*} & H^{n-k}_{d^\omega_c}(U\cap V)^* & \xrightarrow{\delta^*} & H^{n-k-1}_{d^\omega_c}(M)^*
\end{array}$$

So if Poincaré duality holds for U, V and $U\cap V$ it also holds for $U\cup V$ by the five lemma. Finally one chooses a good covering $\mathcal{U}$ such that every $U\in\mathcal{U}$ does only intersect finitely many other sets of $\mathcal{U}$. Then we can write $M = W_1\cup\cdots\cup W_n$ where every W_i is a disjoint union of open balls in $\mathcal{U}$. Since Poincaré duality holds for W_i, W_j and $W_i\cap W_j$ (the latter is also an disjoint union of open balls) it holds also for $W_i\cup W_j$. Proceeding inductively one completes the proof. □

1.6. Example. Let $[f]\in H^0_{d^\omega}(M)$, i.e. $f\in C^\infty(M,\mathbb{R})$ and $d^\omega f = 0$. Consider the set $Z := \{x\in M : f(x) = 0\}$. It is of course closed. We show that it is open too. Let $x\in Z$ and choose a contractible neighborhood U of x. Then $\omega|_U = d(\ln|a|)$ for some nowhere vanishing function a on U and $\frac{1}{a} : H^*_{d^{\omega|_U}}(U)\cong H^*(U)$. So $\frac{1}{a}f|_U$ is a constant function on U and since it vanishes in x we obtain f vanishes on U, that is $U\subseteq Z$. For connected M this shows that $H^0_{d^\omega}(M)$ and similarly $H^0_{d^\omega_c}(M)$ is at most 1-dimensional.

Let M be connected and oriented. Then $i^* : H^0_{d^\omega}(M)\to H^0_{d^{\omega|_U}}(U)$ is injective and by Poincaré duality $i_* : H^n_{d_c^{\omega|_U}}(U)\to H^n_{d^\omega_c}(M)$ is onto. So generators of $H^n_{d^\omega_c}(M)$ can be chosen to have arbitrarily small supports.

Let $M = S^1$ and $\omega = \lambda d\theta$, where $0\neq\lambda\in\mathbb{R}$, be a generator of its first de Rham cohomology. We claim that $H^0_{d^\omega}(S^1) = 0$. So let $f\in\Omega^0(S^1)$ be d^ω-closed.

We consider f as periodic function on $\mathbb{R}$ then $\omega = \lambda dx$. The condition $d^\omega f = 0$ translates to $f' + \lambda f = 0$, but this has no non-trivial periodic solution, hence $f = 0$. So $H^0_{d^\omega}(S^1) = 0$ for every non-exact ω.

Let M be connected and ω a closed 1-form which is not exact. Then there exists a mapping $i : S^1 \to M$ such that $i^*\omega$ is not exact. Now let $f \in \Omega^0(M)$. By the previous paragraph $i^* f = 0$ and hence by connectedness $f = 0$. So we have shown $H^0_{d^\omega}(M) = 0$ and similarly $H^0_{d^\omega_c}(M) = 0$ for every connected M and any non-exact ω. Using Poincaré duality we also obtain $H^n_{d^\omega}(M) = 0$ and $H^n_{d^\omega_c}(M) = 0$ for every oriented, connected, n-dimensional M and every non-exact ω. Using the orientation covering one sees that the assumption of the orientability is superfluous. A different proof of $H^n_{d^\omega}(M) = 0$ can be found in [7].

1.7. Example. Consider $M = \mathbb{R}^2 \setminus \{(-1,0),(1,0)\}$ and let ω resp. η be a generator of $H^1(M)$ supported in $(-\infty,0)\times\mathbb{R}$ resp. $U := (0,\infty)\times\mathbb{R}$. Then obviously $d^\omega\eta = 0$ and $\eta|_U$ cannot be $d^{\omega|_U} = d$-exact. Using Mayer-Vietoris sequence one can show that η generates $H^1_{d^\omega}(M)$.

Suppose we have two manifolds M_1, M_2 and two closed 1-forms ω_1 resp. ω_2 on M_1 resp. M_2. Let $\omega := \mathrm{pr}_1^*\,\omega_1 + \mathrm{pr}_2^*\,\omega_2 \in \Omega^1(M_1\times M_2)$. Then one defines a mapping

$$\Psi : \Omega^k(M_1)\times\Omega^l(M_2)\to\Omega^{k+l}(M_1\times M_2) \qquad (\alpha,\beta)\mapsto \mathrm{pr}_1^*\,\alpha\wedge\mathrm{pr}_2^*\,\beta.$$

It is visible that $d^\omega(\Psi(\alpha,\beta)) = \Psi(d^{\omega_1}\alpha,\beta) + (-1)^{|\alpha|}\Psi(\alpha,d^{\omega_2}\beta)$ and hence we have an induced mapping

$$H^*_{d^{\omega_1}}(M_1)\otimes H^*_{d^{\omega_2}}(M_2)\to H^*_{d^\omega}(M_1\times M_2).$$

As in the ordinary de Rham cohomology one proves, under the assumption that one of the two manifolds has finite dimensional cohomology, that Ψ is an isomorphism. Using this and Example 1.7 one obtains manifolds with arbitrarily complicated d^ω-cohomology and non-exact ω.

Consider $\Omega^*_c(M) = \varinjlim_K \Omega^*_K(M)$ with the inductive limit topology. This is a strict inductive limit of Fréchet spaces (cf. [9] or [15]) and hence a complete separated locally convex vector space. We provide $\operatorname{im} d^\omega_c \subseteq \ker d^\omega_c \subseteq \Omega^*_c(M)$ with the initial topologies and endow $H^*_{d^\omega_c}(M)$ with the quotient topology.

1.8. Theorem. *Let ω be a closed 1-form on a manifold M. Then $H^*_{d^\omega_c}(M)$ is a strict inductive limit of separated finite dimensional topological vector spaces and hence a complete separated locally convex vector space.*

Proof. First we assume that M is oriented. $d^\omega_c : \Omega^*_c(M)\to\Omega^{*+1}_c(M)$ is continuous and hence $\ker d^\omega_c \subseteq \Omega^*_c(M)$ is closed. By Poincaré duality $\sigma\in\ker d^\omega_c$ is contained in $\operatorname{im} d^\omega$ if and only if

$$\int_M \tau\wedge\sigma = 0 \qquad \forall\tau\in\ker\left(d^{-\omega} : \Omega^*(M)\to\Omega^{*+1}(M)\right)$$

but these are continuous conditions and so $\operatorname{im} d^\omega_c \subseteq \ker d^\omega_c$ is closed.

Let $d^\omega_K := d^\omega|_{\Omega^*_K(M)} : \Omega^*_K(M)\to\Omega^{*+1}_K(M)$. It is a general fact that if $E = \varinjlim E_n$ is a strict inductive limit and $F\subseteq E$ is a (not necessarily closed) subspace then $F = \varinjlim(E_n\cap F)$ as strict inductive limit. Applying this twice we obtain

$$\varinjlim_K \ker d^\omega_K = \ker d^\omega_c \quad\text{and}\quad \varinjlim_K\left(\Omega^*_K(M)\cap\operatorname{im} d^\omega_c\right) = \operatorname{im} d^\omega_c.$$

Since $\ker d_K^\omega$ is a Fréchet space and $\operatorname{im} d_c^\omega \subseteq \Omega_c^*(M)$ is closed, $\frac{\ker d_K^\omega}{\Omega_K^*(M)\cap \operatorname{im} d_c^\omega}$ is separated. We claim that it is finite dimensional for nice K.

So assume that K is a $\dim M$-dimensional submanifold with boundary. Let $i : \partial K \hookrightarrow K$ be the inclusion. We let $\Omega^*(K, \partial K) := \{\alpha \in \Omega^*(K) : i^*\alpha = 0\}$ and denote by $H^*_{d^{\omega|_K}}(K, \partial K)$ the corresponding cohomology, i.e. the relative cohomology. As usual we have a long exact sequence

$$\cdots \to H^*_{d^{\omega|_K}}(K, \partial K) \to H^*_{d^{\omega|_K}}(K) \xrightarrow{i^*} H^*_{d^{i^*\omega}}(\partial K) \xrightarrow{\delta} H^{*+1}_{d^{\omega|_K}}(K, \partial K) \to \cdots$$

and so $H^*_{d^{\omega|_K}}(K, \partial K)$ is finite dimensional by Corollary 1.4. We have a mapping $\Omega_K^*(M) \to \Omega^*(K, \partial K)$ and we claim that the induced mapping $H^*_{d_K^\omega}(M) \to H^*_{d^{\omega|_K}}(K, \partial K)$ is injective. To see this let $\alpha \in \Omega_K^*(M)$ be d^ω-closed and such that $\alpha|_K = d^{\omega|_K}\beta$ for some $\beta \in \Omega^*(K, \partial K)$. Next choose a smooth homotopy $g : K \times I \to K$ with $g_0 = \mathrm{id}_K$, $g_t(\partial K) \subseteq \partial K$ and such that there exists an open neighborhood U of ∂K with $g_1(U) \subseteq \partial K$. From Lemma 1.1 we get

$$d^{\omega|_K}\big(\textstyle\int_0^1 a_t \operatorname{inc}_t^* i_{\partial_t} g^*\alpha dt\big) = a_1 g_1^*(\alpha|_K) - a_0 g_0^*(\alpha|_K) = d^{\omega|_K}(a_1 g_1^*\beta) - \alpha|_K$$

By the choice of g we see that $g_1^*\beta$ is zero on U an hence can be extended by 0 to the whole of M. Moreover one sees that $\operatorname{inc}_t^* i_{\partial_t} g^*(\alpha|_K)$ is flat along ∂K and so the integral in the equation above can also be extended to M by 0. But this shows that $[\alpha] = 0 \in H_K^*(M)$. Since we have an injective mapping from $H_K^*(M)$ into the finite dimensional vector space $H^*(K, \partial K)$ the space $H_K^*(M)$ has to be finite dimensional and hence so is $\frac{\ker d_K^\omega}{\Omega_K^*(M)\cap \operatorname{im} d_c^\omega} \subseteq \frac{\ker d_K^\omega}{\operatorname{im} d_K^\omega} = H_K^*(M)$.

Since the inductive limit can be computed via these nice K we obtain

$$H^*_{d_c^\omega}(M) = \ker d_c^\omega / \operatorname{im} d_c^\omega = \frac{\varinjlim_K \ker d_K^\omega}{\varinjlim_K (\Omega_K^*(M)\cap \operatorname{im} d_c^\omega)} = \varinjlim_K \frac{\ker d_K^\omega}{\Omega_K^*(M)\cap \operatorname{im} d_c^\omega}$$

as strict inductive limit and the steps $\frac{\ker d_K^\omega}{\Omega_K^*(M)\cap \operatorname{im} d_c^\omega}$ are separated finite dimensional topological vector spaces.

If M is non-orientable let $\pi : \tilde M \to M$ denote the orientation covering and let $f : \tilde M \to \tilde M$ be the unique non-trivial deck transformation. Then

$$\Omega_c^*(\tilde M) = \Omega_c^{*,\mathrm{even}}(\tilde M) \oplus \Omega_c^{*,\mathrm{odd}}(\tilde M) =: \{\sigma : f^*\sigma = \sigma\} \oplus \{\sigma : f^*\sigma = -\sigma\}$$

It is easily seen that $\pi^* : \Omega_c^*(M) \cong \Omega_c^{*,\mathrm{even}}(\tilde M)$ and hence $H^*_{d_c^\omega}(M) \cong H^{*,\mathrm{even}}_{d_c^{\pi^*\omega}}(\tilde M)$ which is a closed subspace of $H^*_{d_c^{\pi^*\omega}}(\tilde M)$. Since the latter is a strict inductive limit of separated finite dimensional topological vector spaces, so is $H^*_{d_c^\omega}(M)$. □

For every manifold N and every complete locally convex vector space E we define $C_c^\infty(N, E) = \varinjlim_K C_K^\infty(N, E)$. The following is a slight generalization of an argument due to A. Banyaga, see [1] and [3].

1.9. Corollary. *Let N, M be manifolds and ω be a closed 1-form on M. Then every $f \in C_c^\infty(N, \operatorname{im} d_c^\omega)$ can be lifted, i.e. there exists $\tilde f \in C_c^\infty(N, \Omega_c^*(M))$ with $d_c^\omega \circ \tilde f = f$.*

Proof. Since $d_c^\omega : \Omega_c^*(M) \to \operatorname{im} d_c^\omega$ is onto and $\operatorname{im} d_c^\omega$ is complete the mapping

$$d_c^\omega \widehat{\otimes}_\pi \mathrm{id}_{C_c^\infty(N,\mathbb{R})} : \Omega_c^*(M) \widehat{\otimes}_\pi C_c^\infty(N, \mathbb{R}) \twoheadrightarrow \operatorname{im} d_c^\omega \widehat{\otimes}_\pi C_c^\infty(N, \mathbb{R})$$

is surjective. Since $C_c^\infty(N,\mathbb{R})$ is nuclear we obtain

$$\Omega_c^*(M)\widehat{\otimes}_\pi C_c^\infty(N,\mathbb{R}) \cong \Omega_c^*(M)\widehat{\otimes}_\varepsilon C_c^\infty(N,\mathbb{R}) \cong C_c^\infty(N,\Omega_c^*(M))$$

and

$$\operatorname{im} d_c^\omega \widehat{\otimes}_\pi C_c^\infty(N,\mathbb{R}) \cong \operatorname{im} d_c^\omega \widehat{\otimes}_\varepsilon C_c^\infty(N,\mathbb{R}) \cong C_c^\infty(N,\operatorname{im} d_c^\omega)$$

Via these isomorphisms $d_c^\omega \widehat{\otimes}_\pi \operatorname{id}_{C_c^\infty(N,\mathbb{R})}$ corresponds to $(d_c^\omega)_*$ and hence the latter is surjective too. See [8] for the functional analysis involved. $\square$

1.10. Remark. $U \mapsto \mathcal{F}(U) := \{f \in C^\infty(U,\mathbb{R}) : d^{\omega|_U} f = 0\}$ is a locally constant sheaf and

$$0 \to \mathcal{F} \to \Omega^0(\cdot) \xrightarrow{d^\omega} \Omega^1(\cdot) \xrightarrow{d^\omega} \Omega^2(\cdot) \to \cdots$$

is a fine resolution. So $H^*_{d^\omega}(M) \cong H^*(M;\mathcal{F})$, where the latter denotes the cohomology of M with values in the sheaf $\mathcal{F}$, see [5].

2. The Lie Group $\operatorname{Diff}_c^\infty(M)$

It is well known (see [9]) that $\operatorname{Diff}_c^\infty(M) = \varinjlim_K \operatorname{Diff}_K^\infty(M)$ is a Lie group modeled on the complete locally convex vector space $\mathfrak{X}_c(M) = \varinjlim_K \mathfrak{X}_K(M)$. Let $\operatorname{Diff}_c^\infty(M)_\circ$ denote its connected component containing id. Its (kinematic) tangent space at id is $\mathfrak{X}_c(M)$ but the Lie bracket is the negative of the usual Lie bracket of vector fields. The adjoint action of $g \in \operatorname{Diff}_c^\infty(M)$ on $\mathfrak{X}_c(M)$ is given by $(g^{-1})^*$.

For any $X \in C^\infty(\mathbb{R},\mathfrak{X}_c(M))$ there is $g : \mathbb{R} \to \operatorname{Diff}_c^\infty(M)$ such that $\dot g = X$ where

$$\dot g : \mathbb{R} \to \mathfrak{X}_c(M) \qquad \dot g(t)(x) := \tfrac{\partial}{\partial s}|_t (g_s(g_t^{-1}(x))).$$

$\dot g$ is the time-dependent vector field defining uniquely g with $g(0) = id$. Thus there exists a bijective evolution map

$$\operatorname{Evol} : C^\infty\big(\mathbb{R},\mathfrak{X}_c(M)\big) \to C^\infty\big((\mathbb{R},0),(\operatorname{Diff}_c^\infty(M),\operatorname{id})\big)$$

and so $\operatorname{Diff}_c^\infty(M)$ is regular, cf. [9].

Specifically it admits the exponential mapping, namely

$$\exp : \mathfrak{X}_c(M) \to \operatorname{Diff}_c^\infty(M)_\circ \qquad X \mapsto \operatorname{Fl}_1^X,$$

the flow of the vector field at time 1. It is well known that exp is not locally surjective around 0.

Let N be a manifold (possibly infinite dimensional) and $\mathfrak{g}$ a Lie algebra. Then $\Omega^*(N;\mathfrak{g})$, the space of $\mathfrak{g}$-valued forms on N, is a graded Lie algebra with bracket

$$[\Psi,\Theta](X_1,\dots,X_{p+q}) = \frac{1}{p!q!}\sum_{\sigma\in\mathfrak{S}(p+q)} \operatorname{sgn}(\sigma)[\Psi(X_{\sigma(1)},\dots),\Theta(X_{\sigma(p+1)},\dots)]$$

where $\Psi \in \Omega^p(N;\mathfrak{g})$ and $\Theta \in \Omega^q(N;\mathfrak{g})$. For example we have the right Maurer-Cartan form $\kappa^r \in \Omega^1\big(\operatorname{Diff}_c^\infty(M);\mathfrak{X}_c(M)\big)$ given by $\kappa^r_g := T_g(\mu_{g^{-1}})$, where μ_x is the right multiplication by x.

For a smooth mapping $g : N \to \mathrm{Diff}_c^\infty(M)$ let $\delta^r g \in \Omega^1(N; \mathfrak{X}_c(M))$ denote the right logarithmic derivative, i.e. $\delta^r g := g^*\kappa^r$. It satisfies the Maurer-Cartan equation

$$d\delta^r g - \tfrac{1}{2}[\delta^r g, \delta^r g] = 0 \tag{2.1}$$

If N is simply connected then the converse also holds. Given $\alpha \in \Omega^1(N; \mathfrak{X}_c(M))$ satisfying the Maurer-Cartan equation then there exists $g : N \to \mathrm{Diff}_c^\infty(M)$ with $\delta^r g = \alpha$.

For $f, g : N \to \mathrm{Diff}_c^\infty(M)$ then the following Leibniz rule holds:

$$\delta^r(fg)(t) = \delta^r f(t) + \mathrm{Ad}(f(t)) \cdot \delta^r g(t) = \delta^r f(t) + (f(t)^{-1})^*(\delta^r g(t))$$

In case $N = \mathbb{R}$ the right logarithmic derivative is simply $\delta^r g = \dot{g}dt$, and it is the inverse of Evol.

If $\Psi = Xds + Ydt \in \Omega^1(\mathbb{R}^2; \mathfrak{X}_c(M))$, where $X, Y \in C^\infty(\mathbb{R}^2, \mathfrak{X}_c(M))$ then the Maurer-Cartan equation takes the form

$$\tfrac{\partial}{\partial t}X(s,t) - \tfrac{\partial}{\partial s}Y(s,t) = [X(s,t), Y(s,t)]$$

Another formula that will be in use is the following: for $\omega \in C^\infty(N, \Omega^k(M))$, $g \in C^\infty(N, \mathrm{Diff}_c^\infty(M))$ and $X_x \in T_xN$ we have

$$X_x \cdot \big(g(x)^*\omega(x)\big) = g(x)^*\big(L_{\delta^r g(X_x)}\omega(x) + (X_x \cdot \omega)(x)\big)$$

A well known special case is $\frac{\partial}{\partial t}(g_t^*\omega_t) = g_t^*\big(L_{\dot{g}_t}\omega_t + \frac{\partial}{\partial t}\omega_t\big)$ for $g \in C^\infty\big(\mathbb{R}, \mathrm{Diff}_c^\infty(M)\big)$ and $\omega \in C^\infty\big(\mathbb{R}, \Omega^p(M)\big)$. (See [9].)

Let G be a subgroup of $\mathrm{Diff}_c^\infty(M)$. Then we denote by $G_\circ$ the group of all elements in G that can be joined with the identity by a smooth path in G. We call G connected by smooth arcs iff $G = G_\circ$. In this case we denote by $\widetilde{G}$ the group of all smooth paths in G starting at the identity modulo smooth homotopies relative endpoints, with the pointwise multiplication. Next $\mathrm{ev}_1 = \pi : \widetilde{G} \to G$ stands for the canonical projection and $\pi_1(G) := \ker \pi \subseteq \widetilde{G}$ is the first homotopy group of G.

3. Locally Conformal Symplectic Manifolds

An almost symplectic manifold (M, Ω) is called *locally conformal symplectic* (l.c.s.) if there exists an open covering $\{U_i\}_{i \in I}$ and a family of positive functions $\alpha_i \in C^\infty(U_i, \mathbb{R})$ such that $d(\alpha_i\Omega) = 0$ on U_i. It was first observed by H.C.Lee in [10] that then $d \ln \alpha_i$ glue up to a closed 1-form ω, provided $\dim(M) > 2$. So equivalently, and this will be our working definition, a l.c.s. manifold is a triple (M, Ω, ω) where ω is a closed 1-form and Ω is a non-degenerate 2-form satisfying $d^\omega\Omega = d\Omega + \omega \wedge \Omega = 0$. Throughout we will assume that M is connected $2n$-dimensional manifold (unless otherwise stated). Since Ω is non-degenerate we get a canonical vector bundle isomorphism $\flat : TM \cong T^*M$ given by $X \mapsto i_X\Omega$. By $\sharp$ we denote the inverse of $\flat$.

If $\dim M > 2$ then ω is uniquely determined by Ω. Otherwise there would exist $\omega' \neq 0$ with $\omega' \wedge \Omega = 0$. Let $x \in M$ with $\omega'(x) \neq 0$. Then $\Omega(x) = \omega'(x) \wedge \eta$ for some $\eta \in \bigwedge^1 T_x^*M$. This would yield $\Omega^2(x) = 0$, a contradiction.

If (M, Ω, ω) is a l.c.s. manifold and a is a nowhere vanishing function on M then $(M, \frac{1}{a}\Omega, \omega + \frac{da}{a})$ is again a l.c.s. manifold. Two l.c.s. manifolds (M, Ω, ω) and

(M,Ω',ω') are called conformally equivalent (denoted $(M,\Omega,\omega) \sim (M,\Omega',\omega')$) iff there exists a nowhere vanishing function a on M with $\Omega' = \frac{1}{a}\Omega$ and $\omega' = \omega + \frac{da}{a} = \omega + d(\ln|a|)$. So (M,Ω,ω) is *globally conformal symplectic* (g.c.s.) manifold iff $[\omega] = 0 \in H^1(M)$.

A submanifold $i : L \hookrightarrow M$ is called Lagrangian iff $\dim L = n$ and $i^*\Omega = 0$. Notice that the Lagrangian submanifolds remain the same if we change (M,Ω,ω) conformally.

3.1. Example. Let N be an n-dimensional manifold and let ω be a closed 1 form on N. Let Θ denote the canonical 1-form on T^*N. Recall that for $\alpha \in \Omega^1(N)$ considered as mapping $\alpha : N \to T^*N$ one has $\alpha^*\Theta = \alpha$. Define $\omega' := \pi^*\omega$, $\Omega' := d^{\omega'}\Theta$, where $\pi : T^*N \to N$ is the canonical projection. Then (T^*N,Ω',ω') is a l.c.s. manifold and for $\alpha \in \Omega^1(N)$ we have $\alpha^*\Omega' = d^\omega\alpha$. So $\mathrm{im}(\alpha)$ is a Lagrangian submanifold of (T^*N,Ω',ω') if and only if $d^\omega\alpha = 0$. (T^*N,Ω',ω') is conformally equivalent to a symplectic manifold if and only if $[\omega] = 0 \in H^1(N)$.

3.2. Example. On S^3 there exists a global frame of 1-forms $\alpha,\beta,\gamma \in \Omega^1(S^3)$ satisfying $d\alpha = \beta\wedge\gamma$, $d\beta = \gamma\wedge\alpha$, $d\gamma = \alpha\wedge\beta$. This is because S^3 is a Lie group with Lie algebra $\mathfrak{so}(3,\mathbb{R})$ and the latter has a basis $\{A,B,C\}$ satisfying $[A,B] = C$, $[B,C] = A$, $[C,A] = B$. Let $\omega := dt \in \Omega^1(S^1)$ and $\Omega := d^\omega\alpha \in \Omega^2(S^1\times S^3)$. We have $\Omega^2 = 2dt\wedge\alpha\wedge\beta\wedge\gamma$, so Ω is non-degenerated and $(S^1\times S^3,\Omega,\omega)$ is a l.c.s. manifold.

However $S^1\times S^3$ does not admit a symplectic structure, for this would be exact since $H^2(S^1\times S^3) = 0$ and hence would give rise to an exact volume form on $S^1\times S^3$, a contradiction since $S^1\times S^3$ is compact.

3.3. Example. A Jacobi structure ([7]) on a manifold M is given by a Lie bracket $\{\cdot,\cdot\}$ on $C^\infty(M,\mathbb{R})$ which is local in the sense that

$$\mathrm{supp}(\{u,v\}) \subseteq \mathrm{supp}(u)\cap\mathrm{supp}(v) \qquad \forall u,v \in C^\infty(M,\mathbb{R}).$$

One can show that such brackets are in a one-to-one correspondence with pairs (Λ,E), where Λ is a skew symmetric bivector field and E is an ordinary vector field on M satisfying the following relations

$$[\Lambda,\Lambda] = 2E\wedge\Lambda \qquad L_E\Lambda = [E,\Lambda] = 0. \tag{3.1}$$

Here $[\cdot,\cdot]$ denotes the Schouten-Nijenhuis bracket. The bracket is then given by:

$$\{u,v\} = \Lambda(du,dv) + udv(E) - vdu(E)$$

The Hamiltonian vector fields $X_u = \Lambda(du) + uE$ span a generalized distribution which is integrable and on every leaf of the resulting foliation there exists a unique induced Jacobi structure. So one is led to the study of so called transitive Jacobi manifolds, that is with the foliation consisting of only one leaf. Now if a transitive Jacobi manifold is odd dimensional then it is a contact manifold and if it is even dimensional one can identify T^*M with TM via Λ. If we let Ω be the 2-form corresponding to Λ and ω the 1-form corresponding to E the equations (3.1) are equivalent to $d\omega = 0$ and $d^\omega\Omega = 0$. So we see that the l.c.s. manifolds are exactly the transitive, even dimensional Jacobi manifolds. Moreover for the Hamiltonian we have $X_u = \sharp d^\omega u$ in this case.

Suppose $\{\cdot,\cdot\}$ is a bracket as above and a is a nowhere vanishing function on M. Then one can define a new Jacobi bracket by $\{f,g\}_a := \frac{1}{a}\{af, ag\}$. We then have $\Lambda_a = a\Lambda$, $E_a = aE + \Lambda(da)$ and if $\{\cdot,\cdot\}$ is even dimensional and transitive so is $\{\cdot,\cdot\}_a$. Then we have $\Omega_a = \frac{1}{a}\Omega$, $\omega_a = \omega + \frac{da}{a}$. So the deformation of the bracket in the way described above corresponds exactly to the conformal change of (M,Ω,ω).

If g is a diffeomorphism of M then $(M, g^*\Omega, g^*\omega)$ is again a l.c.s. manifold. We write $\mathrm{Diff}^\infty_c(M,\Omega,\omega)$ for the group of all compactly supported diffeomorphisms that preserve the l.c.s. structure (Ω,ω) up to conformal equivalence, i.e.

$$\mathrm{Diff}^\infty_c(M,\Omega,\omega) := \big\{g \in \mathrm{Diff}^\infty_c(M) : (M, g^*\Omega, g^*\omega) \sim (M,\Omega,\omega)\big\}$$

More explicitly, $g \in \mathrm{Diff}^\infty_c(M,\Omega,\omega)$ iff there exists $a \in C^\infty(M, \mathbb{R}\setminus 0)$ such that $g^*\Omega = \frac{1}{a}\Omega$ and $g^*\omega = \omega + d(\ln|a|)$. If $\dim M > 2$ then the first equation implies the second since ω is unique. Next we define

$$\mathfrak{X}_c(M,\Omega,\omega) := \big\{X \in \mathfrak{X}_c(M) : \exists u \in C^\infty(M,\mathbb{R}) \text{ with } L_X\Omega = -u\Omega, L_X\omega = du\big\}$$

Again, if $\dim M > 2$ then the equation $L_X\Omega = -u\Omega$ implies the equation $L_X\omega = du$. It is easily checked that $\mathfrak{X}_c(M,\Omega,\omega)$ is a Lie algebra, see Lemma 4.1. Notice that any Hamiltonian vector field $X_u = \sharp d^\omega u \in \mathfrak{X}_c(M,\Omega,\omega)$ for $u \in C^\infty_c(M,\mathbb{R})$, by the description of $\mathfrak{X}_c(M,\Omega,\omega)$ in Lemma 4.1. $\mathfrak{X}_c(M,\Omega,\omega)$ is the Lie algebra of $\mathrm{Diff}^\infty_c(M,\Omega,\omega)$ in the following sense:

3.4. Lemma. *Let $g \in C^\infty\big((\mathbb{R},0),(\mathrm{Diff}^\infty_c(M),\mathrm{id})\big)$. Then we have:*

$$g \in C^\infty\big(\mathbb{R}, \mathrm{Diff}^\infty_c(M,\Omega,\omega)\big) \Leftrightarrow \delta^r g \in \Omega^1\big(\mathbb{R}; \mathfrak{X}_c(M,\Omega,\omega)\big) \Leftrightarrow \dot g_t \in \mathfrak{X}_c(M,\Omega,\omega)$$

Especially $\mathrm{Fl}^X \in C^\infty\big(\mathbb{R}, \mathrm{Diff}^\infty_c(M,\Omega,\omega)\big)$ *iff* $X \in \mathfrak{X}_c(M,\Omega,\omega)$.

Proof. Suppose we have $g : \big(\mathbb{R},0\big) \to \big(\mathrm{Diff}^\infty_c(M,\Omega,\omega), \mathrm{id}\big)$. Then there exists $a \in C^\infty(\mathbb{R}\times M, \mathbb{R})$ with $g_t^*\Omega = \frac{1}{a_t}\Omega$ and $g_t^*\omega = \omega + d(\ln|a_t|)$. Differentiating these equations with respect to t we obtain $L_{\dot g_t}\Omega = -(g_t^{-1})^*(\frac{\dot a_t}{a_t})\Omega$ and $L_{\dot g_t}\omega = d((g_t^{-1})^*\frac{\dot a_t}{a_t})$, where $\dot a_t := \frac{\partial}{\partial t}a_t$. Hence $\dot g_t \in \mathfrak{X}_c(M,\Omega,\omega)$ with $f_{\dot g_t} = (g_t^{-1})^*\frac{\dot a_t}{a_t}$.

Suppose conversely $L_{\dot g_t}\Omega = -u_t\Omega$ and $L_{\dot g_t}\omega = du_t$. Then we define $a_t := \exp(\int_0^t g_s^* u_s ds)$. It satisfies $g_t^* u_t = \frac{\dot a_t}{a_t}$ and $a_0 = 1$. So we obtain the following differential equation for $g_t^*\Omega$:

$$\tfrac{\partial}{\partial t}(g_t^*\Omega) = -\tfrac{\dot a_t}{a_t}(g_t^*\Omega) \quad \text{with initial condition} \quad g_0^*\Omega = \Omega.$$

This equation has a solution namely $\frac{1}{a_t}\Omega$ and since the solution is unique (evaluate everything at points $x \in M$ and obtain differential equations in a finite dimensional space) we obtain $g_t^*\Omega = \frac{1}{a_t}\Omega$. Similarly one checks $g_t^*\omega = \omega + d(\ln|a_t|)$. □

We define also the group of strict automorphisms of (M,Ω,ω) by:

$$\mathcal{D}_c(M,\Omega,\omega) := \{g \in \mathrm{Diff}^\infty_c(M) : g^*\Omega = \Omega\}.$$

If $\dim(M) > 2$ this is a subgroup of $\mathrm{Diff}^\infty_c(M,\Omega,\omega)$. Moreover we define a Lie algebra:

$$\mathcal{X}_c(M,\Omega,\omega) := \{X \in \mathfrak{X}_c(M) : L_X\Omega = 0\}$$

Again, if $\dim(M) > 2$ then this is a subalgebra of $\mathfrak{X}_c(M, \Omega, \omega)$. For a curve $g \in C^\infty\big((\mathbb{R}, 0), (\mathrm{Diff}^\infty_c(M), \mathrm{id})\big)$ we also have

$$g \in C^\infty\big(\mathbb{R}, \mathcal{D}_c(M, \Omega, \omega)\big) \Leftrightarrow \delta^r g \in \Omega^1\big(\mathbb{R}; \mathcal{X}_c(M, \Omega, \omega)\big) \Leftrightarrow \dot g_t \in \mathcal{X}_c(M, \Omega, \omega)$$

which is an immediate consequence of the formula $\frac{d}{dt}(g_t^*\Omega) = g_t^* L_{\dot g_t}\Omega$. Notice that these concepts are not conformally invariant. Of course, $X_u = \sharp d^\omega u \in \mathcal{X}_c(M, \Omega, \omega)$ iff $\omega(X_u) = 0$. In general, a vector field X is called horizontal if $\omega(X) = 0$. A l.c.s. manifold (M, Ω, ω) is said to be of the first kind ([19]) if there is $X \in \mathcal{X}_c(M, \Omega, \omega)$ which is not horizontal.

Observe that for any $X \in \mathcal{X}_c(M, \Omega, \omega)$ we have $L_X\omega = 0$ and, consequently, $\omega(X) = const$. In particular, if $X, Y \in \mathcal{X}_c(M, \Omega, \omega)$ then $[X, Y]$ is horizontal. It is easily seen that for (M, Ω, ω) of the first kind one has $\omega(x) \neq 0$, $\forall x \in M$, and we get the horizontal foliation (i.e. spanned by the horizontal vector fields).

The proof of the following is due to P. Michor and C. Vizman, see [9].

3.5. Lemma. *Let M be a connected manifold, $\mathfrak{g} \subseteq \mathfrak{X}_c(M)$ a Lie algebra of vector fields and $G \subseteq \mathrm{Diff}^\infty_c(M)$ such that $X \in \mathfrak{g}$ implies $\mathrm{Fl}^X_t \in G$ for all $t \in \mathbb{R}$. If for all $x \in X$ there exist $X_1, \dots, X_n \in \mathfrak{g}$ such that $X_1(x), \dots, X_n(x)$ is a basis of T_xM then $G_\circ$ acts transitively on M.*

Proof. For $x \in M$ consider that mapping

$$f : \mathbb{R}^n \to M \qquad f(t_1, \dots, t_n) := \big(\mathrm{Fl}^{X_1}_{t_1} \circ \dots \circ \mathrm{Fl}^{X_n}_{t_n}\big)(x)$$

Then we have $f(0) = x$ and $f(0, \dots, t_i, \dots, 0) = \mathrm{Fl}^{X_i}_{t_i}(x)$ hence $\frac{\partial}{\partial t_i}|_0 f(t) = X_i(x)$ and so $T_0 f$ is an isomorphism. This shows that the $G_\circ$-orbits are open. Consequently the $G_\circ$-orbits are closed too, and so there exists only one such orbit since M is connected. □

By using a main result of [18] one can strengthen the above as well as pioneer results of Lefebvre [12]. Let us recall some concept from [18]. A diffeomorphism group $G(M)$ is *pseudo-k-transitive* with respect to a foliation $\mathcal{F}$ if it preserves $\mathcal{F}$ and for any two k-tuples of pairwise distinct points $(x_1, \dots, x_k)$ and $(y_1, \dots, y_k)$ of M such that x_i, y_i belong to the same leaf and each leaf of dimension ≤ 1 contains at most one x_i there exists $f \in G(M)$ satisfying $f(x_i) = y_i$, $i = 1, \dots, k$. Observe that for a foliation with a one leaf only this concept coincides with the k-transivity.

The following facts can be deduced from [18].

3.6. Proposition. *The group $\mathrm{Diff}^\infty_c(M, \Omega, \omega)$ act k-transitively on M for each $k \geq 1$. Furthermore, if (M, Ω, ω) is of the first kind the group $\mathcal{D}_c(M, \Omega, \omega)$ is pseudo-k-transitive for each $k \geq 1$ with respect to the horizontal foliation. The leaves of this foliation coincide with the orbits of isotopies g in $\mathcal{D}_c(M, \Omega, \omega)$ such that $\delta^r g$ is horizontal.*

Let us only mention one important consequence of 3.6. In view of results of [17] it is visible that the group $\mathrm{Diff}^\infty_c(M, \Omega, \omega)$ determines uniquely the manifold M and up to conformal equivalence also the l.c.s. structure (Ω, ω). Under some assumptions (e.g. no leaves of dimension 0 in the horizontal foliation) the same may be stated about $\mathcal{D}_c(M, \Omega, \omega)$. Thus the l.c.s. structures belong to a class of geometric structures which fulfills a modern version of the "Erlanger Programm", cf. [3]. The details of the proof will be presented in a forthcoming paper on Jacobi structures.

In the sequel it will be important to know the homotopy type of $\mathrm{Diff}^\infty_c(M,\Omega,\omega)$. By using both Frobenius theorem and implicit function theorem in the case of ILH-Lie groups H.Omori ([15], IX,7.2) proved essentially the following result.

3.7. Theorem. *For K compact $\mathrm{Diff}^\infty_K(M,\Omega,\omega)$ is an ILH-group (and consequently a Lie group).*

3.8. Proposition. *Let $G(M)$ be the kernel of a continuous, surjective homomorphism from $\mathrm{Diff}^\infty_c(M,\Omega,\omega)$ onto a finite dimensional Lie group. Then $G(M)$ admits a structure of a Lie group.*

In fact, it follows from Theorem III,2.5 in [15] that $G_K(M)$ is a Lie group. Then $\mathrm{Diff}^\infty_c(M,\Omega,\omega) = \varinjlim_K \mathrm{Diff}^\infty_K(M,\Omega,\omega)$ is a Lie group modeled on the locally convex vector space $\mathfrak{g} = \varinjlim_K \mathfrak{g}_K$ where $\mathfrak{g}_K$ is the Lie algebra of $G_K(M)$.

As a consequence we have

3.9. Proposition. *The first homotopy group of $\mathrm{Diff}^\infty_c(M,\Omega,\omega)$ or of any G as in Proposition 3.8 is countable.*

Proof. We appeal to well known papers of J. Milnor [14] and R. Palais [16]. By 3.7 and 3.8 the groups in question are infinite dimensional manifolds. Since compactly supported they are metrizable and second countable. So by Theorem 5 [16] they must be absolute neighborhood retracts. Theorem 1 in [14] states that any absolute neighborhood retract is dominated by a countable CW-complex. This completes the proof. $\square$

4. The Extended Lee Homomorphism

Parts of the following lemma can be found in [7] or [19].

4.1. Lemma. *Let X be a compactly supported vector field on M. Then $X \in \mathfrak{X}_c(M,\Omega,\omega)$ if and only if there exists a locally constant function $c_X \in C^\infty(M,\mathbb{R})$ with $d^\omega(\flat X) = c_X\Omega$. In this case c_X is unique and we have $c_X = i_X\omega - u_X$ where u_X is the function satisfying $L_X\Omega = -u_X\Omega$ and $L_X\omega = du_X$. Moreover $\mathfrak{X}_c(M,\Omega,\omega)$ is a Lie algebra and the mapping*

$$\varphi : \mathfrak{X}_c(M,\Omega,\omega) \to H^0_c(M) \qquad X \mapsto [c_X]$$

is a Lie algebra homomorphism (called the extended Lee homomorphism). If M is compact it is surjective iff Ω is d^ω-exact.

If $(M,\Omega,\omega) \sim (M,\Omega',\omega')$ then $\mathfrak{X}_c(M,\Omega,\omega) = \mathfrak{X}_c(M,\Omega',\omega')$ and $\varphi' = \varphi$. Let $g \in \mathrm{Diff}^\infty_c(M)$ and $(M,\Omega'',\omega'') := (M, g^*\Omega, g^*\omega)$. Then $g^* : \mathfrak{X}_c(M,\Omega,\omega) \cong \mathfrak{X}_c(M,\Omega'',\omega'')$ and $\varphi'' \circ g^* = g^* \circ \varphi$. If $g \in \mathrm{Diff}^\infty_c(M,\Omega,\omega)$ then $\mathfrak{X}_c(M,\Omega,\omega) = \mathfrak{X}_c(M,\Omega'',\omega'')$, $\varphi \circ g^* = g^* \circ \varphi$ and if $g \in \mathrm{Diff}^\infty_c(M,\Omega,\omega)_\circ$ have even get $\varphi \circ g^* = \varphi$.

Proof. For any vector field we have $d^\omega(\flat X) = L_X\Omega + i_X\omega \wedge \Omega$ which yields immediately the first statement. For $X, Y \in \mathfrak{X}_c(M,\Omega,\omega)$ one easily shows

$$\flat[X,Y] = d^\omega(i_X i_Y \Omega) - c_X \flat Y + c_Y \flat X. \tag{4.1}$$

Hence $d^\omega(\flat[X,Y]) = 0$ and so $c_{[X,Y]} = 0$.

If $\Omega' = \frac{1}{a}\Omega$, $\omega' = \omega + \frac{da}{a}$ then $\flat' = \frac{1}{a}\flat$ and $d^{\omega'}\frac{1}{a} = \frac{1}{a}d^\omega$. So the equation $d^\omega(\flat X) = c_X\Omega$ is equivalent to $d^{\omega'}(\flat' X) = c_X\Omega'$.

Let $g \in \operatorname{Diff}^\infty_c(M)$. Then $g^* \circ \flat = \flat'' \circ g^*$ and hence the equation $d^\omega(\flat X) = c_X\Omega$ is equivalent to $d^{\omega''}(\flat''(g^*X)) = (g^*c_X)\Omega''$. But this gives $g^* : \mathfrak{X}_c(M,\Omega,\omega) \cong \mathfrak{X}_c(M,\Omega'',\omega'')$ and $\varphi'' \circ g^* = g^* \circ \varphi$. If $g \in \operatorname{Diff}^\infty_c(M,\Omega,\omega)$ then $(M,\Omega'',\omega'') \sim (M,\Omega,\omega)$ and everything follows from the previous paragraph. The last statement is due to the fact that $g^* = \mathrm{id} : H^0_c(M) \to H^0_c(M)$ if g is homotopic to the identity. □

4.2. Remark. Notice that the homomorphism φ vanishes if (M,Ω,ω) is conformally equivalent to a symplectic structure, since $H^0_c(M) \neq 0$ only if M is compact but in this case Ω is not d^ω-exact since an exact symplectic structure can only exist on non-compact manifolds. But φ does not vanish in general. For example let $T^4 = S^1\times S^1\times S^1\times S^1$ be the 4-dimensional torus and let dx, dy, dx', dy' denote the generators of $H^1(T^4)$. We take $\omega := dx$, $\alpha := \sin(y)dx' + \cos(y)dy'$ and $\Omega := d^\omega\alpha$. An easy calculation shows $\Omega^2 = 2dx\wedge dy\wedge dx'\wedge dy'$, so (T^4,Ω,ω) is a compact, d^ω-exact l.c.s. manifold, and φ is non-trivial by Lemma 4.1.

Another example with non-vanishing φ is $S^1 \times S^3$ with the structure from 3.2.

4.3. Proposition. *The Lie algebra homomorphism φ integrates to a group homomorphism*

$$\tilde\Phi : \widetilde{\operatorname{Diff}}^\infty_c(M,\Omega,\omega)_\circ \to H^0_c(M)$$

i.e. $\tilde\Phi\circ\exp = \exp\circ\varphi$, or $\tilde\Phi(\mathrm{Fl}^X) = \varphi(X)$. If M is compact then $\tilde\Phi$ is surjective iff Ω is d^ω-exact. We have the following formulas:

$$\tilde\Phi(g) = \int_I \varphi_*(\delta^r g) = \int_0^1 \varphi(\dot g_t)dt = \big[\int_0^1 c_{\dot g_t}dt\big] = \big[\int_0^1 g_t^* c_{\dot g_t}dt\big]$$

If $(M,\Omega,\omega) \sim (M,\Omega',\omega')$ then $\widetilde{\operatorname{Diff}}^\infty_c(M,\Omega,\omega)_\circ = \widetilde{\operatorname{Diff}}^\infty_c(M,\Omega',\omega')_\circ$ and $\tilde\Phi' = \tilde\Phi$.

Proof. Notice that $\varphi_*(\delta^r g) \in \Omega^1(I; H^0_c(M))$ where $H^0_c(M)$ is a separated complete locally convex vector space and hence integration is well defined. Obviously the various formulas for $\tilde\Phi$ are equal. We have to check that they do only depend on the homotopy type relative endpoints of g. So let $G : D^2 \to \operatorname{Diff}^\infty_c(M,\Omega,\omega)$ and denote by $i : S^1 \hookrightarrow D^2$ the inclusion. Using Stokes theorem and the Maurer-Cartan equation (2.1) for $\delta^r G$ we obtain

$$\int_{S^1}\varphi_*(\delta^r(i^*G)) = \int_{S^1} i^*\varphi_*(\delta^r G) = \int_{D^2} d\varphi_*(\delta^r G) = \int_{D^2}\varphi_*(\tfrac12[\delta^r G,\delta^r G])$$

but the right hand side is zero since φ vanishes on brackets.

Let $f, g : (I,0) \to (\operatorname{Diff}^\infty_c(M,\Omega,\omega),\mathrm{id})$. Using the Leibniz rule, Lemma 4.1 and the fact that $f(t) \in \operatorname{Diff}^\infty_c(M,\Omega,\omega)_\circ$ for every $t \in I$ we obtain

$$\varphi_*(\delta^r(fg))(t) = \varphi(\dot f_t) + \varphi((f_t^{-1})^*\dot g_t) = \varphi(\dot f_t) + \varphi(\dot g_t) = \big(\varphi_*(\delta^r f) + \varphi_*(\delta^r g)\big)(t)$$

So $\varphi_*(\delta^r(fg)) = \varphi_*(\delta^r f) + \varphi_*(\delta^r g)$ and hence $\tilde\Phi(fg) = \tilde\Phi(f) + \tilde\Phi(g)$. The rest follows from $\delta^r(\mathrm{Fl}^X) = Xdt$. □

The homomorphism $\tilde\Phi$ has the following geometrical interpretation:

4.4. Proposition. *Let* $g : (I, 0) \to (\mathrm{Diff}_c^\infty(M, \Omega, \omega), \mathrm{id})$ *and denote by* a_t *the functions satisfying* $g_t^*\Omega = \frac{1}{a_t}\Omega$, $g_t^*\omega = \omega + d(\ln|a_t|)$*. Then for* $x \in M$ *we have*

$$\tilde{\Phi}(g)(x) = \textstyle\int_I (g^x)^*\omega - \ln|a_1(x)|$$

where $g^x : I \to M$ *is the path* $t \mapsto g_t(x)$*.*

Proof. Differentiating the equation $g_t^*\Omega = \frac{1}{a_t}\Omega$ with respect to t we get $\frac{\partial}{\partial t}(\ln|a_t|) = g_t^* f_{\dot{g}_t}$, where $f_{\dot{g}_t}$ are the functions satisfying $L_{\dot{g}_t}\Omega = -f_{\dot{g}_t}\Omega$ and $L_{\dot{g}_t}\omega = df_{\dot{g}_t}$, and therefore

$$\ln|a_1| = \ln|a_1| - \ln|a_0| = \textstyle\int_0^1 \frac{\partial}{\partial t}(\ln|a_t|)dt = \int_0^1 g_t^* f_{\dot{g}_t} dt$$

Next we have

$$\textstyle\int_I (g^x)^*\omega = \int_0^1 \omega(\frac{\partial}{\partial s}|_t g_s(x))dt = \int_0^1 \omega(\dot{g}_t(g_t(x))dt = \int_0^1 (g_t^* i_{\dot{g}_t}\omega)dt(x)$$

Combining these two equations we obtain

$$\textstyle\int_I (g^x)^*\omega - \ln|a_1(x)| = \int_0^1 g_t^*(i_{\dot{g}_t}\omega - f_{\dot{g}_t})dt(x) = \int_0^1 g_t^* c_{\dot{g}_t} dt(x) = \tilde{\Phi}(g)(x)$$

and we are done. □

We define $\Delta := \tilde{\Phi}\big(\pi_1(\mathrm{Diff}_c^\infty(M, \Omega, \omega)_\circ\big)$. Then $\tilde{\Phi}$ factors to a homomorphism

$$\Phi : \mathrm{Diff}_c^\infty(M, \Omega, \omega)_\circ \to H_c^0(M)/\Delta$$

If M is compact then, by Lemma 4.1, Φ is surjective iff Ω is d^ω-exact. If $(M, \Omega, \omega) \sim (M, \Omega', \omega')$ then $\mathrm{Diff}_c^\infty(M, \Omega, \omega)_\circ = \mathrm{Diff}_c^\infty(M, \Omega', \omega')_\circ$, $\Delta = \Delta'$ and $\Phi = \Phi'$.

4.5. Corollary. *If* M *is compact then* $H_c^0(M) \cong \mathbb{R}$ *and*

$$\Delta \subseteq \mathrm{Per}(\omega) := \{\langle\omega, c\rangle : c \in H_1(M; \mathbb{Z})\} \subseteq \mathbb{R}$$

Especially $\Delta \subseteq H_c^0(M)$ *is always countable.*

4.6. Corollary. *Let* $g \in C^\infty\big((\mathbb{R}, 0), (\mathrm{Diff}_c^\infty(M, \Omega, \omega), \mathrm{id})\big)$*. Then we have:*

$$g \in C^\infty(\mathbb{R}, \ker\Phi) \quad \Leftrightarrow \quad \delta^r g \in \Omega^1(\mathbb{R}; \ker\varphi) \quad \Leftrightarrow \quad \dot{g}_t \in \ker\varphi$$

In particular, $\mathrm{Fl}^X \in C^\infty(\mathbb{R}, \ker\Phi)$ *iff* $X \in \ker\varphi$*.*

Proof. By Lemma 3.4 we may assume that g has values in $\mathrm{Diff}_c^\infty(M, \Omega, \omega)$ and $\delta^r g \in \Omega^1\big(\mathbb{R}; \mathfrak{X}_c(M, \Omega, \omega)\big)$. For $s \in \mathbb{R}$ let $\mu_s : I \to \mathbb{R}$, $\mu_s(t) := ts$. We then have

$$\Phi(g_s) = \tau(\tilde{\Phi}(\mu_s^* g)) = \tau\big(\textstyle\int_I \mu_s^*\varphi_*(\delta^r g)\big) = \tau\big(\int_{\mu_s(I)} \varphi_*(\delta^r g)\big) = \tau\big(\int_0^s \varphi(\dot{g}_t)dt\big).$$

Here τ is the canonical projection $\tau : H_c^0(M) \to H_c^0(M)/\Delta$. The implication $\Leftarrow$ follows immediately. So let us assume that g has values in $\ker\Phi$. Then $\int_0^s \varphi(\dot{g}_t)dt \in \Delta$ for all $s \in I$. Since this depends continuously on s and has values in a countable subset of a separated topological vector space it has to be constant, i.e. $\int_0^s \varphi(\dot{g}_t)dt = 0$ for all $s \in I$. Differentiating with respect to s we obtain $\dot{g}_s \in \ker\varphi$ for all $s \in \mathbb{R}$. □

4.7. Lemma. $\ker\Phi$ *is connected by smooth arcs, and* $\widetilde{\ker\Phi}\cong\ker\tilde\Phi$.

Proof. Since we have Corollary 4.6. the inclusion $\ker\Phi\subseteq \mathrm{Diff}^\infty_c(M,\Omega,\omega)$ induces a well defined mapping $i:\widetilde{\ker\Phi}\to\ker\tilde\Phi$. In order to show that i is onto let $g\in\ker\tilde\Phi$. For any $s\in I$ we define $h_s\in C^\infty\big((I,0),(\mathrm{Diff}^\infty_c(M,\Omega,\omega),\mathrm{id})\big)$ by $\delta^r(h_s)=s\delta^r g$ (cf. Lemma 3.4). Then $h_0(t)=\mathrm{id}$ and $h_1(t)=g(t)$. Moreover

$$\tilde\Phi(h_s)=\textstyle\int_I\varphi_*(\delta^r(h_s))=s\int_I\varphi_*(\delta^r g)=s\tilde\Phi(g)=0$$

and so $\Phi(h_s(1))=0$ for all $s\in I$. So h defines a homotopy relative endpoints from g to $s\mapsto h_s(1)$, which is a curve in $\ker\Phi$. Consequently i is onto. Next we show that i is one-to-one. Let $g\in\ker i$, i.e. there exists $G\in C^\infty\big(I\times I,\mathrm{Diff}^\infty_c(M,\Omega,\omega)\big)$ with $G(0,t)=\mathrm{id}$, $G(1,t)=g(t)$ and $G(s,0)=G(s,1)=\mathrm{id}$. For $(s,u)\in I\times I$ we define $H(s,\cdot,u)\in C^\infty\big((I,0),(\mathrm{Diff}^\infty_c(M,\Omega,\omega),\mathrm{id})\big)$ by $\delta^r H(s,\cdot,u)=u\delta^r G(s,\cdot)$. We have $G(1,t)=g(t)\in\ker\Phi$, so $\delta^r G(1,\cdot)\in\Omega^1(I;\ker\varphi)$, hence $\delta^r H(1,\cdot,u)\in\Omega^1(I;\ker\varphi)$ and thus $H(1,t,u)\in\ker\Phi$ for all $(t,u)\in I\times I$. So g is homotopic relative endpoints in $\ker\Phi$ to $u\mapsto H(1,1,u)$. Moreover $(s,u)\mapsto H(s,1,u)$ is a smooth homotopy relative endpoints from id to $H(1,1,\cdot)$. We claim that it has values in $\ker\Phi$. Indeed, since $\tilde\Phi(G(s,\cdot))=\tilde\Phi(G(0,\cdot))=\tilde\Phi(\mathrm{id})=0$ we have

$$\tilde\Phi\big(H(s,\cdot,u)\big)=\textstyle\int_I\varphi_*\delta^r H(s,\cdot,u)=u\int_I\varphi_*\delta^r G(s,\cdot)=u\tilde\Phi\big(G(s,\cdot)\big)=0$$

and hence $\Phi(H(s,1,u))=0$. Summing up we have seen that g is homotopic relative endpoints in $\ker\Phi$ to id, i.e. i is one-to-one. Remains to show that $\ker\Phi$ is connected by smooth arcs. It is clear that $ev_1:\ker\tilde\Phi\to\ker\Phi$ is onto, hence $ev_1=ev_1\circ i:\widetilde{\ker\Phi}\to\ker\Phi$ is also onto and so $\ker\Phi$ is connected by smooth arcs. □

5. The Flux Homomorphism

In this section we define an analogue of the flux homomorphism. This concept is essentially due to E. Calabi [6].

5.1. Lemma. *We have a surjective Lie algebra homomorphism*

$$\psi:\ker\varphi\to H^1_{d^\omega_c}(M)\qquad X\mapsto[\flat X]$$

If $(M,\Omega,\omega)\sim(M,\Omega',\omega')$ *with* $\Omega'=\frac1a\Omega$, $\omega'=\omega+d(\ln|a|)$ *then* $\ker\varphi=\ker\varphi'$ *and* $\frac1a\psi=\psi'$. *Let* $g\in\mathrm{Diff}^\infty_c(M)$ *and* $(M,\Omega'',\omega''):=(M,g^*\Omega,g^*\omega)$. *Then* $g^*:\ker\varphi\cong\ker\varphi''$ *and* $\psi''\circ g^*=g^*\circ\psi$. *If* $g\in\mathrm{Diff}^\infty_c(M,\Omega,\omega)$ *then* $\ker\varphi=\ker\varphi''$ *and* $\frac1a\psi\circ g^*=g^*\circ\psi$ *and if* $g\in\ker\Phi$ *we even get* $\psi\circ g^*=\psi$.

Proof. ψ is a Lie algebra homomorphism by formula (4.1). If $[\sigma]\in H^1_{d^\omega_c}(M)$ then $\sharp\sigma\in\ker\varphi$ and $\psi(\sharp\sigma)=[\sigma]$, so ψ is onto.

If $(M,\Omega,\omega)\sim(M,\Omega',\omega')$ then $\varphi=\varphi'$ and $\flat'=\frac1a\flat$. Hence $\ker\varphi=\ker\varphi'$ and $\frac1a\psi=\psi'$.

Let $g\in\mathrm{Diff}^\infty_c(M)$. From Lemma 4.1 we get $g^*:\ker\varphi\cong\ker\varphi''$ and since we have $\flat''\circ g^*=g^*\circ\flat$ we also obtain $\psi''\circ g^*=g^*\circ\psi$. Combining this with the previous paragraph we obtain the statements about $g\in\mathrm{Diff}^\infty_c(M,\Omega,\omega)$. So it remains to show that $ag^*=\mathrm{id}:H^1_{d^\omega_c}(M)\to H^1_{d^\omega_c}(M)$ for $g\in\ker\Phi$. Choose a curve g_t in $\ker\Phi$

from the identity to g and define a_t by $g_t^*\Omega = \frac{1}{a_t}\Omega$ and $g_t^*\omega = \omega + d(\ln|a_t|)$. So we have $g_1 = g$, $a_1 = a$, $g_0 = \mathrm{id}$ and $a_0 = 1$. Moreover since $g_t \in \ker\Phi$ we have $\dot g_t \in \ker\varphi$ hence $f_{\dot g_t} = i_{\dot g_t}\omega$ and $\dot a_t = a_t g_t^* i_{\dot g_t}\omega = a_t\, \mathrm{inc}_t^*\, i_{\partial_t} g^*\omega$. So a_t satisfy the same differential equation as the a_t of Lemma 1.1 and therefore they are the same. But now Lemma 1.1 yields $ag^* = a_1 g_1^* = \mathrm{id}$. $\square$

5.2. Proposition. *The Lie algebra homomorphism ψ integrates to a surjective group homomorphism*

$$\tilde\Psi : \widetilde{\ker\Phi} \to H^1_{d_c^\omega}(M)$$

i.e. $\tilde\Psi\circ\exp = \psi\circ\exp$, *or* $\tilde\Psi(\mathrm{Fl}^X) = \psi(X)$. *We have the following formulas:*

$$\tilde\Psi(g) = \textstyle\int_I \psi_*(\delta^r g) = \int_0^1 \psi(\dot g_t)dt = \big[\int_0^1 i_{\dot g_t}\Omega dt\big] = \big[\int_0^1 a_t g_t^* i_{\dot g_t}\Omega dt\big]$$

where $g_t^*\Omega = \frac{1}{a_t}\Omega$. If $(M,\Omega,\omega)\sim(M,\Omega',\omega')$ with $\Omega' = \frac{1}{a}\Omega$ and $\omega' = \omega + d(\ln|a|)$ then $\widetilde{\ker\Phi'} = \widetilde{\ker\Phi}$ and $\frac{1}{a}\tilde\Psi' = \tilde\Psi$.

Proof. The proof is exactly the same as the proof of Proposition 4.3. $\square$

We let $\Gamma := \tilde\Psi(\pi_1(\ker\Phi))$. Then $\tilde\Psi$ factors to a surjective homomorphism

$$\Psi : \ker\Phi \to H^1_{d_c^\omega}(M)/\Gamma$$

If $(M,\Omega,\omega)\sim(M,\Omega',\omega')$ with $\Omega' = \frac{1}{a}\Omega$ and $\omega' = \omega + d(\ln|a|)$ then $\ker\Phi' = \ker\Phi$, $\frac{1}{a}\Gamma = \Gamma'$ and $\frac{1}{a}\Psi' = \Psi$. Notice that Γ is countable in view of Proposition 3.9 and Theorem 1.8. Similarly to the proof of Corollary 4.6 one shows

5.3. Lemma. *Let $g \in C^\infty\big((\mathbb{R},0),(\mathrm{Diff}_c^\infty(M,\Omega,\omega),\mathrm{id})\big)$. Then*

$$g \in C^\infty(\mathbb{R},\ker\Psi) \quad\Leftrightarrow\quad \delta^r g \in \Omega^1(\mathbb{R};\ker\psi) \quad\Leftrightarrow\quad \dot g_t \in \ker\psi$$

Specifically, $\mathrm{Fl}^X \in C^\infty(\mathbb{R},\ker\Psi)$ *iff* $X \in \ker\psi$.

5.4. Remark. The above lemma gives a characterization of Hamiltonian isotopies on (M,Ω,ω). Namely, an isotopy $g \in C^\infty\big((\mathbb{R},0),(\mathrm{Diff}_c^\infty(M),\mathrm{id})\big)$ is by definition Hamiltonian if $\delta^r g \in \Omega^1(\mathbb{R},\ker\psi)$ (cf. [13] in the symplectic case). It follows then from 5.3 that any isotopy in $\ker\Psi$ is Hamiltonian. For an essentially more complicated reasoning in the symplectic case, see [13], p. 318-320.

5.5. Lemma. $\ker\Psi$ *is connected by smooth arcs, and* $\widetilde{\ker\Psi} \cong \ker\tilde\Psi$.

Proof. The proof is the same as the proof of Lemma 4.7. $\square$

A l.c.s. manifold (M,Ω,ω) is said to be *exact* if there is a 1-form α such that $\Omega = d^\omega\alpha$.

5.6. Proposition. *Suppose (M,Ω,ω) is an exact l.c.s. manifold with $\Omega = d^\omega\alpha$. Then for $g \in \widetilde{\ker}\ \Phi$ we have*

$$\tilde\Psi(g) = [a_1 g_1^*\alpha - \alpha] \in H^1_{d_c^\omega}(M)$$

where $g_t^*\Omega = \frac{1}{a_t}\Omega$ and $g_t^*\omega = \omega + d(\ln|a_t|)$. In particular, $\Gamma = 0$.

Proof. First of all we have

$$i_{\dot g_t}\Omega = i_{\dot g_t}d^\omega\alpha = L_{\dot g_t}\alpha + i_{\dot g_t}\omega\wedge\alpha - d^\omega(i_{\dot g_t}\alpha).$$

Since $g_t \in \ker \Phi$ the a_t are the same as the a_t of Lemma 1.1. So we get

$$\psi(\dot g_t) = [i_{\dot g_t}\Omega] = [L_{\dot g_t}\alpha + i_{\dot g_t}\omega \wedge \alpha] = [a_t g_t^*(L_{\dot g_t}\alpha + i_{\dot g_t}\omega \wedge \alpha)].$$

Since $\dot g_t \in \ker \varphi$ we have $\dot a_t = a_t g_t^* f_{\dot g_t} = a_t g_t^* i_{\dot g_t}\omega$ and hence

$$a_t g_t^*(L_{\dot g_t}\alpha + i_{\dot g_t}\omega \wedge \alpha) = a_t \tfrac{\partial}{\partial t}(g_t^*\alpha) + (\tfrac{\partial}{\partial t}a_t)g_t^*\alpha = \tfrac{\partial}{\partial t}(a_t g_t^*\alpha).$$

Putting all together we obtain

$$\tilde\Psi(g) = \textstyle\int_0^1 \psi(\dot g_t)dt = \big[\int_0^1 \tfrac{\partial}{\partial t}(a_t g_t^*\alpha)dt\big] = [a_1 g_1^*\alpha - a_0 g_0^*\alpha] = [a_1 g^*\alpha - \alpha].$$

□

5.7. Lemma. *Let $\mathcal{U}$ be an open cover of M. Then any $g \in C^\infty\big((I,0),(\ker\Psi, \mathrm{id})\big)$ has a decomposition $g = g_1 \cdots g_n$, where each g_i is supported in some $U_i \in \mathcal{U}$ and $g_i \in C^\infty\big((I,0),(\ker\Psi_{U_i}, \mathrm{id})\big)$*

Proof. Fix a compact set $K \subseteq M$ and define

$$H_K : C^\infty\big(I, \Omega^0_K(M)\big) \to C^\infty\big((I,0),(\ker\Psi,\mathrm{id})\big) \qquad \alpha \mapsto \mathrm{Evol}\,\big((\sharp \circ d^\omega)_*\alpha\big)$$

that is the defining equation for $g = H_K(\alpha)$ is $\flat \dot g_t = d^\omega \alpha_t$ with initial condition $g_0 = \mathrm{id}$. We define the structure of a topological group on the left hand side space such that H_K becomes a continuous homomorphism. Namely we set

$$(\alpha\beta)(t) := \alpha(t) + (H_K(\alpha)(t)^{-1})^*(\tfrac{1}{a_t}\beta(t))$$

where $H_K(\alpha)(t)^*\Omega = \frac{1}{a_t}\Omega$. One easily checks that this is a topological group and H_K is a continuous homomorphism. By Lemma 5.3 and Corollary 1.9 we see that $\bigcup_K \mathrm{im}\, H_K = C^\infty\big((I,0),(\ker\Psi,\mathrm{id})\big)$ and so we only have to show that every $g \in \mathrm{im}(H_K)$ has the desired decomposition.

Now choose $U_1, \ldots, U_n \in \mathcal{U}$ covering K, open sets V_i, W_i with $\bar W_i \subseteq V_i \subseteq \bar V_i \subseteq U_i$ such that W_i still cover K and a partition of unity $\{\lambda_0, \ldots, \lambda_n\}$ subordinated to $\{M \setminus K, W_1, \ldots, W_n\}$. Consider the open neighborhoods $\mathcal{W}_i$ of the identity

$$\mathcal{W}_i := \big\{g \in C^\infty\big((I,0),(\mathrm{Diff}^\infty_c(M,\Omega,\omega),\mathrm{id})\big) : g_t(M \setminus \bar V_i) \subseteq M \setminus \bar W_i \quad \forall t \in I\big\}$$

and define an open neighborhood of $0 \in C^\infty\big(I, \Omega^0_K(M)\big)$ by

$$\mathcal{W}_K := \big\{\alpha \in C^\infty\big(I,\Omega^0_K(M)\big) : H_K(\textstyle\sum_{j=0}^{i-1}\lambda_j\alpha) \in \mathcal{W}_i \quad \forall 1 \le i \le n\big\}$$

Since $\mathcal{W}_K$ is open $H_K(\mathcal{W}_K)$ generates $\mathrm{im}\, H_K$ and so it suffices to show that every $g \in H_K(\mathcal{W}_K)$ has the desired decomposition.

For $\alpha \in \mathcal{W}_K$ we set $f_i := H_K(\sum_{j=0}^{i}\lambda_j\alpha)$. Then we have $f_0 = \mathrm{id}$, $f_n = H_K(\alpha)$, and if we let $g_i := f_{i-1}^{-1}f_i$ we obtain $H_K(\alpha) = g_1 \cdots g_n$. It remains to show that $g_i \in C^\infty\big((I,0),(\ker\Psi_{U_i},\mathrm{id})\big)$, but this follows from

$$g_i = f_{i-1}^{-1}f_i = H_K\big(t \mapsto a_{i-1}(t)f_{i-1}(t)^*(\lambda_i\alpha(t))\big)$$

where, $f_i^*\Omega = \frac{1}{a_i}\Omega$, for we have $\mathrm{supp}\,\big(t \mapsto a_{i-1}(t)f_{i-1}(t)^*(\lambda_i\alpha(t))\big) \subseteq \bar V_i \subseteq U_i$. □

6. The Calabi Invariant

The first trace of the Calabi invariant can be found in [6]. It has been popularized under this name in [1], see also [13],[3].

6.1. Lemma. *Let (M,Ω,ω) be a $2n$-dimensional l.c.s. manifold and assume that $H^0_{d^\omega_c}(M)=0$. Then we have a surjective Lie algebra homomorphism*

$$\rho : \ker\psi \to H^{2n}_{d_c^{(n+1)\omega}}(M) \qquad X \mapsto [u\Omega^n]$$

where u is the unique function on M such that $\flat X = d^\omega u$.

If $(M,\Omega,\omega)\sim(M,\Omega',\omega')$ with $\Omega'=\frac{1}{a}\Omega$, $\omega'=\omega+d(\ln|a|)$ then $\ker\psi=\ker\psi'$ and $\frac{1}{a^{n+1}}\circ\rho=\rho'$. Let $g\in\mathrm{Diff}^\infty_c(M)$ and $(M,\Omega'',\omega''):=(M,g^*\Omega,g^*\omega)$. Then $g^*:\ker\psi\cong\ker\psi''$ and $\rho''\circ g^*=g^*\circ\rho$. If $g\in\mathrm{Diff}^\infty_c(M,\Omega,\omega)$ then $\ker\psi=\ker\psi''$ and $\frac{1}{a^{n+1}}\circ\rho\circ g^*=g^*\circ\rho$ and if $g\in\ker\Psi$ we even get $\rho\circ g^*=\rho$.

Proof. Notice that u is unique since we have the assumption $H^0_{d^\omega_c}(M)=0$. Let $\flat X=d^\omega u$ and $\flat Y=d^\omega v$. By formula (4.1) we get $\flat[X,Y]=d^\omega(i_Xi_Y\Omega)$ and since

$$(i_Xi_Y\Omega)\Omega^n = nd^\omega v\wedge d^\omega u\wedge\Omega^{n-1} = nd^{(n+1)\omega}\big(vd^\omega u\wedge\Omega^{n-1}\big)$$

we see that ρ vanishes on brackets. Given any $[\sigma]\in H^{2n}_{d_c^{(n+1)\omega}}(M)$ we may write $\sigma=u\Omega^n$ for some $u\in C^\infty_c(M,\mathbb{R})$, since Ω is non-degenerate. But then $\sharp(d^\omega u)\in\ker\psi$ and $\rho(\sharp(d^\omega u))=[\sigma]$. So ρ is onto.

If $(M,\Omega,\omega)\sim(M,\Omega',\omega')$ then Lemma 5.1 yields $\ker\psi=\ker\psi'$. Moreover $(n+1)\omega'=(n+1)\omega+d(\ln|a^{n+1}|)$ and so $\frac{1}{a^{n+1}}: H^{2n}_{d_c^{(n+1)\omega}}(M)\cong H^{2n}_{d_c^{(n+1)\omega'}}(M)$. If $\flat X=d^\omega u$ then $\flat' X=\frac{1}{a}\flat X=\frac{1}{a}d^\omega u=d^{\omega'}(\frac{1}{a}u)$ and so $\rho'(X)=[\frac{1}{a}u\Omega'^n]=[\frac{1}{a^{n+1}}u\Omega^n]=\frac{1}{a^{n+1}}\rho(X)$.

Let $g\in\mathrm{Diff}^\infty_c(M)$. From Lemma 5.1 we get $g^*:\ker\psi\cong\ker\psi''$. If $\flat X=d^\omega u$ then $\flat''(g^*X)=d^{\omega''}(g^*u)$ and hence $\rho''(g^*X)=[g^*u\Omega''^n]=[g^*(u\Omega^n)]=g^*\rho(X)$. The statements about $g\in\mathrm{Diff}^\infty_c(M,\Omega,\omega)$ follow now from the previous paragraph. So it remains to show that $a^{n+1}\circ g^*=\mathrm{id}: H^{2n}_{d_c^{(n+1)\omega}}(M)\to H^{2n}_{d_c^{(n+1)\omega}}(M)$ for $g\in\ker\Phi$, but the proof is similar to the proof of the corresponding statement on ψ (cf. proof of Lemma 5.1). □

6.2. Remark. Suppose (M,Ω,ω) is not conformally equivalent to a symplectic manifold. By Example 1.6 we have $H^0_{d^\omega_c}(M)=0$ and $H^{2n}_{d_c^{(n+1)\omega}}(M)=0$. So ρ is always defined, but it is identically 0.

6.3. Proposition. *Let (M,Ω,ω) be a $2n$-dimensional l.c.s. manifold and assume that $H^0_{d^\omega_c}(M)=0$. Then the Lie algebra homomorphism ρ integrates to a surjective group homomorphism*

$$\tilde R : \widetilde{\ker\Psi} \to H^{2n}_{d_c^{(n+1)\omega}}(M)$$

i.e. $\tilde R\circ\exp=\rho\circ\exp$, i.e. $\tilde R(\mathrm{Fl}^X)=\rho(X)$. We have the following formulas:

$$\tilde R(g)=\textstyle\int_I\rho_*(\delta^r g)=\int_0^1\rho(\dot g_t)dt=\big[\int_0^1 u_t\Omega^n dt\big]=\big[\int_0^1 a_t(g_t^*u_t)\Omega^n dt\big]$$

where $g_t^*\Omega=\frac{1}{a_t}\Omega$ and $d^\omega u_t=\flat\dot g_t$. If $(M,\Omega,\omega)\sim(M,\Omega',\omega')$ with $\Omega'=\frac{1}{a}\Omega$ and $\omega'=\omega+d(\ln|a|)$ then $\widetilde{\ker\Psi}=\widetilde{\ker\Psi'}$ and $\frac{1}{a^{n+1}}\tilde R=\tilde R'$.

Proof. The proof is exactly the same as the proof of Proposition 4.3. □

We let $\Lambda := \tilde{R}\big(\pi_1(\ker\Psi)\big)$. Note that Λ is countable in view of Proposition 3.9. The homomorphism $\tilde{R}$ factors to a surjective homomorphism

$$R : \ker\Psi \to H^{2n}_{d_c^{(n+1)\omega}}(M)/\Lambda$$

If $(M,\Omega,\omega) \sim (M,\Omega',\omega')$ with $\Omega' = \frac{1}{a}\Omega$ and $\omega' = \omega + d(\ln|a|)$ then $\ker\Psi = \ker\Psi'$, $\frac{1}{a^{n+1}}\Lambda = \Lambda'$ and $\frac{1}{a^{n+1}}\circ R = R'$. Notice that Λ is countable in view of Proposition 3.9 and Theorem 1.8. Similarly to the proof of Corollary 4.6 one shows:

6.4. Lemma. *Let (M,Ω,ω) be a $2n$-dimensional l.c.s. manifold with $H^0_{d_c^\omega}(M) = 0$. Let $g \in C^\infty\big((\mathbb{R},0),(\mathrm{Diff}_c^\infty(M,\Omega,\omega),\mathrm{id})\big)$. Then*

$$g \in C^\infty(\mathbb{R},\ker R) \quad\Leftrightarrow\quad \delta^r g \in \Omega^1(\mathbb{R};\ker\rho) \quad\Leftrightarrow\quad \dot{g}_t \in \ker\rho$$

Especially $\mathrm{Fl}^X \in C^\infty(\mathbb{R},\ker R)$ iff $X \in \ker\rho$.

6.5. Lemma. *$\ker R$ is connected by smooth arcs, and $\widetilde{\ker R} \cong \ker\tilde{R}$.*

Proof. The proof is the same as the proof of Lemma 4.7. □

6.6. Proposition. *Let (M,Ω,ω) be a $2n$-dimensional exact l.c.s. manifold, $\Omega = d^\omega\alpha$. Then $H^0_{d_c^\omega}(M) = 0$ and for $g \in \widetilde{\ker\Psi}$ we have*

$$\tilde{R}(g) = \tfrac{1}{n+1}[u\Omega^n] = \tfrac{1}{n+1}[(a_1 g_1^*\alpha)\wedge\alpha\wedge\Omega^{n-1}]$$

where $u \in C_c^\infty(M,\mathbb{R})$ is the unique function satisfying $d^\omega u = a_1 g_1^*\alpha - \alpha$ (cf. Proposition 5.6). Especially we have $\Lambda = 0$.

Proof. If ω is not exact then by Example 1.6 $H^0_{d_c^\omega}(M) = 0$, and if ω is exact then (M,Ω,ω) is conformally equivalent to a symplectic structure and it is well known that this can only happen if M is not compact, i.e. $0 = H^0_c(M) \cong H^0_{d_c^\omega}(M)$. So we always have $H^0_{d_c^\omega}(M) = 0$ and thus u is unique.

Let v_t be the functions satisfying $\flat\dot{g}_t = d^\omega v_t$ and recall the homotopy operator from Lemma 1.1. Then we have

$$a_1 g_1^*\alpha - \alpha = H(d^\omega\alpha) + d^\omega H(\alpha) = H(\Omega) + d^\omega H(\alpha)$$

and

$$H(\Omega) = \textstyle\int_0^1 a_t g_t^* i_{\dot{g}_t}\Omega dt = \int_0^1 a_t g_t^*(d^\omega v_t) dt = \int_0^1 d^\omega(a_t g_t^* v_t) dt.$$

Together this yields

$$a_1 g_1^*\alpha - \alpha = d^\omega\big(\textstyle\int_0^1 a_t g_t^* v_t dt\big) + d^\omega\big(\int_0^1 a_t g_t^* i_{\dot{g}_t}\alpha dt\big)$$

and so

$$u = \textstyle\int_0^1 a_t g_t^* v_t dt + \int_0^1 a_t g_t^* i_{\dot{g}_t}\alpha dt =: u_1 + u_2.$$

Next we have

$$\begin{aligned}(a_t g_t^* i_{\dot{g}_t}\alpha)\wedge\Omega^n &= a_t^{n+1} g_t^*(i_{\dot{g}_t}\alpha\wedge\Omega^n) = n a_t^{n+1} g_t^*(\alpha\wedge i_{\dot{g}_t}\Omega\wedge\Omega^{n-1})\\ &= n a_t^{n+1} g_t^*(\alpha\wedge d^\omega v_t\wedge\Omega^{n-1})\\ &= n a_t^{n+1} g_t^*(v_t\Omega^n) - n a_t^{n+1} g_t^* d^{(n+1)\omega}(\alpha v_t\Omega^{n-1})\\ &= n a_t^{n+1} g_t^*(v_t\Omega^n) - d^{(n+1)\omega}(n a_t^{n+1} g_t^*(\alpha v_t\Omega^{n-1})).\end{aligned}$$

and therefore $[u_2\Omega^n] = n[u_1\Omega^n] \in H^{2n}_{d_c^{(n+1)\omega}}(M)$. So

$$\tfrac{1}{n+1}[u\Omega^n] = \int_0^1 [a_t^{n+1} g_t^*(v_t\Omega^n)]dt = \int_0^1 [v_t\Omega^n]dt = \int_0^1 \rho(\dot g_t)dt = \tilde R(g).$$

The second expression follows now easily

$$\begin{aligned}[u\Omega^n] &= [ud^\omega\alpha\wedge\Omega^{n-1}] = [d^{(n+1)\omega}(u\alpha\wedge\Omega^{n-1}) - (d^\omega u)\wedge\alpha\wedge\Omega^{n-1}] \\ &= [(a_1 g_1^*\alpha - \alpha)\wedge\alpha\wedge\Omega^{n-1}] = [(a_1 g_1^*\alpha)\wedge\alpha\wedge\Omega^{n-1}].\end{aligned}$$

Finally, in view of this formula $\tilde R(g)$ does only depend on the endpoint of g, so $\tilde R$ vanishes on closed loops, i.e. $\Lambda = 0$. □

6.7. Lemma. *Let $\mathcal U$ be a covering by open balls of a l.c.s. manifold (M,Ω,ω) with $H^0_{d_c^\omega}(M) = 0$. Then every $g \in C^\infty\big((I,0),(\ker R,\mathrm{id})\big)$ has a decomposition $g = g_1\cdots g_n$, where every g_i is supported in $U_i \in \mathcal U$ and $g_i \in C^\infty\big((I,0),(\ker R_{U_i},\mathrm{id})\big)$.*

Proof. Fix a compact set $K \subseteq M$ and define

$$H_K : C^\infty\big(I,\Omega_K^{2n-1}(M)\big) \to C^\infty\big((I,0),(\ker R,\mathrm{id})\big) \qquad \alpha \mapsto \mathrm{Evol}\big((\sharp\circ d^\omega)_* u\big)$$

where $u \in C^\infty\big(I,\Omega^0_K(M)\big)$ is the unique function satisfying $d^\omega\alpha_t = u_t\Omega^n$. So the defining equation for $g = H_K(\alpha)$ is $\flat\dot g_t = d^\omega u_t$ with initial condition $g_0 = \mathrm{id}$. We define the structure of a topological group on the left hand side space such that H_K becomes a continuous homomorphism. Namely we set

$$(\alpha\beta)(t) := \alpha(t) + (H_K(\alpha)(t)^{-1})^*(\tfrac{1}{a_t}\beta(t))$$

where $H_K(\alpha)(t)^*\Omega = \frac{1}{a_t}\Omega$. One easily checks that this is a topological group and H_K is a continuous homomorphism. By Lemma 6.4 and Corollary 1.9 we see that $\bigcup_K \mathrm{im}\, H_K = C^\infty\big((I,0),(\ker R,\mathrm{id})\big)$ and so we only have to show that every $g \in \mathrm{im}(H_K)$ has the desired decomposition. From now on the proof is completely similar to the proof of 5.7. □

6.8. Lemma. *Let $\mathcal U$ be a covering by open balls of a l.c.s. manifold (M,Ω,ω) with $H^0_{d_c^\omega}(M) \neq 0$. Then every $g \in C^\infty\big((I,0),(\ker\Psi,\mathrm{id})\big)$ has a decomposition $g = g_1\cdots g_n$, where every g_i is supported in $U_i \in \mathcal U$ and $g_i \in C^\infty\big((I,0),(\ker R_{U_i},\mathrm{id})\big)$.*

Proof. By Example 1.6 ω has to be exact and so (M,Ω,ω) is conformally equivalent to a compact, symplectic manifold. But in this case the statement is well known and can be found in [1] and [3]. □

7. Simplicity Theorem

We start with the following result due to W. Thurston (see [3]).

7.1 Lemma. *Let X be a Hausdorff topological space, $\mathcal U$ be a basis of the topology and $G \subseteq \mathrm{Homeo}(X)$ be a subgroup of homeomorphisms on X. Assume we have for all $U \in \mathcal U$ a perfect subgroup $G_U \subseteq G \cap \mathrm{Homeo}_U(X)$, satisfying:*
(1) every G-orbit is dense in X (weak transitivity)
(2) $\mathcal V \subseteq \mathcal U$ is a covering of X then $\bigcup_{V\in\mathcal V} G_V$ generates G (fragmentation)
(3) $U, V \in \mathcal U$, $g \in G$, $g(U) \subseteq V$ then $gG_Ug^{-1} \subseteq G_V$

Then G is simple.

Proof. Let $\mathrm{id} \neq g \in G$. We want to show $N(g) = G$, where $N(g)$ denotes the normal subgroup in G generated by g. Since $\mathrm{id} \neq g$ we find $x \in X$ with $g(x) \neq x$ and by 1 we also find $h \in G$ with $h(x) \neq x$ and $h(x) \neq g(x)$. Since X is Hausdorff we can separate $x, g(x), h(x)$ by open neighborhoods W_1, W_2, W_3 of $x, g(x), h(x)$ respectively. We let $U := W_1 \cap g^{-1}(W_2) \cap h^{-1}(W_3)$. Then U is an open neighborhood of x and $U, g(U), h(U)$ are pairwise disjoint. We claim

$$[u, v] = [[u, g], [v, h]] \qquad \forall u, v \in \mathrm{Homeo}_U(X) \tag{7.1}$$

Since $U \cap g(U) = \emptyset$ and $U \cap h(U) = \emptyset$ we have

$$[u, g] = \begin{cases} u & \text{on } U \\ gu^{-1}g^{-1} & \text{on } g(U) \\ \mathrm{id} & \text{elsewhere} \end{cases} \qquad [v, h] = \begin{cases} v & \text{on } U \\ hv^{-1}h^{-1} & \text{on } h(U) \\ \mathrm{id} & \text{elsewhere} \end{cases} \tag{7.2}$$

and so (7.1) holds on $M \setminus (g(U) \cup h(U))$. It remains to check $[[u, g], [v, h]]|_{g(U) \cup h(U)} = \mathrm{id}$ but this follows again from (7.2) and the fact $g(U) \cap h(U) = \emptyset$.

¿From (7.1) and the perfectness of G_U we obtain

$$G_U = [G_U, G_U] \subseteq [[G_U, g], [G_U, h]] \subseteq [N(g), G] \subseteq N(g)$$

Now let $y \in X$ be arbitrary. From 1 we get $U_y \in \mathcal{U}$ and $\alpha_y \in G$ with $\alpha_y(U_y) \subseteq U$ and hence using 3

$$G_{U_y} \subseteq \alpha_y^{-1} G_U \alpha_y \subseteq \alpha_y^{-1} N(g) \alpha_y \subseteq N(g)$$

Since $\{U_y : y \in X\}$ covers X, $\bigcup_{y \in X} G_{U_y}$ generates G by 2 and so $G \subseteq N(g)$. □

Now we make use of the following theorem due to A. Banyaga (see [1]).

7.2. Theorem. *Let U be an open ball and Ω a symplectic form on U. Then $\ker R$ is a perfect group, i.e. $\ker R = [\ker R, \ker R]$.*

Our main result is the following:

7.3. Theorem. *Let (M, Ω, ω) be a connected l.c.s. manifold such that $H^0_{d^\omega_c}(M) \neq 0$. Then $\ker \Psi$ is simple and if $H^0_{d^\omega_c}(M) = 0$ then $\ker R$ is simple.*

Proof. We will apply Lemma 7.1, so we have to check its assumptions. If $H^0_{d^\omega_c}(M) \neq 0$ we let $G := \ker \Psi$, $\mathcal{U}$ be the totality of open balls in M and $G_U := \ker R_U$. Theorem 7.2 yields $G_U = [G_U, G_U]$, Lemma 3.5 gives property 1, Lemma 6.8 gives property 2 and property 3 is obvious.

If $H^0_{d^\omega_c}(M) = 0$ we let $G := \ker R$, $\mathcal{U}$ be the open balls in M and $G_U := \ker R_U$. Again Theorem 7.2 yields $G_U = [G_U, G_U]$, Lemma 3.5 gives property 1, Lemma 6.7 gives property 2 and property 3 is obvious. □

7.4. Corollary. *Let (M, Ω, ω) be a connected l.c.s. manifold. If $H^0_{d^\omega_c}(M) \neq 0$ then $\ker \Psi = [\ker \Phi, \ker \Phi]$ is perfect and if $H^0_{d^\omega_c}(M) = 0$ then $\ker R = [\ker \Psi, \ker \Psi]$ is perfect.*

Proof. Everything follows immediately from Theorem 7.3. □

7.5. Corollary. *Let (M, Ω, ω) be a connected l.c.s. manifold which is not g.c.s. (i.e. ω is not exact). Then $\ker \Psi = [\ker \Phi, \ker \Phi]$ is simple (and hence perfect).*

Proof. As mentioned several times we have in this case $R = 0$. □

8. Cohomological Invariants of $[(\Omega, \omega)]$

By $\mathcal{C} = [(\Omega, \omega)]$ we will denote the conformal equivalence class of a l.c.s. (Ω, ω) on M. We will assume that $\dim(M) > 2$ and consider the Ω only.

We follow some ideas of Banyaga [2] (see also references therein). For $g \in \mathrm{Diff}^\infty_c(M, \Omega, \omega)$ we denote $g^*\Omega = \frac{1}{a_g}\Omega$ and for $X \in \mathfrak{X}_c(M, \Omega, \omega)$ we write $L_X\Omega = -u_X\Omega$.

8.1. Proposition. *(i) The mapping*

$$\xi_\Omega : \mathrm{Diff}^\infty_c(M, \Omega, \omega) \to C^\infty(M, \mathbb{R}) \quad g \mapsto \ln(a_{g^{-1}})$$

is a 1-cocycle on the group $\mathrm{Diff}^\infty_c(M, \Omega, \omega)$ *with values in* $C^\infty(M, \mathbb{R})$. *Here we consider* $C^\infty(M, \mathbb{R})$ *as a* $\mathrm{Diff}^\infty_c(M, \Omega, \omega)$*-module:* $g \cdot a = a \circ g^{-1}$. *The class* $\Xi_{\mathcal{C}} := [\xi_\Omega] \in H^1\big(\mathrm{Diff}^\infty_c(M, \Omega, \omega); C^\infty(M, \mathbb{R})\big)$ *is independent of the choice of a representant in* $\mathcal{C}$.

(ii) The mapping

$$\zeta_\Omega : \mathfrak{X}_c(M, \Omega, \omega) \to C^\infty(M, \mathbb{R}) \quad X \mapsto -u_X$$

is a 1-cocycle on the Lie algebra $\mathfrak{X}_c(M, \Omega, \omega)$ *with values in its module* $C^\infty(M, \mathbb{R})$. *Its cohomology class* $Z_{\mathcal{C}} = [\zeta_\Omega]$ *depends on* $\mathcal{C}$ *only.*

The proof is straightforward, see, e.g., [2].

8.2. Proposition. *The following statements are equivalent:*

(1) $\mathcal{C}$ *is g.c.s.*

(2) $\Xi_{\mathcal{C}} = 0$

(3) $Z_{\mathcal{C}} = 0$

Proof. First we show that (1) $\Rightarrow$ (2). Choose a symplectic structure Ω on M representing $\mathcal{C}$, i.e. $\omega = 0$. If $g \in \mathrm{Diff}^\infty_c(M, \Omega, 0)$ with $g^*\Omega = \frac{1}{a_g}\Omega$ we get $0 = dg^*\Omega = -\frac{1}{a_g^2} da_g \wedge \Omega$ and so a_g is constant, since Ω is non-degenerated and $\dim(M) > 2$. If $K := \mathrm{supp}(g)$ we obtain

$$\mathrm{vol}(K) = \textstyle\int_K \Omega^n = \int_{g^*K} g^*\Omega^n = \int_K \frac{1}{a_g^n}\Omega^n = \frac{1}{a_g^n}\,\mathrm{vol}(K)$$

and so we must have $a_g = 1$. Consequently $\xi_\Omega(g) = 0$ and so $\Xi_{\mathcal{C}} = 0$. (2) $\Rightarrow$ (3) follows immediately from the formula:

$$\zeta_\Omega(X) = \tfrac{d}{dt}|_0\big(\xi_\Omega(\mathrm{Fl}^X_{-t})\big) \tag{8.1}$$

Indeed if $\Xi_{\mathcal{C}} = 0$ then there exists $u \in C^\infty(M, \mathbb{R})$ with $\xi_\Omega(g) = (g^{-1})^*u - u$ for all $g \in \mathrm{Diff}^\infty_c(M, \Omega, \omega)$, and so (8.1) yields $\zeta_\Omega(X) = \frac{d}{dt}|_0\big((\mathrm{Fl}^X_t)^*u - u\big) = L_X u$ for all $X \in \mathfrak{X}_c(M, \Omega, \omega)$, i.e. $Z_{\mathcal{C}} = 0$. Finally we show (3) $\Rightarrow$ (1). Since $Z_{\mathcal{C}} = 0$ there exists $u \in C^\infty(M, \mathbb{R})$ such that $\zeta_\Omega(X) = L_X u$ for all $X \in \mathfrak{X}_c(M, \Omega, \omega)$. Consider the l.c.s. structure (Ω', ω'), where $\Omega' := \frac{1}{e^{-u}}\Omega$. We claim that $\omega' = 0$. An easy calculation shows $L_X\Omega' = 0$ for all $X \in \mathfrak{X}_c(M, \Omega', \omega') = \mathfrak{X}_c(M, \Omega, \omega)$. From this we get $L_X\omega' = 0$, that is $i_X\omega'$ is constant for all $X \in \mathfrak{X}_c(M, \Omega, \omega)$. If ω' would be non-zero then we could easily construct $X = \sharp d^\omega v \in \mathfrak{X}_c(M, \Omega, \omega)$ with $i_X\omega'$ non-constant, a contradiction. So $\omega' = 0$ and (M, Ω, ω) is conformally equivalent to the symplectic manifold (M, Ω', ω'). $\square$

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