

## On the Group of Isometries on a Locally Compact Metric Space

Antonios Manoussos and Polychronis Strantzalos\*

Communicated by W. A. F. Ruppert

**Abstract.** In the present paper we study conditions under which the group of isometries on a locally compact metric space is locally compact, or acts properly.

*Keywords:* Ellis semigroup, isometry, pointwise convergence, proper action.

*Mathematics Subject Classification:* Primary: 54H20, 37B; Secondary: 54E15, 54H15.

### 1. Introduction

It is long known from the work of van Dantzig and van der Waerden ([1], cf. also [2, Ch.I, Th.4.7]) that if  $(X, d)$  is a connected locally compact metric space then its group of isometries  $I(X, d)$ , when endowed with the topology of pointwise convergence, is always locally compact and acts properly on  $X$ . More recently it was shown by one of the authors ([6]) that the pointwise closure of  $I(X, d)$  is locally compact if the space  $\Sigma(X)$  of the connected components of  $X$  is quasicompact (compact but not necessarily Hausdorff) with respect to the quotient topology. The question whether  $I(X, d)$  is closed in  $C(X, X)$  (the space of all continuous selfmaps of  $X$  endowed with the topology of pointwise convergence) remained open. In this note we fill this gap (cf. also [4]), i.e., we show that if  $\Sigma(X)$  is quasicompact then  $I(X, d)$  coincides with its Ellis' semigroup, completing the proof of the following:

**Theorem.** *Let  $(X, d)$  be a locally compact metric space. Denote by  $I(X, d)$  its group of isometries, with the topology of pointwise convergence, and by  $\Sigma(X)$  the space of the connected components of  $X$ , endowed with the quotient topology. Then*

1. *If  $\Sigma(X)$  is not quasicompact, then  $I(X, d)$  need not be locally compact, nor act properly on  $X$ .*
2. *If  $\Sigma(X)$  is quasicompact then*

---

\* The authors want to thank the referee for his useful and constructive comments.

- (a)  $I(X, d)$  is locally compact,
- (b) the action  $(I(X, d), X)$  is not always proper, and
- (c) the action  $(I(X, d), X)$  is proper if  $X$  is connected.

For the sake of completeness, we give short and slightly improved proofs of some of the previously published partial results of the authors, these are crucial for a unified proof of the above theorem. Our treatment is based on the sets  $(x, V_x) = \{g \in I(X, d) : g(x) \in V_x\}$ , where  $V_x$  is a neighborhood of  $x \in X$ . These sets form a neighborhood subbasis at the identity with respect to the topology of pointwise convergence, the natural topology of  $I(X, d)$ .

## 2. Generalities

**2.1.** The following simple examples establish 1 and 2(b) of the above theorem.

**Example.** Let  $X = \mathbb{Z}$  with the discrete metric. Obviously  $\Sigma(X)$  is not quasi-compact. It can be easily seen that  $I(X, d)$  is the group of all bijections of  $\mathbb{Z}$ , which is not locally compact with respect to the topology of pointwise convergence, therefore it cannot act properly on a locally compact space.

**Example.** Let  $X = Y \cup \{(1, 0)\} \subset \mathbb{R}^2$  where  $Y = \{(0, y) : y \in \mathbb{R}\}$ , and  $d = \min\{1, \delta\}$ , where  $\delta$  denotes the Euclidean metric. As we shall see in §3, by Theorem 3.7,  $I(X, d)$  is locally compact; however the action  $(I(X, d), X)$  is not proper, because the isotropy group of  $(1, 0)$  is not compact, since it contains the translations of  $Y$ . So, the action of  $I(X, d)$  on  $X$  is not proper, even if  $X$  has two components.

Since the sets  $(x, V_x)$  as above form a neighborhood subbasis at the identity in  $I(X, d)$ , the following condition is necessary for the local compactness of  $I(X, d)$ :

- (a) There exist  $x_i \in X$ ,  $i = 1, \dots, m$  such that  $\bigcap_{i=1}^m (x_i, V_{x_i})$  is relatively compact in  $C(X, X)$ .

This condition becomes also sufficient if, in addition, the following condition is satisfied:

- (b)  $I(X, d)$  is closed in  $C(X, X)$ .

So, to prove that  $I(X, d)$  is locally compact, we have to ensure that both of the above conditions are satisfied.

## 3. The local compactness of $I(X, d)$

The following is crucial for the investigation of the conditions 2.1(a) and (b):

**3.1. Lemma.** *Let  $(X, d)$  be a locally compact metric space,  $F \subseteq I(X, d)$ , and*

$$K(F) = \{x \in X : F(x) = \{f(x) : f \in F\} \text{ is relatively compact}\}.$$

Then  $K(F)$  is an open and closed subset of  $X$ .

**Proof.** Since  $F$  is an equicontinuous family of selfmaps of  $X$  we see that  $K(F)$  is open. It remains to prove that  $K(F)$  is closed.

We write  $S(x, \eta) = \{y \in X \mid d(x, y) < \eta\}$  for any  $x \in X$  and  $\eta > 0$ , and  $S(M, \eta) = \bigcup\{S(x, \eta) \mid x \in M\}$  for subsets  $M \subseteq X$ . Let  $x$  be a cluster point of  $K(F)$  and let  $\eta$  be a positive real such that  $S(x, 5\eta)$  is relatively compact. Choose a point  $k \in K(F) \cap S(x, \eta)$ . Then  $\overline{F(k)} \subseteq S(F(S(x, \eta)), \eta) = F(S(x, 2\eta))$ , and by the compactness of  $\overline{F(k)}$  we can find a finite subset  $L \subseteq F$  such that  $\overline{F(k)} \subseteq L(S(x, 2\eta))$ . We show that  $F(x)$  is contained in the relatively compact set  $L(S(x, 5\eta))$ . To see this, pick  $f \in F$  and let  $g \in L$  such that  $f(k) \in g(S(x, 2\eta))$ . Then

$$\begin{aligned} d(f(x), g(k)) &\leq d(f(x), f(k)) + d(f(k), g(x)) + d(g(x), g(k)) \\ &= d(x, k) + d(f(k), g(x)) + d(x, k) \leq 4\eta \end{aligned}$$

and therefore

$$f(x) \in S(g(k), 4\eta) = g(S(k, 4\eta)) \subseteq g(S(x, 5\eta)) \subseteq L(S(x, 5\eta)).$$

Thus  $x \in K(F)$  and the proof is finished.

**3.2. Remark.** In the sequel we assume that  $\Sigma(X)$  is *quasicompact* in the quotient topology via the natural map  $q : X \rightarrow \Sigma(X)$ . Note that  $\Sigma(X)$  is a  $T_1$ -space, and need not be Hausdorff. Nevertheless

$X$  is separable, hence second countable; so sequences are adequate in  $C(X, X)$ .

The proof is similar to the lengthy one in [5] (see also [2, Appendix 2]).

**3.3. Lemma.** Let  $(X, d)$  be a locally compact metric space with a quasicompact space of connected components  $\Sigma(X)$ . Then condition 2.1(a) is satisfied.

**Proof.** Let  $V_x$  be a relatively compact neighborhood of  $x \in X$ . Then

$$(x, V_x) = \{g \in I(X, d) : g(x) \in V_x\}$$

is a neighborhood of the identity in  $I(X, d)$ . Since  $x \in K((x, V_x))$ ,  $K((x, V_x))$  is not empty, and by Lemma 3.1 is open and contains entire components of  $X$ . Therefore  $q(K((x, V_x)))$  is an open subset of  $\Sigma(X)$ . Since  $\Sigma(X)$  is quasicompact, there are  $x_i$ ,  $i = 1, \dots, m$ , such that the corresponding  $q(K((x_i, V_{x_i})))$ 's cover  $\Sigma(X)$ . This means that  $X = \bigcup_{i=1}^m K((x_i, V_{x_i}))$ , i.e., the neighborhood  $F = \bigcap_{i=1}^m (x_i, V_{x_i})$  of the identity has the property: for every  $x \in X$  the set  $F(x)$  is relatively compact in  $X$ . Therefore, by Ascoli's theorem,  $F$  is relatively compact in  $C(X, X)$ .

**3.4.** Now we prove that if  $\Sigma(X)$  is quasicompact then  $I(X, d)$  is a closed subspace of  $C(X, X)$ . Because of Remark 3.2, the elements  $f$  of the boundary of  $I(X, d)$  in  $C(X, X)$  are limits of sequences  $\{f_n \in I(X, d), n \in \mathbb{N}\}$ . Obviously, such an  $f$  preserves  $d$ ; so the question is whether  $f$  is surjective. If  $\Sigma(X)$  is not quasicompact then this is not always true:

**Example.** Let  $X = \mathbb{Z}$  with the discrete metric. If  $f_n(z) = z$  for  $-n < z < 0$ ,  $f_n(-n) = 0$ , and  $f_n(z) = z+1$  otherwise, then  $f_n \rightarrow f$ , where  $f(z) = z$  for  $z < 0$ , and  $f(z) = z+1$  for  $z \geq 0$ . Hence each  $f_n$  is an isometry, but  $f$  is not surjective since  $0 \notin f(\mathbb{Z})$ .

**3.5. Lemma.** *If  $\Sigma(X)$  is quasicompact and  $\{(f_n) : f_n \in I(X, d)\}$  is a sequence such that  $f_n \rightarrow f$  for some selfmap  $f$  of  $X$  with respect to the topology of pointwise convergence, then  $f(X)$  is open and closed in  $X$ .*

**Proof.** By Lemma 3.1, it suffices to show that  $f(X) = K(F)$ , where  $F = \{f_n^{-1}, n \in \mathbb{N}\}$ . Indeed, since  $d(f_n(x), f(x)) = d(x, f_n^{-1}(f(x)))$ , we have  $f_n^{-1}(f(x)) \rightarrow x$ , so (since  $X$  is locally compact)  $f(x) \in K(F)$ , for every  $x \in X$ . Now, if  $y \in K(F)$ , we may assume  $f_{n_k}^{-1}(y) \rightarrow x$  for some  $x \in X$ , because  $F(y)$  is relatively compact in  $X$ , hence  $f(x) = y$ .

**3.6. Proposition.** *If  $(X, d)$  is a locally compact metric space, and  $\Sigma(X)$  is quasicompact, then  $I(X, d)$  is closed in  $C(X, X)$ .*

**Proof.** Let  $\{(f_n) : f_n \in I(X, d)\}$  be a sequence such that  $f_n \rightarrow f$  for some selfmap  $f$  of  $X$  with respect to the topology of pointwise convergence. We prove that  $f$  is surjective. Let  $y \in X$ . We denote by  $S_x$  the connected component containing  $x \in X$ , and by  $S_n$  the component of  $f_n^{-1}(y)$ . If  $\{S_n, n \in \mathbb{N}\}$  has a constant subnet  $\{S_{n_i}, i \in I\}$ , then  $S_{n_i} = S_0$ , for some  $S_0 \in \Sigma(X)$ . Hence  $S_{f_{n_i}^{-1}(y)} = S_0$ , so  $f_{n_i}(S_0) = S_y$ , for every  $i \in I$ . Pick an  $x \in S_0$ , then  $f_{n_i}(x) \in S_y$ . But  $f_{n_i}(x) \rightarrow f(x)$ , so  $f(x) \in S_y$ . By Lemma 3.5  $S_y \subseteq f(X)$ , hence  $y \in f(X)$ .

Suppose that  $\{S_n, n \in \mathbb{N}\}$  has no constant subnet. By the quasicompactness of  $\Sigma(X)$ , there exists a subnet  $\{S_{n_i}, i \in I\}$  of  $\{S_n, n \in \mathbb{N}\}$  such that  $S_{n_i} \rightarrow S$ , for some  $S \in \Sigma(X)$ . With the above notation, the following is true:

**Claim.** *There exists a subsequence  $\{S_k, k \in \mathbb{N}\}$  of  $\{S_n, n \in \mathbb{N}\}$  such that there are  $x_k \in S_k$  with  $x_k \rightarrow x_0$ , for some  $x_0 \in X$ .*

*Proof.* If not,  $R = (\bigcup_{n=1}^{\infty} S_n) \setminus S$  is closed in  $X$ . Indeed, let  $\{(y_m) : y_m \in R\}$  be a sequence such that  $y_m \rightarrow y \in X$ . If  $y_m \in (\bigcup_{n=1}^{n_0} S_n) \setminus S$  for  $m > m_0$ , then a subsequence of  $\{y_m, m \in \mathbb{N}\}$  is contained in some  $S_i$  for some  $i \in \{1, \dots, n_0\}$ , therefore  $y \in S_i \subseteq R$ , as required. If this is not the case, we construct a subsequence  $\{y_{m_p}, p \in \mathbb{N}\}$  of  $\{y_m, m \in \mathbb{N}\}$  in the following way: For  $S_1$  we choose a point  $y_{m_1} \in S_{n_1}$  with  $n_1 > 1$  and  $d(y_{m_1}, y) < 1$ , for  $(\bigcup_{n=1}^{n_1} S_n) \setminus S$  a point  $y_{m_2} \in S_{n_2}$  with  $n_2 > n_1$  and  $d(y_{m_2}, y) < \frac{1}{2}$ , and so on. Obviously,  $y_{m_p} \in S_{n_p}$  and  $y_{m_p} \rightarrow y$ , a contradiction.

Since  $S$  does not meet  $R$ , then  $S \subseteq X \setminus R$ . On the other hand  $X \setminus R$  is open (since  $R$  is closed in  $X$ ) and contains entire components (recall that  $R$  is a union of components), so  $S_{n_i} \subseteq X \setminus R$ , eventually. Therefore  $S_{n_i} = S$ , a contradiction, since we have assumed that  $\{S_n, n \in \mathbb{N}\}$  has no constant subnet.

According to the Claim, there exists a sequence  $\{(x_k) : x_k \in S_k\}$  such that  $x_k \rightarrow x_0 \in X$ , where  $S_k = S_{f_k^{-1}(y)} = f_k^{-1}(S_y)$ , from which follows  $x_k = f_k^{-1}(y_k)$  for

some  $y_k \in S_y$ . Then

$$\begin{aligned} d(y_k, f(x_0)) &\leq d(y_k, f_k(x_0)) + d(f_k(x_0), f(x_0)) \\ &= d(f_k^{-1}(y_k), x_0) + d(f_k(x_0), f(x_0)) \rightarrow 0, \end{aligned}$$

therefore  $f(x_0) \in S_y$ , which means that  $S_y \cap f(X) \neq \emptyset$  and, by Lemma 3.5,  $S_y \subseteq f(X)$ , hence  $y \in f(X)$ , and  $f$  is surjective.

**3.7. Theorem.** *If  $\Sigma(X)$  is quasicompact, then  $I(X, d)$  is locally compact.*

**Proof.** This assertion follows from Lemma 3.3 and Proposition 3.6, since both conditions 2.1(a) and (b) are satisfied.

#### 4. The properness of the action $(I(X, d), X)$

In this short section, applying the methods used previously, we give a complete proof of the following:

**Proposition.** *If  $(X, d)$  is locally compact and connected, then  $I(X, d)$  is locally compact and the action  $(I(X, d), X)$  is proper.*

**Proof.** Since  $X$  is connected  $G = I(X, d)$  is locally compact by Theorem 3.7. So, we have to show that, for every  $x, y \in X$ , there are neighborhoods  $U_x, U_y$  of  $x$  and  $y$  respectively such that

$$(U_x, U_y) := \{g \in G : (gU_x) \cap U_y \neq \emptyset\}$$

is relatively compact in  $G$ . Let  $U_x = S(x, \varepsilon)$  and  $U_y = S(y, \varepsilon)$  be such that  $S(y, 2\varepsilon)$  is relatively compact. Then, for  $g \in (U_x, U_y)$  and  $z \in U_x$  with  $g(z) \in U_y$ , we have

$$d(g(x), (y)) \leq d(g(x), g(z)) + d(g(z), y) = d(x, z) + d(g(z), y) < 2\varepsilon,$$

therefore  $g \in F = \{g \in G : g(x) \in S(y, 2\varepsilon)\}$ . Then  $x \in K(F)$ , and, according to Lemma 3.1,  $K(F)$  coincides with the connected space  $X$ . From this and Ascoli's theorem it follows that  $F$  is relatively compact in  $C(X, X)$ . So  $(U_x, U_y) \subseteq F$  is relatively compact in  $C(X, X)$ , hence in  $G$ , because  $G$  is closed (cf. Proposition 3.6).

This proves the Proposition and completes the proof of the Theorem in the Introduction.

#### 5. Final Remark

Using the same arguments we can prove that if  $X$  is a locally compact metrizable space, then  $I(X, d)$  is locally compact for all admissible metrics  $d$ , provided that the space  $Q(X)$  of the quasicomponents of  $X$  is compact with respect to the quotient topology (note that  $Q(X)$  is always Hausdorff) (cf. [3]). Recall that the

quasicomponent of a point is the intersection of all open and closed sets which contain it. Our exposition is given via  $\Sigma(X)$  because we regard the condition " $\Sigma(X)$  is quasicompact" as a topologically more natural condition than " $Q(X)$  is compact", although it is more restrictive: There are locally compact metric spaces with compact  $Q(X)$  and non quasicompact  $\Sigma(X)$  as the following example shows:

**Example.** The space of all connected components of the locally compact space

$$X = \left( \bigcup_{n=1}^{\infty} \left\{ \left( \frac{1}{n}, y \right) : y \in [-1, 1] \right\} \right) \cup \left\{ (0, y) : y \in [-1, 0] \right\} \cup \left( \bigcup_{k=1}^{\infty} I_k \right) \subseteq \mathbb{R}^2,$$

where

$$I_k = \left\{ (0, y) : y \in \left( \frac{1}{k+1}, \frac{1}{k} \right) \right\}, \quad k \in \mathbb{N}^*,$$

is not quasicompact, because the sequence  $\{I_k\} \subseteq \Sigma(X)$  does not have a convergent subsequence in  $\Sigma(X)$ . On the contrary,  $Q(X)$  is compact, because the quasicomponent of the point  $(0, -1)$  consists of the set  $\{(0, y) : y \in [-1, 0]\}$  and the intervals  $I_k$ ,  $k \in \mathbb{N}^*$ .

So the quasicompactness of  $\Sigma(X)$  is not necessary for the local compactness of  $I(X, d)$ .

### References

- [1] van Dantzig, D., and B. L. van der Waerden, *Über metrisch homogene Räume*, Abh. Math. Sem. Hamburg **6** (1928), 367–376.
- [2] Kobayashi, S., and K. Nomizu, "Foundations of Differential Geometry, Vol. I," Interscience, New York, 1963.
- [3] Manoussos, A., *Contribution to the study of D-stable actions*, Ph.D. Thesis, University of Athens, Greece (1993) (in Greek).
- [4] Manoussos, A., *The Ellis semigroup of the full group of isometries of a locally compact metric space*, Preprint.
- [5] Sierpinsky, W., *Sur les espaces métriques localement séparables*, Fund. Math. **21** (1933), 107–113.
- [6] Strantzalos, P., *Actions by Isometries*, Lecture Notes in Math. **1375** (1989), 319–325.

Antonios Manoussos  
123 Sapphos Street  
176 75 Kallithea  
Athens  
Greece  
amanou@cc.uoa.gr

Polychronis Strantzalos  
Department of Mathematics  
University of Athens  
Athens, GR – 157 84  
Greece  
pstrantz@math.uoa.gr

Received August 25, 2000  
and in final form March 9, 2002