

On the group of self-homotopy equivalences of H -spaces of rank 2

By

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Dedicated to Professor T. Kudo on his 60-th birthday

§1. Introduction.

The set $\mathcal{E}(X)$ of homotopy classes of homotopy equivalences of a based space X to itself forms a group under composition of maps. This group is called the group of self-homotopy equivalences of X . The group $\mathcal{E}(X)$ has been studied by several authors (e.g. [2], [5], [10], [12], [14]).

In the present paper, we study the group $\mathcal{E}(X)$ for a simply connected, finite H -complex X of rank 2. The classification of simply connected, finite H -complexes of rank 2 has been given in [8] as follows: X is homotopy equivalent to one of $S^3 \times S^3$, $SU(3)$, E_k ($k=0, 1, 3, 4, 5$), $S^7 \times S^7$ and $G_{2,b}$ ($-2 \leq b \leq 5$). Here E_k is the principal S^3 -bundle over S^7 with the characteristic class $k\omega \in \pi_7(BS^3) \cong \mathbb{Z}_{12}$, ω a generator. For example $E_0 = S^3 \times S^7$, $E_1 = Sp(2)$ and $G_{2,b}$ is the principal S^3 -bundle over the Stiefel manifold $V_{7,2} = SO(7)/SO(5)$ induced by a suitable map $f_b: V_{7,2} \rightarrow BS^3$ (see §3 for details) such that $G_{2,0} = G_2$ is the compact, exceptional Lie group G_2 of rank 2.

For torsion free, finite H -complexes X of rank 2 which have been classified in [4], [17], the group $\mathcal{E}(X)$ is already known, that is, for $S^i \times S^j$ ($i, j=1, 3, 7$) in [13], [14], for $SU(3)$ and $Sp(2)$ in [10], and for E_k ($k \neq 0, 1$) in [12]. So we will determine $\mathcal{E}(G_{2,b})$ for $-2 \leq b \leq 5$.

In a short exact sequence: $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 1$, we write the group composition in A as addition, and it in B and C as multiplication.

Then the following is our main result obtained in §4.

Main Theorem. *We have the following exact sequences:*

$$(i) \quad 0 \longrightarrow D(\mathbb{Z}_{168} \oplus \mathbb{Z}_2) \longrightarrow \mathcal{E}(G_{2,b}) \longrightarrow \mathbb{Z}_2 \longrightarrow 1 \quad (b = -1, 0, 2, 3, 5),$$

$$(ii) \quad 0 \longrightarrow D(\mathbb{Z}_{168} \oplus \mathbb{Z}_6) \longrightarrow \mathcal{E}(G_{2,b}) \longrightarrow \mathbb{Z}_2 \longrightarrow 1 \quad (b = 1, 4),$$

$$(iii) \quad 0 \longrightarrow \mathbb{Z}_{168} \oplus \mathbb{Z}_6 \longrightarrow \mathcal{E}(G_{2,-2}) \longrightarrow G \longrightarrow 1,$$

$$0 \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 1,$$

where, for an abelian group H , $D(H)$ is a group given by the split exact se-

quence: $0 \rightarrow H \rightarrow D(H) \rightarrow \mathbf{Z}_2 \rightarrow 1$, with the splitting action \mathbf{Z}_2 on H given by $(-1) \cdot h = -h$ for $-1 \in \mathbf{Z}_2$ and $h \in H$.

The paper is organized as follows. The Barcus-Barratt theorem is introduced in §2. In §3 some results on homotopy of $G_{2,b}$, which will be needed in §4, are prepared. In §4 we study the group $\mathcal{E}(G_{2,b})$ by making use of the results of the previous sections and we obtain the main theorem. In the last section, §5, we give a proof of the lemma used in §4.

Throughout the paper, all spaces have homotopy types of CW-complexes with base points and all (continuous) maps and homotopies preserve the base points. For given spaces X and Y , we denote by $[X, Y]$ the set of (based) homotopy classes of maps from X to Y , and by the same letter f a map $f: X \rightarrow Y$ and its homotopy class $f \in [X, Y]$. The integral coefficient of the homology is omitted: $H_i(X) = H_i(X; \mathbf{Z})$. $X^{(n)}$ stands for the n -skeleton of X and $\pi_i(X; p)$ the p -component of $\pi_i(X)$ and \mathbf{Q}_p the ring of those fractions, whose denominators, in the lowest form, are prime to p .

§2. The theorem of Barcus-Barratt.

Let K be a simply connected CW-complex of finite dimension. Let

$$S^q \xrightarrow{\alpha} K \xrightarrow{i} K \cup_{\alpha} e^{q+1} \xrightarrow{p} S^{q+1}, \quad q > \dim K,$$

be the sequence of induced cofiberings. The coaction

$$l: K \cup e^{q+1} \rightarrow (K \cup e^{q+1}) \vee S^{q+1}$$

is defined by shrinking the equator $S^q \times \{1/2\}$ of e^{q+1} . We define a map

$$(2.1) \quad A: \pi_{q+1}(K \cup e^{q+1}) \rightarrow [K \cup e^{q+1}, K \cup e^{q+1}]$$

by $A(\xi) = \nabla \circ (1 \vee \xi) \circ l$, where ∇ is the folding map and 1 is the class of identity map of $K \cup e^{q+1}$. Furthermore, since $q > \dim K$, by the restriction A on $i_* \pi_{q+1}(K)$ we can define a homomorphism (cf. Lemmas 1.4 and 1.8 of [10])

$$\lambda: i_* \pi_{q+1}(K) \rightarrow \mathcal{E}(K \cup e^{q+1}).$$

Since $q > \dim K$, the induced maps

$$(2.2) \quad \begin{aligned} i_*: [K, K] &\rightarrow [K, K \cup e^{q+1}] \quad \text{and} \\ p_*: [S^{q+1}, S^{q+1}] &\rightarrow [K \cup e^{q+1}, S^{q+1}] \end{aligned} \quad \text{are both bijective.}$$

So the maps $\Phi: [K \cup e^{q+1}, K \cup e^{q+1}] \rightarrow [K, K]$ and $\Psi: [K \cup e^{q+1}, K \cup e^{q+1}] \rightarrow [S^{q+1}, S^{q+1}]$ can be defined by the following homotopy commutative diagram:

$$\begin{array}{ccccc} K & \xrightarrow{i} & K \cup e^{q+1} & \xrightarrow{p} & S^{q+1} \\ \downarrow \Phi(h) & & \downarrow h & & \downarrow \Psi(h) \\ K & \xrightarrow{i} & K \cup e^{q+1} & \xrightarrow{p} & S^{q+1}. \end{array}$$

Since these two maps preserve composition of maps, a homomorphism

$$\phi \times \psi : \mathcal{E}(K \cup e^{q+1}) \longrightarrow \mathcal{E}(K) \times \mathcal{E}(S^{q+1})$$

can be defined by the restriction of $\Phi \times \Psi$. Let $g : (S^q, s_0) \rightarrow (K, k_0)$ be a map representing $\alpha \in \pi_q(K)$. Let X^Y be the function space of maps: $(Y, y_0) \rightarrow (X, x_0)$. Then Barcus-Barratt defined in [2] a homomorphism

$$\alpha_i : \pi_1(X^K, i) \xrightarrow{g^*} \pi_1(X^{S^q}, i \circ g) \xrightarrow{(i \circ g)_!} \pi_{q+1}(X).$$

Here $X = K \cup e^{q+1}$, $g^*(F) = F \circ (g \times 1_I)$, 1_I is the identity of $I = [0, 1]$, $(i \circ g)_!(F) = d(F, (i \circ g)^b)$ where $(i \circ g)^b : S^q \times I \rightarrow K \cup e^{q+1}$ is the map defined by $(i \circ g)^b(s, t) = i \circ g(s)$, and $d(F, (i \circ g)^b)$ is the separation element of F and $(i \circ g)^b$ (see [2] for the definition). Then we have the following theorem due to Barcus-Barratt.

Theorem 2.1. (Theorem 6.1 of [2]) *The following sequence is exact:*

$$0 \longrightarrow \Delta \longrightarrow i_* \pi_{q+1}(K) \xrightarrow{\lambda} \mathcal{E}(K \cup e^{q+1}) \xrightarrow{\phi \times \psi} G \longrightarrow 1.$$

Here $\Delta = i_* \pi_{q+1}(K) \cap \alpha_i \pi_1((K \cup e^{q+1})^K, i)$, the subgroup G of $\mathcal{E}(K) \times \mathcal{E}(S^{q+1})$ is isomorphic to

$G_1 = \{h \in \mathcal{E}(K) \mid h_* \alpha = \varepsilon \alpha, \varepsilon = \pm 1, \text{ in } \pi_q(K)\}$, if $2\alpha \neq 0$,
and G is isomorphic to $G_1 \times \mathbb{Z}_2$, if $2\alpha = 0$.

The next corollary will be used in the later section.

Corollary 2.2. (cf. Remark in p. 304 of [12]) *If $K \cup e^{q+1}$ has a multiplication, the homomorphism λ and the group Δ are given as follows:*

$$\begin{aligned} \lambda(\xi) &= 1 + \xi \circ p, & \xi &\in i_* \pi_{q+1}(K), \\ \Delta &= (S\alpha)^* [SK, K \cup e^{q+1}]. \end{aligned}$$

Therefore we have the exact sequence:

$$(2.3) \quad 0 \longrightarrow H \longrightarrow \mathcal{E}(K \cup e^{q+1}) \longrightarrow G \longrightarrow 1,$$

where G is given in the above theorem, and H is given as follows:

$$(2.4) \quad H = i_* \pi_{q+1}(K) / (S\alpha)^* [SK, K \cup e^{q+1}].$$

Proof. By the definition of λ , we have a homotopy commutative diagram:

$$\begin{array}{ccccccc} K \cup e^{q+1} & \xrightarrow{l} & (K \cup e^{q+1}) \vee S^{q+1} & \xrightarrow{1 \vee \xi} & (K \cup e^{q+1}) \vee (K \cup e^{q+1}) & \xrightarrow{\nabla} & K \cup e^{q+1} \\ \downarrow d & & \searrow j & & \searrow j & & \uparrow m \\ (K \cup e^{q+1}) \times (K \cup e^{q+1}) & \xrightarrow{1 \times p} & (K \cup e^{q+1}) \times S^{q+1} & \xrightarrow{1 \times \xi} & (K \cup e^{q+1}) \times (K \cup e^{q+1}), & & \end{array}$$

where m is a multiplication on $K \cup e^{q+1}$, j is the inclusion and d is the diagonal map. Hence we have $\lambda(\xi)=1+\xi \circ p$ and so we have

$$\lambda^{-1}(1)=p^{*-1}(0) \cap i_* \pi_{q+1}(K).$$

Consider the following commutative diagram consisting of the Puppe exact sequence :

$$\begin{array}{ccccc} [SK, K \cup e^{q+1}] & \xrightarrow{(S\alpha)^*} & [S^{q+1}, K \cup e^{q+1}] & \xrightarrow{p^*} & [K \cup e^{q+1}, K \cup e^{q+1}] \\ & & \downarrow p_* & & \downarrow p_* \\ & & [S^{q+1}, S^{q+1}] & \xrightarrow{p^*} & [K \cup e^{q+1}, S^{q+1}] \end{array}$$

where the lower p^* is bijective by (2.2). We have

$$(S\alpha)^*[SK, K \cup e^{q+1}] \subset \text{Ker} \{p_* : \pi_{q+1}(K \cup e^{q+1}) \longrightarrow \pi_{q+1}(S^{q+1})\} = i_* \pi_{q+1}(K)$$

since $\pi_{q+1}(K) \xrightarrow{i_*} \pi_{q+1}(K \cup e^{q+1}) \xrightarrow{p_*} \pi_{q+1}(S^{q+1}) \longrightarrow \dots$ is exact. Therefore we have

$$p^{*-1}(0)=(S\alpha)^*[SK, K \cup e^{q+1}] \subset i_* \pi_{q+1}(K).$$

So, we have $\Delta=\lambda^{-1}(1)=(S\alpha)^*[SK, K \cup e^{q+1}]$. q.e.d.

§ 3. Some homotopy of $G_{2,b}$.

Let G_2 be the compact, exceptional Lie group of rank 2. Let $f: V_{7,2} \rightarrow BS^3$ be the classifying map of G_2 . Let $\phi: V_{7,2} \rightarrow V_{7,2} \vee S^{11}$ be the map shrinking the equator $S^{10} \times \{1/2\}$ in $V_{7,2} = M^6 \cup CS^{10}$. Let α be a generator of $\pi_{11}(BS^3) \cong \pi_{10}(S^3) \cong \mathbf{Z}_{15}$ which corresponds to 8ω under the monomorphism: $\pi_{10}(S^3) \rightarrow \pi_{10}(G_2^{(9)})$ (see Lemma 3.9). For each integer b , let $g_b: S^{11} \rightarrow BS^3$ represent $b\alpha$ and let $G_{2,b}$ be the principal S^3 -bundle over $V_{7,2}$ induced by the composition

$$f_b = \nabla \circ (f \vee g_b) \circ \phi : V_{7,2} \longrightarrow V_{7,2} \vee S^{11} \longrightarrow BS^3 \vee BS^3 \longrightarrow BS^3.$$

For example, $G_2 = G_{2,0}$.

Recall the following

Theorem 3.1. (Theorem 5.1 of [8]) *Let X be a 1-connected, finite H-complex of rank 2 such that $H_*(X; \mathbf{Z})$ has 2-torsion. Then X is homotopy equivalent to $G_{2,b}$ for some b . There are just 8 homotopy types of such H-complexes: $G_{2,b}$ for $-2 \leq b \leq 5$.*

By making use of the exact sequence associated with the fibering $SU(3) \xrightarrow{i} G_2 \xrightarrow{p} S^6$, one can compute $\pi_i(G_2; 2)$ (the odd primary components of $\pi_i(G_2)$ are computed by the killing-homotopy method).

Lemma 3.2. ([6]) $\pi_i(G_2)$ for $i \leq 14$ are as follows:

i	1	2	3	4	5	6	7	8	9	10
$\pi_i(G_2)$	0	0	\mathbf{Z}	0	0	\mathbf{Z}_8	0	\mathbf{Z}_2	\mathbf{Z}_6	0
gen. of 2-comp.			$i_*\iota_3$					$\langle \eta_6^2 \rangle$	$\langle \eta_6^2 \rangle \circ \eta_8$	

11	12	13	14
$\mathbf{Z} \oplus \mathbf{Z}_2$	0	0	$\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_{21}$
$\langle 2\Delta\iota_{13} \rangle, i_*[\nu_5^2]$			$\langle \bar{\nu}_6 + \epsilon_6 \rangle, i_*[\nu_5^2] \circ \nu_{11}$

where the notation $[\alpha]$ means such an element of $\pi_i(SU(3): 2)$ that $q_*[\alpha] = \alpha \in \pi_i(S^5: 2)$ for the projection $q: SU(3) \rightarrow S^5 = SU(3)/SU(2)$, and the notation $\langle \beta \rangle$ means such an element of $\pi_i(G_2: 2)$ that $p_*\langle \beta \rangle = \beta \in \pi_i(S^6: 2)$ for the projection $p: G_2 \rightarrow S^6$.

By [8, §6] we have

$$G_{2,b} \underset{p}{\simeq} G_2 \text{ for } p \neq 3, 5,$$

$$G_{2,b} \underset{3}{\simeq} \begin{cases} G_2 & (b = -1, 0, 2, 3, 5) \\ S^3 \times S^{11} & (b = -2, 1, 4), \end{cases}$$

$$G_{2,b} \underset{5}{\simeq} \begin{cases} G_2 & (b = -1, 0, 1, 2, 4, 5) \\ S^3 \times S^{11} & (b = -2, 3). \end{cases}$$

By Lemma 3.2 and by the results in Toda's book [15] these p -equivalences give

Lemma 3.3. (i) $\pi_{10}(G_{2,-2}) \cong \mathbf{Z}_{15}, \pi_{10}(G_{2,b}) \cong \mathbf{Z}_3$ ($b = 1, 4$), $\pi_{10}(G_{2,3}) \cong \mathbf{Z}_5$.

(ii) $\pi_{13}(G_{2,b}) \cong \mathbf{Z}_3$ ($b = -2, 1, 4$).

(iii) $\pi_{14}(G_{2,b}) \cong \mathbf{Z}_{168} \oplus \mathbf{Z}_6$ ($b = -2, 1, 4$).

(iv) The other homotopy groups $\pi_i(G_{2,b})$ for $0 \leq i \leq 14$ ($-2 \leq b \leq 5$) are isomorphic to $\pi_i(G_2)$ given in Lemma 3.2.

By Theorem 2.2 of [8] we have

Theorem 3.4. (i) $H^*(G_{2,b}; \mathbf{Z}_2) = \mathbf{Z}_2[x_3]/(x_3^4) \otimes \Lambda(Sq^2x_3)$.

(ii) $H^*(G_{2,b}; \mathbf{Z}_p) \cong \Lambda(x_3, x_{11})$ for each prime $p \geq 3$, where $\deg x_i = i$.

Therefore $G_{2,b}$ has a cell structure:

$$G_{2,b} \simeq S^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14}.$$

Let $M^n = S^{n-1} \cup_2 e^n$ be the mapping cone of a map: $S^{n-1} \rightarrow S^{n-1}$ of degree 2.

Then we have two cofiberings:

$$(3.1) \quad S^3 \longrightarrow G_{2,b}^{(6)} \longrightarrow M^6, \quad G_{2,b}^{(6)} \longrightarrow G_{2,b}^{(9)} \longrightarrow M^9,$$

which are equivalent to induced cofiberings by some maps $f_1: M^5 \rightarrow S^3$ and $f_2: M^8 \rightarrow G_{2,b}^{(6)}$ respectively by [3].

Lemma 3.5. (i) $[M^5, G_{2,b}^{(6)}] = [M^6, G_{2,b}^{(6)}] = [M^7, G_{2,b}^{(9)}] = 0$.

(ii) $[M^9, G_{2,b}^{(9)}] \cong [M^9, G_{2,b}] \cong \mathbf{Z}_4$ generated by an extension of a non-trivial element of $\pi_8(G_{2,b}) \cong \mathbf{Z}_2$.

(iii) $[M^{10}, G_{2,b}] \cong \mathbf{Z}_2$ generated by an extension of a non-trivial element of $\pi_9(G_{2,b}: 2) \cong \mathbf{Z}_2$.

Proof. (i) Consider the Puppe exact sequence

$$(3.2) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & [S^n, G_{2,b}] & \longrightarrow & [S^n, G_{2,b}] & \xrightarrow{p^*} & [M^n, G_{2,b}] \\ & & & & \downarrow i^* & & \\ & & & & [S^{n-1}, G_{2,b}] & \longrightarrow & [S^{n-1}, G_{2,b}] \longrightarrow \cdots \end{array}$$

We have $\pi_4(G_{2,b}) = \pi_5(G_{2,b}) = \pi_7(G_{2,b}) = 0$ and $\pi_6(G_{2,b}) \cong \mathbf{Z}_3$ by Lemmas 3.2 and 3.3. Therefore we have that $[M^5, G_{2,b}] = [M^6, G_{2,b}] = [M^7, G_{2,b}] = 0$, since $[M^n, X]$ is a \mathbf{Z}_4 -group by [1].

For dimensional reasons, we have $[M^5, G_{2,b}^{(6)}] = [M^6, G_{2,b}^{(6)}] = [M^7, G_{2,b}^{(9)}] = 0$.

(ii) Since $\pi_8(G_{2,b}) \cong \mathbf{Z}_2$ and $\pi_9(G_{2,b}: 2) \cong \mathbf{Z}_2$, by the above exact sequence for $n=9$, we have an exact sequence:

$$0 \longrightarrow \mathbf{Z}_2 \xrightarrow{p^*} [M^9, G_{2,b}] \xrightarrow{i^*} \mathbf{Z}_2 \longrightarrow 0.$$

Since $G_{2,b}$ is 2-equivalent to G_2 , we may verify it in the case $b=0$. Let $\text{Ext} \langle \eta_6^2 \rangle$ be an extension of $\langle \eta_6^2 \rangle \in \pi_8(G_2)$. Since $2 \cdot 1_{M^9} = i \circ \eta_8 \circ p$ by [16], we have $2 \text{Ext} \langle \eta_6^2 \rangle = \text{Ext} \langle \eta_6^2 \rangle \circ i \circ \eta_8 \circ p = \langle \eta_6^2 \rangle \circ \eta_8 \circ p$ and so $2 \text{Ext} \langle \eta_6^2 \rangle \neq 0$ in $[M^9, G_2]$ by Lemma 3.2 (Recall that $\langle \eta_6^2 \rangle \circ \eta_8 \neq 0$ and that p^* is monic). Therefore we have $[M^9, G_{2,b}] \cong \mathbf{Z}_4$ generated by an extension of a non-trivial element of $\pi_8(G_{2,b}) \cong \mathbf{Z}_2$.

(iii) By Lemmas 3.2 and 3.3 we have $\pi_9(G_{2,b}: 2) \cong \mathbf{Z}_2$ and $\pi_{10}(G_{2,b}: 2) = 0$. Therefore we have immediately $[M^{10}, G_{2,b}] \cong \pi_9(G_{2,b}: 2) \cong \mathbf{Z}_2$ by (3.2). q.e.d.

Lemma 3.6. (i) $i_*: [S^3, S^3] \longrightarrow [S^3, G_{2,b}^{(6)}]$ and

$$i^*: [G_{2,b}^{(6)}, G_{2,b}^{(6)}] \longrightarrow [S^3, G_{2,b}^{(6)}] \text{ are both bijective.}$$

(ii) $[SG_{2,b}^{(6)}, G_{2,b}^{(9)}] = [SG_{2,b}^{(6)}, G_{2,b}] = 0$.

Proof. (i) For dimensional reasons we see easily that $i_*: [S^3, S^3] \rightarrow [S^3, G_{2,b}^{(6)}]$ is bijective. Consider the Puppe exact sequence associated with (3.1):

$$\longrightarrow [M^6, G_{2,b}^{(6)}] \longrightarrow [G_{2,b}^{(6)}, G_{2,b}^{(6)}] \xrightarrow{i^*} [S^3, G_{2,b}^{(6)}] \xrightarrow{f_1^*} [M^5, G_{2,b}^{(6)}] \longrightarrow .$$

By (i) of Lemma 3.5 $[M^n, G_{2,b}^{(6)}] = 0$ for $n=5, 6$. Therefore i^* is bijective.

(ii) Consider the Puppe exact sequence associated with (3.1):

$$\cdots \longrightarrow [M^7, G_{2,b}^{(9)}] \longrightarrow [SG_{2,b}^{(6)}, G_{2,b}^{(9)}] \longrightarrow [S^4, G_{2,b}^{(9)}] \longrightarrow \cdots$$

Since $[M^7, G_{2,b}^{(9)}]=[S^4, G_{2,b}^{(9)}]=0$ by (i) of Lemma 3.5 and Lemmas 3.2 and 3.3 for dimensional reasons, we have

$$[SG_{2,b}^{(6)}, G_{2,b}] = [SG_{2,b}^{(6)}, G_{2,b}^{(9)}] = 0. \quad \text{q.e.d.}$$

Lemma 3.7. (i) $\pi_7(G_{2,b}^{(6)}) \cong \mathbf{Z}_2$.

(ii) $\pi_8(G_{2,b}^{(6)}) = \mathbf{Z}_4$.

Proof. Let F be the 3-connective fibre space over $G_{2,b}^{(6)}$. Then we have a fibering :

$$F \xrightarrow{i} G_{2,b}^{(6)} \xrightarrow{\pi} K(\mathbf{Z}, 3).$$

Since $H^*(G_{2,b}^{(6)}; \mathbf{Z}_2) = \{1, x_3, x_5 = Sq^2 x_3, x_3^2\}$, we see that $\pi^*: H^*(\mathbf{Z}, 3; \mathbf{Z}_2) \rightarrow H^*(G_{2,b}^{(6)}; \mathbf{Z}_2)$ is an epimorphism with $\text{Ker } \pi^* = \sum_{i \geq 8} H^i(\mathbf{Z}, 3; \mathbf{Z}_2)$. Therefore, there exists a transgressive element $y_7 \in H^7(F; \mathbf{Z}_2)$ whose transgression image is $\tau(y_7) = uSq^2u$ where $u \in H^8(\mathbf{Z}, 3; \mathbf{Z}_2)$ is the fundamental class. Then $\tau(Sq^1 y_7) = Sq^1 \tau(y_7) = Sq^1(uSq^2u) = u^3$. So there exists a transgressive element $y_8 \in H^8(F; \mathbf{Z}_2)$ such that $\tau(y_8) = Sq^4 Sq^2 u$. Then $\tau(Sq^1 y_8) = Sq^1 \tau(y_8) = Sq^1 Sq^4 Sq^2 u = Sq^5 Sq^2 u = (Sq^2 u)^2$ and $\tau(Sq^2 y_7) = Sq^2 \tau(y_7) = Sq^2(uSq^2u) = (Sq^2 u)^2$, whence $Sq^1 y_8 = Sq^2 y_7$. Thus we have

$$H^*(F; \mathbf{Z}_2) = \{1, y_7, Sq^1 y_7, y_8, Sq^1 y_8, Sq^2 Sq^1 y_8, \dots\}.$$

Take a CW-complex L with minimum cells 2-equivalent to F , and so we may take $L = ((S^7 \cup e^8) \vee S^8) \cup e^9 \cup e^{11} \cup \dots$. The attaching class of the 9-cell in L is $i \circ \eta_7 \vee 2\iota_8: S^8 \rightarrow (S^7 \cup e^8) \vee S^8$ where $i: S^7 \rightarrow S^7 \cup e^8$ is the inclusion. Consider the exact sequence of the pair $(L, S^7 \cup e^8 \vee S^8)$:

$$\pi_9(L, S^7 \cup e^8 \vee S^8) \xrightarrow{\partial} \pi_8(S^7 \cup e^8 \vee S^8) \rightarrow \pi_8(L) \rightarrow \pi_8(L, S^7 \cup e^8 \vee S^8) \rightarrow,$$

where $\pi_9(L, S^7 \cup e^8 \vee S^8) \cong \pi_9(S^9) \cong \mathbf{Z}$ generated by ι_9 , $\pi_8(L, S^7 \cup e^8 \vee S^8) = 0$, $\pi_8(S^7 \cup e^8 \vee S^8) \cong \mathbf{Z}_2 \oplus \mathbf{Z}$ generated by $i \circ \eta_7$ and ι_8 , and $\partial(\iota_9) = i \circ \eta_7 + 2\iota_8$. Therefore $\pi_8(L) \cong \mathbf{Z}_4$. Also, we have immediately $\pi_7(L) \cong \mathbf{Z}_2$. Since $\pi_i(L; 2)$ for $i=7, 8$ is isomorphic to $\pi_i(G_{2,b}^{(6)}; 2)$, $\pi_7(G_{2,b}^{(6)}; 2)$ and $\pi_8(G_{2,b}^{(6)}; 2)$ are isomorphic to \mathbf{Z}_2 and \mathbf{Z}_4 respectively. Since $i^*: H^*(G_{2,b}^{(6)}; \mathbf{Z}_p) \rightarrow H^*(S^3; \mathbf{Z}_p)$ is isomorphic for any odd prime p , the inclusion $i: S^3 \rightarrow G_{2,b}^{(6)}$ is a p -equivalence. So $\pi_7(G_{2,b}^{(6)}; p) = \pi_8(G_{2,b}^{(6)}; p) = 0$ for any odd prime p , since $\pi_7(S^3) \cong \mathbf{Z}_2$ and $\pi_8(S^3) \cong \mathbf{Z}_2$ by [15]. Therefore, we have $\pi_7(G_{2,b}^{(6)}) \cong \mathbf{Z}_2$ and $\pi_8(G_{2,b}^{(6)}) \cong \mathbf{Z}_4$. q.e.d.

Lemma 3.8. Let β be a generator of $[M^9, G_{2,b}^{(9)}] \cong \mathbf{Z}_4$ and let $\pi: G_{2,b}^{(9)} \rightarrow M^9 = G_{2,b}^{(9)}/G_{2,b}^{(6)}$ be the projection. Then we have $\pi_* \beta = 2 \cdot 1_{M^9} \in [M^9, M^9] \cong \mathbf{Z}_4$.

Proof. Consider the following commutative diagram of the exact sequence :

$$\begin{array}{ccccccccc} [S^8, G_{2,b}^{(6)}] & \xrightarrow{i_*} & [S^8, G_{2,b}^{(9)}] & \xrightarrow{\pi_*} & [S^8, M^9] & \xrightarrow{\partial} & [S^7, G_{2,b}^{(6)}] & \xrightarrow{i_*} & [S^7, G_{2,b}^{(9)}] \\ \uparrow j_* & & \uparrow j_* & & \uparrow j_* & & \uparrow j_* & & \uparrow j_* \\ [M^9, G_{2,b}^{(6)}] & \xrightarrow{i_*} & [M^9, G_{2,b}^{(9)}] & \xrightarrow{\pi_*} & [M^9, M^9] & \xrightarrow{\partial} & [M^8, G_{2,b}^{(6)}] & \xrightarrow{i_*} & [M^8, G_{2,b}^{(9)}] \end{array}$$

where $j: S^n \rightarrow M^{n+1}$ and $i: G_{2,b}^{(6)} \rightarrow G_{2,b}^{(9)}$ are the natural inclusions. Recall that $\pi_8(G_{2,b}^{(6)}) \cong \mathbf{Z}_4$ and $\pi_7(G_{2,b}^{(6)}) \cong \mathbf{Z}_2$ by Lemma 3.7, $\pi_8(G_{2,b}^{(9)}) \cong \mathbf{Z}_2$ and $\pi_7(G_{2,b}^{(9)}) = 0$ by Lemmas 3.2 and 3.3 and $\pi_8(M^9) \cong \mathbf{Z}_2$ generated by j , by [9].

Clearly the upper π_* is trivial and so $j^* \pi_* \beta = \pi_* j^* \beta = 0$. Therefore $\pi_* \beta$ is not a generator of $[M^9, M^9] \cong \mathbf{Z}_4$, since $[M^9, M^9] \cong \mathbf{Z}_4$ generated by 1_{M^9} , by [9]. We have $\text{Im}\{j^*: [M^9, G_{2,b}^{(6)}] \rightarrow [S^8, G_{2,b}^{(6)}]\} = \text{Tor}(\pi_8(G_{2,b}^{(6)}), \mathbf{Z}_2) = \mathbf{Z}_2$ by (ii) of Lemma 3.7. It follows from this fact that $j^* i_* = i_* j^* = 0$ in the left diagram and that a generator $\beta \in [M^9, G_{2,b}^{(9)}]$ is not contained in $\text{Im}\{i_*: [M^9, G_{2,b}^{(6)}] \rightarrow [M^9, G_{2,b}^{(9)}]\}$, since $j^* \beta$ is non-trivial by (ii) of Lemma 3.5. Therefore $\pi_* \beta$ is non-trivial and is not a generator of $[M^9, M^9]$. Thus $\pi_* \beta = 2 \cdot 1_{M^9}$. q.e.d.

The following lemma is a summary of Lemmas 4.3, 5.2 and 5.3 in [8].

Lemma 3.9. (i) $G_{2,b}^{(9)} \simeq G_2^{(9)}$ and $\pi_{10}(G_{2,b}^{(9)}) \cong \mathbf{Z}_{120}$.

(ii) The attaching class of the 11-cell in $G_{2,b}^{(11)} = G_{2,b}^{(9)} \cup e^{11}$ is $(1+8b)\omega$ with ω a generator of $\pi_{10}(G_{2,b}^{(9)}) \cong \mathbf{Z}_{120}$.

(iii) Let $\pi: G_2^{(9)} \rightarrow M^9 = G_2^{(9)}/G_2^{(6)}$ be the projection. Then $\pi_*(\omega) = \gamma$ is a generator of $\pi_{10}(M^9) \cong \mathbf{Z}_4$.

Let $i: G_{2,b}^{(9)} \rightarrow G_{2,b}^{(11)}$ be the inclusion.

Lemma 3.10. $i_* \pi_{11}(G_{2,b}^{(9)}) \cong \mathbf{Z}_2$.

Proof. Consider the exact sequence of the pair $(G_{2,b}^{(11)}, G_{2,b}^{(9)})$:

$$\cdots \longrightarrow \pi_{11}(G_{2,b}^{(9)}) \xrightarrow{i_*} \pi_{11}(G_{2,b}^{(11)}) \longrightarrow \pi_{11}(G_{2,b}^{(11)}, G_{2,b}^{(9)}) \xrightarrow{\partial} \pi_{10}(G_{2,b}^{(9)}) \longrightarrow \cdots,$$

where $\pi_{11}(G_{2,b}^{(11)}) \cong \pi_{11}(G_{2,b}) = \mathbf{Z} \oplus \mathbf{Z}_2$ by Lemmas 3.2 and 3.3, $\pi_{11}(G_{2,b}^{(11)}, G_{2,b}^{(9)}) \cong \pi_{11}(S^{11}) \cong \mathbf{Z}$ by the Blakers-Massey theorem, and $\pi_{10}(G_{2,b}^{(9)}) \cong \mathbf{Z}_{120}$ by (i) of Lemma 3.9. Then we have immediately

$$i_* \pi_{11}(G_{2,b}^{(9)}) \cong \mathbf{Z}_2. \quad \text{q.e.d.}$$

Lemma 3.11. For ω a generator of $\pi_{10}(G_{2,b}^{(9)})$, the homomorphism

$$(S\omega)^*: [SG_{2,b}^{(9)}, G_{2,b}] \longrightarrow [S^{11}, G_{2,b}]$$

is trivial for $-2 \leq b \leq 5$.

Proof. Since the suspension homomorphism: $\pi_{10}(M^9) \rightarrow \pi_{11}(M^{10})$ is clearly isomorphic, we have $\pi_* S\omega = \gamma$ by (iii) of Lemma 3.9, where $\pi: SG_{2,b}^{(9)} \rightarrow M^{10}$ is the projection and γ is a generator of $\pi_{11}(M^{10})$. We have the following commutative diagram:

$$(3.3) \quad \begin{array}{ccccc} \longrightarrow & [M^{10}, G_{2,b}] & \xrightarrow{\pi^*} & [SG_{2,b}^{(9)}, G_{2,b}] & \longrightarrow & [SG_{2,b}^{(6)}, G_{2,b}] \\ & \searrow \gamma^* & & \downarrow (S\omega)^* & & \\ & & & [S^{11}, G_{2,b}] & & \end{array}$$

where the horizontal sequence is the Puppe exact sequence associated with the cofibering (3.1).

Since $[SG_{2,b}^{(6)}, G_{2,b}] = 0$ by (ii) of Lemma 3.6, we see that

$$(3.4) \quad \pi^* : [M^{10}, G_{2,b}] \longrightarrow [SG_{2,b}^{(9)}, G_{2,b}] \text{ is epimorphic.}$$

Next we will show that

$$(3.5) \quad \gamma^* = 0 : [M^{10}, G_{2,b}] \longrightarrow [S^{11}, G_{2,b}].$$

As $[M^n, X]$ is a \mathbf{Z}_2 -group and $G_{2,b}$ is 2-equivalent to G_2 , it is sufficient to show it for the case $b=0$. By Lemma 3.2 and (iii) of Lemma 3.5 we have $[M^{10}, G_2] \cong \mathbf{Z}_2$ generated by an extension $\text{Ext} \{ \langle \eta_8^2 \rangle \circ \eta_8 \}$ of $\langle \eta_8^2 \rangle \circ \eta_8 \in \pi_9(G_2)$. Since γ is a coextension of $\eta_{10} \in \pi_{11}(S^{10})$, we have

$$\text{Ext} \{ \langle \eta_8^2 \rangle \circ \eta_8 \} \circ \gamma \in \{ \langle \eta_8^2 \rangle \circ \eta_8, 2\iota_9, \eta_9 \}.$$

By (5.4) and (5.5) of [15] and Lemma 3.2 we have

$$\begin{aligned} \langle \eta_8^2 \rangle \circ \eta_8, 2\iota_9, \eta_9 &\supset \langle \eta_8^2 \rangle \circ \{ \eta_8, 2\iota_9, \eta_9 \} \\ &= \langle \eta_8^2 \rangle \circ \{ S^5\nu', -S^5\nu' \} \\ &= \langle \eta_8^2 \rangle \circ \{ 2\nu_8, -2\nu_8 \} = 0 \end{aligned}$$

$$\text{modulo } \langle \eta_8^2 \rangle \circ \eta_8 \circ \pi_{11}(S^9) + \pi_{10}(G_2) \circ \eta_{10} = \{ \langle \eta_8^2 \rangle \circ \eta_8 \} = \{ \langle \eta_8^2 \rangle \circ 4\nu_8 \} = 0.$$

Therefore we have $\text{Ext} \{ \langle \eta_8^2 \rangle \circ \eta_8 \} \circ \gamma = 0$, and so (3.5) was proved. By (3.4) and (3.5) and by the commutativity of (3.3) we have

$$\text{Im}(S\omega)^* = \text{Im}(S\omega)^* \circ \pi^* = \text{Im } \gamma^* = 0. \quad \text{q.e.d.}$$

Lemma 3.12. (i) For $b = -1, 0, 2, 3, 5$, $\pi_{13}(G_{2,b}^{(11)}) \cong \mathbf{Z}$ generated by the attaching class of the 14-cell in $G_{2,b} = G_{2,b}^{(11)} \cup e^{14}$.

(ii) For $b = -2, 1, 4$, the short exact sequence

$$0 \longrightarrow \pi_{14}(G_{2,b}, G_{2,b}^{(11)}) \xrightarrow{\partial} \pi_{13}(G_{2,b}^{(11)}) \longrightarrow \pi_{13}(G_{2,b}) \longrightarrow 0$$

has a splitting homomorphism $\rho : \pi_{13}(G_{2,b}) \rightarrow \pi_{13}(G_{2,b}^{(11)})$, where $\pi_{14}(G_{2,b}, G_{2,b}^{(11)}) \cong \mathbf{Z}$ and $\pi_{13}(G_{2,b}) \cong \mathbf{Z}_3$, and so we have

$$\pi_{13}(G_{2,b}^{(11)}) \cong \rho \pi_{13}(G_{2,b}) \oplus \partial \pi_{14}(G_{2,b}, G_{2,b}^{(11)}) = \mathbf{Z}_3 \oplus \mathbf{Z}.$$

Here the free part of $\pi_{13}(G_{2,b}^{(11)})$ is generated by the attaching class of the 14-cell in $G_{2,b} = G_{2,b}^{(11)} \cup e^{14}$.

(iii) Let $i : G_{2,b}^{(11)} \rightarrow G_{2,b}$ be the inclusion. Then the homomorphism $i_* : \pi_{14}(G_{2,b}^{(11)}) \rightarrow \pi_{14}(G_{2,b})$ is epimorphic for $-2 \leq b \leq 5$.

Proof. Consider the exact sequence of the pair $(G_{2,b}, G_{2,b}^{(11)})$:

$$(3.6) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & \pi_{14}(G_{2,b}^{(11)}) & \xrightarrow{i_*} & \pi_{14}(G_{2,b}) & \xrightarrow{j_*} & \pi_{14}(G_{2,b}, G_{2,b}^{(11)}) \\ & & \partial & & & & \\ & & \longrightarrow & \pi_{13}(G_{2,b}^{(11)}) & \longrightarrow & \pi_{13}(G_{2,b}) & \longrightarrow \cdots, \end{array}$$

where $\pi_{14}(G_{2,b}, G_{2,b}^{(11)}) \cong \pi_{14}(S^{14}) \cong \mathbf{Z}$ by the Blakers-Massey theorem. Recall from Lemmas 3.2 and 3.3 we have $\pi_{14}(G_{2,b}) \cong \mathbf{Z}_{168} \oplus \mathbf{Z}_2$ and $\pi_{13}(G_{2,b}) = 0$ for $b = -1, 0, 2, 3, 5$. Therefore we see that $\pi_{13}(G_{2,b}^{(11)}) \cong \mathbf{Z}$ generated by the attaching class of the 14-cell in $G_{2,b} = G_{2,b}^{(11)} \cup e^{14}$. For $b = -2, 1, 4$, we have $\pi_{13}(G_{2,b}) \cong \mathbf{Z}_3$ by (ii) of Lemma 3.3 and $G_{2,b}^{(11)}$ is 3-equivalent to $S^3 \vee S^{11}$ by [8, § 6], where $\pi_{13}(S^3 \vee S^{11}) \cong \pi_{13}(S^3) \oplus \pi_{13}(S^{11}) \oplus \pi_{14}(S^3 \times S^{11}, S^3 \vee S^{11}) = \mathbf{Z}_{12} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}$ by [15]. Therefore we see that $\pi_{13}(G_{2,b}^{(11)}) \cong \mathbf{Z}_3 \oplus \mathbf{Z}$, and so by (3.6) we have the split exact sequence in the lemma and the desired results.

It follows from the above argument that the homomorphism $j_* : \pi_{14}(G_{2,b}) \rightarrow \pi_{14}(G_{2,b}, G_{2,b}^{(11)})$ is trivial, and so by (3.6) we have (iii). q.e.d.

Lemma 3.13. *Let f be the attaching class of the 14-cell in $G_{2,b} = G_{2,b}^{(11)} \cup e^{14}$. Then the homomorphism*

$$(Sf)^* : [SG_{2,b}^{(11)}, G_{2,b}] \longrightarrow [S^{14}, G_{2,b}]$$

is trivial for $-2 \leq b \leq 5$.

Proof. Let $\pi : SG_{2,b}^{(9)} \rightarrow SG_{2,b}^{(9)}/SG_{2,b}^{(6)} = M^{10}$ be the projection. Let γ be a generator of $\pi_{11}(M^{10})$. Then by (iii) of Lemma 3.9 we have a commutative diagram

$$\begin{array}{ccccccc}
 \pi_{14}(SG_{2,b}^{(9)}) & \xrightarrow{i_*} & \pi_{14}(SG_{2,b}^{(11)}) & \longrightarrow & \pi_{14}(SG_{2,b}^{(11)}, SG_{2,b}^{(9)}) & \xrightarrow{\partial} & \pi_{13}(SG_{2,b}^{(9)}) \\
 & & \cong \swarrow & & \searrow & \nearrow (S\omega)_* & \downarrow \pi_* \\
 & & \pi_{14}(S^{12}) & \xleftarrow[S \cong]{} & \pi_{13}(S^{11}) & \xrightarrow{\gamma_*} & \pi_{13}(M^{10}),
 \end{array}$$

where the horizontal sequence is exact. Since we can see that $\pi_*(S\omega)_* \eta_{11}^2 = \gamma_* \eta_{11}^2$ is a generator of $\pi_{13}(M^{10}) \cong \mathbf{Z}_2$ by [9], the homomorphism $\partial : \pi_{14}(SG_{2,b}^{(9)}, SG_{2,b}^{(9)}) \rightarrow \pi_{13}(SG_{2,b}^{(9)})$ is monomorphic, and so $i_* : \pi_{14}(SG_{2,b}^{(9)}) \rightarrow \pi_{14}(SG_{2,b}^{(11)})$ is epimorphic. Therefore for $Sf \in \pi_{14}(SG_{2,b}^{(11)})$ there exists an element $\bar{f} \in \pi_{14}(SG_{2,b}^{(9)})$ such that $Sf = i_* \bar{f}$.

Consider the Puppe exact sequence associated with cofibering (3.1):

$$\cdots \longrightarrow [M^{10}, G_{2,b}] \xrightarrow{\pi^*} [SG_{2,b}^{(9)}, G_{2,b}] \longrightarrow [SG_{2,b}^{(6)}, G_{2,b}] \longrightarrow \cdots$$

Since $[SG_{2,b}^{(6)}, G_{2,b}] = 0$ by (ii) of Lemma 3.6, $\pi^* : [M^{10}, G_{2,b}] \rightarrow [SG_{2,b}^{(9)}, G_{2,b}]$ is epimorphic. Also we have $\pi \circ \bar{f} \in \pi_{14}(M^{10}) = 0$ by [9]. Therefore we have

$$\text{Im}(Sf)^* = \text{Im}(i_* \bar{f})^* \subset \text{Im} \bar{f}^* = \text{Im} \bar{f}^* \circ \pi^* = 0. \quad \text{q.e.d.}$$

§ 4. Self-homotopy equivalences of $G_{2,b}^{(k)}$.

In this section, we study the group $\mathcal{E}(G_{2,b}^{(k)})$ for the k -skeleton $G_{2,b}^{(k)}$ of $G_{2,b}$ for $-2 \leq b \leq 5$ by making use of the results of the previous section, and obtain

our main result of this paper.

Lemma 4.1. (i) $\mathcal{E}(G_{2,b}^{(k)}) \cong \mathbf{Z}_2$ for $k=3, 6$.

(ii) $\mathcal{E}(G_{2,b}^{(9)}) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$.

Proof. (i) Clearly, $\mathcal{E}(G_{2,b}^{(3)}) = \mathcal{E}(S^3) \cong \mathbf{Z}_2$. Let $i: S^3 \rightarrow G_{2,b}^{(6)}$ be the inclusion. Then, by (i) of Lemma 3.6 the composition: $[S^3, S^3] \xrightarrow{i_*} [S^3, G_{2,b}^{(6)}] \xrightarrow{i_*^{-1}} [G_{2,b}^{(6)}, G_{2,b}^{(6)}]$ is a bijection which preserves the composition. Hence we have $\mathcal{E}(G_{2,b}^{(6)}) \cong \mathcal{E}(S^3) \cong \mathbf{Z}_2$.

(ii) Let p be an odd prime and let \mathbf{P} be the set of all primes. Let $\sigma_1 = -\iota_8 \times -\iota_{11}: S^3 \times S^{11} \rightarrow S^3 \times S^{11}$ and let σ_2 be the inversion of G_2 , then the localizations $(\sigma_1)_{(p)}$ and $(\sigma_2)_{p-(p)}$ are two side inversions of H -complexes $(S^3 \times S^{11})_{(p)}$ and $(G_2)_{p-(p)}$ such that $(\sigma_1)_{(p)} \circ (\sigma_1)_{(p)} = 1$ and $(\sigma_2)_{p-(p)} \circ (\sigma_2)_{p-(p)} = 1$, respectively. Furthermore the localization $(\sigma_1)_q$ is homotopic to $(\sigma_2)_q$.

Therefore $G_{2,b}$ has the two side inversion σ as a pull-back of $(\sigma_1)_{(p)}$ and $(\sigma_2)_{p-(p)}$ such that $\sigma \circ \sigma = 1$ by [5].

Let $\sigma^{(k)}: G_{2,b}^{(k)} \rightarrow G_{2,b}^{(k)}$ be the restriction of σ for $k=3, 6, 9, 11$. Then we define two maps

$$\lambda \text{ and } \bar{\lambda}: [M^9, G_{2,b}^{(9)}] \longrightarrow [G_{2,b}^{(9)}, G_{2,b}^{(9)}]$$

by $\lambda(\xi) = \nabla \circ (1 \vee \xi) \circ l$ and $\bar{\lambda}(\xi) = \nabla \circ (\sigma^{(9)} \vee \xi) \circ l$ respectively, where $l: G_{2,b}^{(9)} \rightarrow G_{2,b}^{(9)} \vee M^9$ is the map shrinking $M^9 \times \{1/2\}$ in $G_{2,b}^{(9)} = G_{2,b}^{(6)} \cup CM^9$. Since $i_*: [G_{2,b}^{(6)}, G_{2,b}^{(6)}] \rightarrow [G_{2,b}^{(6)}, G_{2,b}^{(9)}]$ is bijective for dimensional reasons, we can define a homomorphism

$$\psi: \mathcal{E}(G_{2,b}^{(9)}) \longrightarrow \mathcal{E}(G_{2,b}^{(6)})$$

by the restriction of the composition $[G_{2,b}^{(9)}, G_{2,b}^{(9)}] \xrightarrow{i_*} [G_{2,b}^{(6)}, G_{2,b}^{(9)}] \xrightarrow{i_*^{-1}} [G_{2,b}^{(6)}, G_{2,b}^{(6)}]$. By

(i) we have $\mathcal{E}(G_{2,b}^{(6)}) \cong \mathbf{Z}_2$ generated by $\sigma^{(6)}$. If $h \in [G_{2,b}^{(9)}, G_{2,b}^{(9)}]$ satisfies $i_*^{-1}i_*(h) = 1$ or $i_*^{-1}i_*(h) = \sigma^{(6)}$, then there exists an element $\xi \in [M^9, G_{2,b}^{(9)}]$ such that $\lambda(\xi) = h$ or $\bar{\lambda}(\xi) = h$ by p. 326 of [11]. Therefore we have

$$(4.1) \quad \{\lambda(\xi), \bar{\lambda}(\xi), \xi \in [M^9, G_{2,b}^{(9)}]\} \supset \mathcal{E}(G_{2,b}^{(9)}),$$

since $i_*^{-1}i_*(\gamma) = 1$ or $\sigma^{(6)}$ for any $\gamma \in \mathcal{E}(G_{2,b}^{(9)})$.

We have the following homotopy commutative diagram for any $\xi \in [M^9, G_{2,b}^{(9)}]$ by the definition of λ :

$$\begin{array}{ccccccc} G_{2,b}^{(9)} & \xrightarrow{l} & G_{2,b}^{(9)} \vee M^9 & \xrightarrow{1 \vee \xi} & G_{2,b}^{(9)} \vee G_{2,b}^{(9)} & \xrightarrow{\nabla} & G_{2,b}^{(9)} \\ \downarrow d & & \downarrow j & & \downarrow j & & \downarrow j \\ G_{2,b}^{(9)} \times G_{2,b}^{(9)} & \xrightarrow{1 \times \pi} & G_{2,b}^{(9)} \times M^9 & \xrightarrow{j \times j \circ \xi} & G_{2,b} \times G_{2,b} & \xrightarrow{m} & G_{2,b} \end{array}$$

where m is a multiplication on $G_{2,b}$. Let $j + \pi^*$ and $\sigma \circ j + \pi^*: [M^9, G_{2,b}] \rightarrow [G_{2,b}^{(9)}, G_{2,b}]$ be the maps defined by $(j + \pi^*)(\xi) = j + \pi^*(\xi)$ and $(\sigma \circ j + \pi^*)(\xi)$

$=\sigma \circ j + \pi^*(\xi)$ respectively, where $+$ is the multiplication induced by a multiplication on $G_{2,b}$. Then we have the following commutative diagram :

$$(4.2) \quad \begin{array}{ccccc} [G_{2,b}^{(9)}, G_{2,b}^{(9)}] & \xleftarrow{\lambda} & [M^9, G_{2,b}^{(9)}] & \xrightarrow{\bar{\lambda}} & [G_{2,b}^{(9)}, G_{2,b}^{(9)}] \\ \downarrow j_* & & \downarrow j_* & & \downarrow j_* \\ [G_{2,b}^{(9)}, G_{2,b}] & \xleftarrow{j+\pi^*} & [M^9, G_{2,b}] & \xrightarrow{\sigma \circ j + \pi^*} & [G_{2,b}^{(9)}, G_{2,b}] \end{array}$$

where all j_* are bijective for dimensional reasons. If $\lambda(\xi_1)=\lambda(\xi_2)$, then we have $(j+\pi^*)j_*(\xi_1)=(j+\pi^*)j_*(\xi_2)$, and so $\pi^*j_*(\xi_1)=\pi^*j_*(\xi_2)$. We have $\xi_1=\xi_2$, since $\pi^*: [M^9, G_{2,b}] \rightarrow [G_{2,b}^{(9)}, G_{2,b}]$ is monomorphic by (ii) of Lemma 3.6 and since j_* is bijective. Thus we have that λ is injective. Quite similarly we have that $\bar{\lambda}$ is injective. Let β be a generator of $[M^9, G_{2,b}^{(9)}] \cong Z_t$ and $0 \leq t \leq 3$. Then by the definition of λ , we have that $\lambda(t\beta) \in G_{2,b}^{(9)}$ is the inclusion $j: G_{2,b}^{(9)} \rightarrow G_{2,b}^{(9)}$. Also, by Lemma 3.8 we have the following homotopy commutative diagram :

$$\begin{array}{ccccccc} G_{2,b}^{(9)} & \xrightarrow{l} & G_{2,b}^{(9)} \vee M^9 & \xrightarrow{1 \vee t\beta} & G_{2,b}^{(9)} \vee G_{2,b}^{(9)} & \xrightarrow{\nabla} & G_{2,b}^{(9)} \\ \downarrow \pi & & \downarrow \pi \vee 1 & & \downarrow \pi \vee \pi & & \downarrow \pi \\ M^9 & \xrightarrow{l} & M^9 \vee M^9 & \xrightarrow{1 \vee 2t1} & M^9 \vee M^9 & \xrightarrow{\nabla} & M^9. \end{array}$$

$j_*: H_i(G_{2,b}^{(9)}) \rightarrow H_i(G_{2,b}^{(9)})$ is isomorphic for $0 \leq i \leq 7$ and $\pi_*: H_i(G_{2,b}^{(9)}) \rightarrow H_i(M^9)$ is isomorphic for $i \geq 8$. Therefore by the above diagram, $\lambda(t\beta)_*: H_i(G_{2,b}^{(9)}) \rightarrow H_i(G_{2,b}^{(9)})$ is isomorphic for all i and so we have

$$(4.3) \quad \lambda(t\beta) \in \mathcal{E}(G_{2,b}^{(9)}).$$

We have $\sigma^{(9)}_*(\beta) = -\beta$, since $j_*: [M^9, G_{2,b}^{(9)}] \rightarrow [M^9, G_{2,b}]$ is bijective and $\sigma_* = -1: [M^9, G_{2,b}] \rightarrow [M^9, G_{2,b}]$. So the diagram

$$\begin{array}{ccccccc} G_{2,b}^{(9)} & \xrightarrow{l} & G_{2,b}^{(9)} \vee M^9 & \xrightarrow{1 \vee (-t\beta)} & G_{2,b}^{(9)} \vee G_{2,b}^{(9)} & \xrightarrow{\nabla} & G_{2,b}^{(9)} \\ & & \searrow \sigma^{(9)} \vee t\beta & & \downarrow \sigma^{(9)} \vee \sigma^{(9)} & & \downarrow \sigma^{(9)} \\ & & & & G_{2,b}^{(9)} \vee G_{2,b}^{(9)} & \xrightarrow{\nabla} & G_{2,b}^{(9)} \end{array}$$

homotopy commutes and leads us that $\bar{\lambda}(t\beta) = \sigma^{(9)} \circ \lambda(-t\beta)$. Hence we have

$$(4.4) \quad \bar{\lambda}(t\beta) \in \mathcal{E}(G_{2,b}^{(9)})$$

by the fact that $\sigma^{(9)} \in \mathcal{E}(G_{2,b}^{(9)})$ and $\lambda(-t\beta) \in \mathcal{E}(G_{2,b}^{(9)})$. Thus by (4.1), (4.3) and (4.4)

we have

$$(4.5) \quad \mathcal{E}(G_{2,b}^{(9)}) = \{\lambda(t\beta), \bar{\lambda}(t\beta); 0 \leq t \leq 3\} \text{ as a set}$$

with β a generator of $[M^9, G_{2,b}^{(9)}] \cong \mathbf{Z}_4$.

Remark that the self homotopy equivalences f_t and \bar{f}_t of $G_{2,b}^{(9)}$ for $0 \leq t \leq 3$ have been defined in (5.1) of [8] and $f_t = \lambda(t\beta)$ and $\bar{f}_t = \bar{\lambda}(t\beta)$. It is also shown in p. 623 of [8] that $f_{t*}(\omega) = \omega + t\beta_*\gamma$ and $\bar{f}_{t*}(\omega) = -\omega + t\beta_*\gamma$ where ω is a generator of $\pi_{10}(G_{2,b}^{(9)}) \cong \mathbf{Z}_{120}$, $\gamma = \pi_*\omega$ is a generator of $\pi_{10}(M^9)$ (see Lemma 3.9) and $\beta_*\gamma = \pm 30\omega$. By taking a suitable generator β such that $\beta_*\gamma = 30\omega$, we have

$$(4.6) \quad \lambda(t\beta)_*(\omega) = (1+30t)\omega \quad \text{and} \quad \bar{\lambda}(t\beta)_*(\omega) = (-1+30t)\omega \quad \text{for } t=0, 1, 2, 3.$$

It follows from this that the natural homomorphism

$$\mathcal{E}(G_{2,b}^{(9)}) \longrightarrow \text{Aut } \pi_{10}(G_{2,b}^{(9)})$$

is monomorphic by (4.5). This fact and the equality

$$\lambda(t\beta)_*\lambda(t\beta)_*(\omega) = (1+30t)^2\omega = \omega$$

lead to a conclusion that $\lambda(t\beta)^2 = 1$. By a similar calculation we have $\bar{\lambda}(t\beta)^2 = 1$.

Thus we have $\mathcal{E}(G_{2,b}^{(9)}) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$. q.e.d.

By Lemma 4.1 and Theorem 2.1, we have

- Lemma 4.2.** (i) $\mathcal{E}(G_{2,b}^{(11)}) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$ for $-1 \leq b \leq 5$.
 (ii) There is an exact sequence :

$$0 \longrightarrow \mathbf{Z}_2 \oplus \mathbf{Z}_2 \longrightarrow \mathcal{E}(G_{2,-2}^{(11)}) \longrightarrow \mathbf{Z}_2 \longrightarrow 1.$$

Proof. We apply Theorem 2.1 to the cell structure

$$G_{2,b}^{(11)} = G_{2,b}^{(9)} \cup e^{11}$$

in which the attaching class of the 11-cell is $(1+8b)\omega$ by (ii) of Lemma 3.9.

First we compute the group

$$G_1 = \{h \in \mathcal{E}(G_{2,b}^{(9)}), h_*(1+8b)\omega = \varepsilon(1+8b)\omega, \varepsilon = \pm 1\}$$

given in Theorem 2.1, since $2(1+8b)\omega \neq 0$. By (4.6) the conditions

$$\lambda(t\beta)_*(1+8b)\omega = \varepsilon(1+8b)\omega \quad \text{and} \quad \bar{\lambda}(t\beta)_*(1+8b)\omega = \varepsilon(1+8b)\omega$$

are equivalent to

$$(1+8b)(1+30t)\omega = \varepsilon(1+8b)\omega \quad \text{and} \quad (1+8b)(-1+30t)\omega = \varepsilon(1+8b)\omega$$

respectively. For $-1 \leq b \leq 5$, we have easily

$$\begin{aligned} (1+8b)(1+30t)\omega &= (1+8b)\omega && \text{if and only if } t=0; \\ (1+8b)(1+30t)\omega &\neq -(1+8b)\omega && \text{if } t=0, 1, 2, 3; \\ (1+8b)(-1+30t)\omega &= -(1+8b)\omega && \text{if and only if } t=0; \end{aligned}$$

$$(1+8b)(-1+30t)\omega \neq (1+8b)\omega \quad \text{if } t=0, 1, 2, 3.$$

Therefore, if a non-trivial element $h \in \mathcal{E}(G_{2,b}^{(9)})$ satisfies the condition $h_*(1+8b)\omega = \varepsilon(1+8b)\omega$ with $\varepsilon = \pm 1$, then $h = \bar{\lambda}(0) = \sigma^{(9)}$. Thus we have

$$(4.7) \quad G_1 \cong \mathbf{Z}_2 \text{ generated by } \sigma^{(9)}.$$

For $b = -2$, we have the following

$$(1+8b)(1+30t)\omega = (1+8b)\omega \quad \text{if and only if } t=0;$$

$$(1+8b)(1+30t)\omega = -(1+8b)\omega \quad \text{if and only if } t=1;$$

$$(1+8b)(-1+30t)\omega = -(1+8b)\omega \quad \text{if and only if } t=0;$$

$$(1+8b)(-1+30t)\omega = (1+8b)\omega \quad \text{if and only if } t=3.$$

Therefore by the definition of G_1 and by Lemma 4.1 we have

$$G_1 = \{\lambda(0)=1, \lambda(\beta), \bar{\lambda}(0)=\sigma^{(9)}, \bar{\lambda}(3\beta)\} \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2.$$

Next we consider the homomorphism

$$(4.8) \quad \lambda: i_*\pi_{11}(G_{2,b}^{(9)}) \longrightarrow \mathcal{E}(G_{2,b}^{(11)})$$

given in Theorem 2.1, where $i_*\pi_{11}(G_{2,b}^{(9)}) \cong \mathbf{Z}_2$ by Lemma 3.10. This homomorphism λ is the restriction of the map A defined by (2.1).

Let $j + \pi^*: [S^{11}, G_{2,b}] \rightarrow [G_{2,b}^{(11)}, G_{2,b}]$ be a map defined by $(j + \pi^*)(\xi) = j + \pi^*(\xi)$ where $+$ is a multiplication induced by a multiplication on $G_{2,b}$. Then we have a commutative diagram by the similar way to that in (4.2)

$$\begin{array}{ccc} [S^{11}, G_{2,b}^{(11)}] & \xrightarrow{A} & [G_{2,b}^{(11)}, G_{2,b}^{(11)}] \\ \downarrow j_* & & \downarrow j_* \\ [S^{11}, G_{2,b}] & \xrightarrow{j + \pi^*} & [G_{2,b}^{(11)}, G_{2,b}] \end{array}$$

where both j_* are bijective for dimensional reasons, and hence we have

$$j_*A^{-1}(1) = \pi^{*-1}(0) = (1+8b)(S\omega)^*[SG_{2,b}^{(9)}, G_{2,b}]$$

from the Puppe exact sequence:

$$\cdots \longrightarrow [SG_{2,b}^{(9)}, G_{2,b}] \xrightarrow{(1+8b)(S\omega)^*} [S^{11}, G_{2,b}] \xrightarrow{\pi^*} [G_{2,b}^{(11)}, G_{2,b}] \longrightarrow \cdots$$

Therefore $A^{-1}(1) = 0$ by Lemma 3.11, since j_* is bijective. By this fact, (4.7), (4.8) and Theorem 2.1 we have a short exact sequence:

$$0 \longrightarrow \mathbf{Z}_2 \longrightarrow \mathcal{E}(G_{2,b}^{(11)}) \longrightarrow G_1 \longrightarrow 1,$$

where G_1 is isomorphic to \mathbf{Z}_2 generated by $\sigma^{(9)}$ for $-1 \leq b \leq 5$ and G_1 is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ generated by $\sigma^{(9)}$ and $\lambda(\beta)$ for $b = -2$. In the above sequence the subgroup \mathbf{Z}_2 generated by $\sigma^{(9)}$ of G_1 splits, since the splitting homomorphism $\rho: \mathbf{Z}_2 \rightarrow \mathcal{E}(G_{2,b}^{(11)})$ can be defined by $\rho(\sigma^{(9)}) = \sigma^{(11)}$. Hence we have $\mathcal{E}(G_{2,b}^{(11)}) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$

for $-1 \leq b \leq 5$ and there is an exact sequence

$$0 \longrightarrow \mathbf{Z}_2 \oplus \mathbf{Z}_2 \longrightarrow \mathcal{E}(G_{2,-2}^{(11)}) \longrightarrow \mathbf{Z}_2 \longrightarrow 1. \quad \text{q.e.d.}$$

Let $\chi: \mathcal{E}(G_{2,b}^{(11)}) \longrightarrow \text{Aut } \pi_{13}(G_{2,b}^{(11)})$ be the natural homomorphism.

Lemma 4.3. For $b = -2, 1, 4$, $\text{Im } \{\chi: \mathcal{E}(G_{2,b}^{(11)}) \rightarrow \text{Aut } \pi_{13}(G_{2,b}^{(11)})\}$ is contained in $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ generated by $-1 \oplus 1$ and $1 \oplus -1: \pi_{13}(G_{2,b}^{(11)}) \cong \mathbf{Z}_8 \oplus \mathbf{Z} \rightarrow \mathbf{Z}_8 \oplus \mathbf{Z} \cong \pi_{13}(G_{2,b}^{(11)})$, where the isomorphism $\pi_{13}(G_{2,b}^{(11)}) \cong \mathbf{Z}_8 \oplus \mathbf{Z}$ is the one given in (ii) of Lemma 3.12. Specifically, for the attaching class f of the 14-cell in $G_{2,b} = G_{2,b}^{(11)} \cup e^{14}$ and for any element h of $\mathcal{E}(G_{2,b}^{(11)})$,

$$h_* f = \varepsilon f, \quad \varepsilon = \pm 1.$$

(A proof will be given in § 5).

Now, we apply Corollary 2.2 to the cell structure

$$G_{2,b} = G_{2,b}^{(11)} \cup e^{14}$$

and we have the following main result.

Theorem 4.4. Let $G_{2,b}$ be an H -complex of type (3, 11) in Theorem 3.1. Then we have the following exact sequences:

- (i) $0 \longrightarrow D(\mathbf{Z}_{168} \oplus \mathbf{Z}_2) \longrightarrow \mathcal{E}(G_{2,b}) \longrightarrow \mathbf{Z}_2 \longrightarrow 1 \quad (b = -1, 0, 2, 3, 5),$
- (ii) $0 \longrightarrow D(\mathbf{Z}_{168} \oplus \mathbf{Z}_6) \longrightarrow \mathcal{E}(G_{2,b}) \longrightarrow \mathbf{Z}_2 \longrightarrow 1 \quad (b = 1, 4),$
- (iii) $0 \longrightarrow \mathbf{Z}_{168} \oplus \mathbf{Z}_6 \longrightarrow \mathcal{E}(G_{2,-2}) \longrightarrow \mathcal{E}(G_{2,-2}^{(11)}) \longrightarrow 1.$

Here the group $\mathcal{E}(G_{2,-2}^{(11)})$ is given in (ii) of Lemma 4.2 and for an abelian group H , $D(H)$ is a group given by the following split exact sequence: $0 \rightarrow H \rightarrow D(H) \rightarrow \mathbf{Z}_2 \rightarrow 1$, where the splitting action \mathbf{Z}_2 on H is given by $(-1) \cdot h = -h$ for $-1 \in \mathbf{Z}_2$ and $h \in H$.

Proof. Since $G_{2,b}$ has a multiplication, we can apply Corollary 2.2 to the cell structure $G_{2,b} = G_{2,b}^{(11)} \cup e^{14}$.

First let H be the group given in (2.4): $H = i_* \pi_{14}(G_{2,b}^{(11)}) / (Sf)^* [SG_{2,b}^{(11)}, G_{2,b}]$, where f is the attaching class of the 14-cell in $G_{2,b}^{(11)} \cup e^{14}$. Then we have

$$(4.9) \quad H \cong \pi_{14}(G_{2,b})$$

by (iii) of Lemma 3.12 and Lemma 3.13.

Next we compute the group

$$G_1 = \{h \in \mathcal{E}(G_{2,b}^{(11)}) \mid h_* f = \varepsilon f, \varepsilon = \pm 1 \text{ in } \pi_{13}(G_{2,b}^{(11)})\}$$

given in Theorem 2.1, since $2f \neq 0$ by Lemma 3.12.

For $b = -1, 0, 2, 3, 5$, f is a generator of $\pi_{13}(G_{2,b}^{(11)}) \cong \mathbf{Z}$ by (i) of Lemma 3.12 and $\text{Aut } \pi_{13}(G_{2,b}^{(11)}) \cong \text{Aut } \mathbf{Z} \cong \mathbf{Z}_2$ generated by -1 . Therefore for any element $h \in \mathcal{E}(G_{2,b}^{(11)})$ we have

$$h_*f=f \text{ or } -f.$$

Hence we have $G_1 \cong \mathcal{E}(G_{2,b}^{(11)})$.

For $b=-2, 1, 4$, by (ii) of Lemma 3.12 and Lemma 4.3, we have the same results $G_1 \cong \mathcal{E}(G_{2,b}^{(11)})$.

Thus, by this results, (4.9) and (2.3), we have the exact sequence

$$(4.10) \quad 0 \longrightarrow \pi_{14}(G_{2,b}) \xrightarrow{\lambda} \mathcal{E}(G_{2,b}) \longrightarrow \mathcal{E}(G_{2,b}^{(11)}) \longrightarrow 1.$$

The subgroup Z_2 generated by $\sigma^{(11)}$ of $\mathcal{E}(G_{2,b}^{(11)})$ splits, since a splitting homomorphism $\rho: Z_2 \rightarrow \mathcal{E}(G_{2,b})$ can be defined by $\rho(\sigma^{(11)}) = \sigma$. We can easily see that $\sigma^* = 1: H^{14}(G_{2,b}; Z_p) \rightarrow H^{14}(G_{2,b}; Z_p)$ for any odd prime p by the ring structure of $H^*(G_{2,b}; Z_p)$ and by the fact that

$$\sigma^* = -1: H^i(G_{2,b}; Z_p) \longrightarrow H^i(G_{2,b}; Z_p) \quad \text{for } i=3, 11.$$

Also $\sigma_* = -1: \pi_{14}(G_{2,b}) \rightarrow \pi_{14}(G_{2,b})$. Therefore we have a homotopy commutative diagram:

$$\begin{array}{ccccccc} G_{2,b} & \xrightarrow{l} & G_{2,b} \vee S^{14} & \xrightarrow{1 \vee \xi} & G_{2,b} \vee G_{2,b} & \xrightarrow{\nabla} & G_{2,b} \\ \downarrow \sigma & & \downarrow \sigma \vee 1 & & \downarrow \sigma \vee \sigma & & \downarrow \sigma \\ G_{2,b} & \xrightarrow{l} & G_{2,b} \vee S^{14} & \xrightarrow{1 \vee (-\xi)} & G_{2,b} \vee G_{2,b} & \xrightarrow{\nabla} & G_{2,b} \end{array}$$

From this diagram we see that the splitting action is given by $\sigma \cdot \xi = -\xi$. Hence the desired results are obtained by (4.10) and Lemmas 3.2, 3.3 and 4.2. q.e.d.

§ 5. A proof of Lemma 4.3.

Let $G_{2,b(3)}^{(11)}$ be the localization of $G_{2,b}^{(11)}$ at 3, and let $l: \mathcal{E}(G_{2,b}^{(11)}) \rightarrow \mathcal{E}(G_{2,b(3)}^{(11)})$ be the natural homomorphism defined by the localization $l(h) = h_{(3)}: G_{2,b(3)}^{(11)} \rightarrow G_{2,b(3)}^{(11)}$ (see [5] and [7]). First we show the lemma for the case $b=-2$. Consider the following exact sequence of the pair $(G_{2,-2}^{(11)}, G_{2,-2}^{(9)})$:

$$\cdots \longrightarrow \pi_{11}(G_{2,-2}^{(9)}) \longrightarrow \pi_{11}(G_{2,-2}^{(11)}) \longrightarrow \pi_{11}(G_{2,-2}^{(11)}, G_{2,-2}^{(9)}) \xrightarrow{\partial} \pi_{10}(G_{2,-2}^{(9)}) \longrightarrow \cdots,$$

where $\pi_{11}(G_{2,-2}^{(11)}) \cong Z \oplus Z_2$ by Lemmas 3.2 and 3.3, $\pi_{11}(G_{2,-2}^{(11)}, G_{2,-2}^{(9)}) \cong \pi_{11}(S^{11}) \cong Z$ generated by ι_{11} , $\pi_{10}(G_{2,-2}^{(9)}) \cong Z_{120}$ generated by ω and $\partial(\iota_{11}) = -15\omega$ by Lemma 3.9. There exists a coextension $\tilde{8}\iota_{11}: S^{11} \rightarrow G_{2,b}^{(11)}$ of $8\iota_{11}: S^{11} \rightarrow S^{11}$, and so we can define a 3-equivalence $q: S^3 \vee S^{11} \rightarrow G_{2,-2}^{(11)}$ by $q = \nabla \circ (i \vee \tilde{8}\iota_{11})$ where $i: S^3 \rightarrow G_{2,-2}^{(11)}$ be the inclusion. Then we have a commutative diagram:

$$(5.1) \quad \begin{array}{ccccc} \mathcal{E}(G_{2,-2}^{(11)}) & \xrightarrow{l} & \mathcal{E}(G_{2,-2(3)}^{(11)}) & \xleftarrow[q']{\cong} & \mathcal{E}(S_{(3)}^3 \vee S_{(3)}^{11}) \\ \downarrow \chi & & \downarrow \chi & & \downarrow \chi \\ \text{Aut } \pi_{13}(G_{2,-2}^{(11)}) & \xrightarrow{l'} & \text{Aut } \pi_{13}(G_{2,-2(3)}^{(11)}) & \xleftarrow[q'']{\cong} & \text{Aut } \pi_{13}(S_{(3)}^3 \vee S_{(3)}^{11}), \end{array}$$

where $l': \text{Aut } \pi_{13}(G_{2,-2}^{(11)}) \rightarrow \text{Aut } \pi_{13}(G_{2,-2(3)}^{(11)})$ is the canonical homomorphism defined

by $l'(h)=h \otimes 1: \pi_{13}(G_{2,-2}^{(11)}) \cong \pi_{13}(G_{2,-2}^{(11)}) \otimes \mathbf{Q}_3 \rightarrow \pi_{13}(G_{2,-2}^{(11)}) \otimes \mathbf{Q}_3 \cong \pi_{13}(G_{2,-2}^{(11)})$ and q' and q'' are isomorphisms induced by the 3-equivalence q .

By (ii) of Lemma 3.12 $\pi_{13}(G_{2,-2}^{(11)}) \cong \mathbf{Z}_3 \oplus \mathbf{Z}$ and $\pi_{13}(G_{2,-2}^{(11)}) \cong \mathbf{Z}_3 \oplus \mathbf{Q}_3$, and so $\text{Aut } \pi_{13}(G_{2,-2}^{(11)})$ is isomorphic to a group of matrices:

$$\left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mid a \in \text{Aut } \mathbf{Z}_3, \quad b \in \text{Aut } \mathbf{Z}, \quad c \in \text{Hom}(\mathbf{Z}, \mathbf{Z}_3) \right\}$$

where $\text{Aut } \mathbf{Z}_3 \cong \mathbf{Z}_2$ generated by -1 , $\text{Aut } \mathbf{Z} = \mathbf{Z}_2$ generated by -1 and $\text{Hom}(\mathbf{Z}, \mathbf{Z}_3) = \mathbf{Z}_3$ generated by the mod 3 reduction. Therefore we see that l' is monomorphic. By (ii) of Lemma 4.2 we have

$$(5.2) \quad \mathcal{E}(G_{2,-2}^{(11)}) = \{1, \lambda(\xi), \sigma^{(11)}, h, \lambda(\xi)\sigma^{(11)}, \lambda(\xi)h, \sigma^{(11)}h, \lambda(\xi)\sigma^{(11)}h\}.$$

Here $\lambda(\xi)$, $\xi \neq 0$ is given in (4.8) and h is an element satisfying the following homotopy commutative diagram:

$$(5.3) \quad \begin{array}{ccccccc} S^{10} & \xrightarrow{-15\omega} & G_{2,-2}^{(9)} & \longrightarrow & G_{2,-2}^{(11)} & \longrightarrow & S^{11} \\ & \downarrow -1 & \downarrow \lambda(\beta) & & \downarrow h & & \downarrow -1 \\ S^{10} & \xrightarrow{-15\omega} & G_{2,-2}^{(9)} & \longrightarrow & G_{2,-2}^{(11)} & \longrightarrow & S^{11}, \end{array}$$

where $\lambda(\beta)$ is given in (4.5).

Since $2\xi \neq 0$, we have a homotopy commutative diagram:

$$\begin{array}{ccccccc} G_{2,-2}^{(11)} & \xrightarrow{l} & G_{2,-2}^{(11)} \vee S^{11} & \xrightarrow{1 \vee \xi} & G_{2,-2}^{(11)} \vee G_{2,-2}^{(11)} & \xrightarrow{\nabla} & G_{2,-2}^{(11)} \\ \downarrow J & & \downarrow J & & \downarrow J & & \downarrow J \\ G_{2,-2}^{(11)} & \xrightarrow{l_{(3)}} & (G_{2,-2}^{(11)} \vee S^{11})_{(3)} & \xrightarrow{(1 \vee \xi)_{(3)}} & (G_{2,-2}^{(11)} \vee G_{2,-2}^{(11)})_{(3)} & \xrightarrow{\nabla_{(3)}} & G_{2,-2}^{(11)}_{(3)} \\ & & \uparrow \cong & & \uparrow \cong & & \\ & & G_{2,-2}^{(11)} \vee S_{(3)}^{11} & \xrightarrow{1 \vee \xi_{(3)}} & G_{2,-2}^{(11)} \vee G_{2,-2}^{(11)} & & \end{array}$$

where $\xi_{(3)} = 0 \in [S_{(3)}^{11}, G_{2,-2}^{(11)}] \cong \pi_{11}(G_{2,-2}^{(11)})$. Hence we have $l(\lambda(\xi)) = 1$. Therefore $l'(\chi(\lambda(\xi))) = \chi l(\lambda(\xi)) = 1$ and so we have $\chi(\lambda(\xi)) = 1: \pi_{13}(G_{2,-2}^{(11)}) \rightarrow \pi_{13}(G_{2,-2}^{(11)})$. Since $\sigma^{(11)}$ is the restriction of $\sigma: G_{2,-2} \rightarrow G_{2,-2}$, we have

$$\begin{array}{ccccccc} H_{14}(G_{2,-2}) & \longrightarrow & H_{14}(G_{2,-2}, G_{2,-2}^{(11)}) & \xleftarrow{\mathcal{E}} & \pi_{14}(G_{2,-2}, G_{2,-2}^{(11)}) & \xrightarrow{\partial} & \pi_{13}(G_{2,-2}^{(11)}) \longrightarrow \pi_{13}(G_{2,-2}) \\ \downarrow \sigma_* & & \downarrow \sigma_* & & \downarrow \sigma_* & & \downarrow \sigma^{(11)*} \quad \downarrow \sigma_* \\ H_{14}(G_{2,-2}) & \longrightarrow & H_{14}(G_{2,-2}, G_{2,-2}^{(11)}) & \xleftarrow{\mathcal{E}} & \pi_{14}(G_{2,-2}, G_{2,-2}^{(11)}) & \xrightarrow{\partial} & \pi_{13}(G_{2,-2}^{(11)}) \longrightarrow \pi_{13}(G_{2,-2}) \end{array}$$

where $\sigma_* = 1: H_{14}(G_{2,-2}) \rightarrow H_{14}(G_{2,-2})$ by the ring structure of $H^*(G_{2,-2}; \mathbf{Z}_p)$ (p : odd prime), $\mathcal{E}: \pi_{14}(G_{2,-2}, G_{2,-2}^{(11)}) \rightarrow H_{14}(G_{2,-2}, G_{2,-2}^{(11)})$ is the Hurewicz isomorphism and $\pi_{13}(G_{2,-2}^{(11)}) \cong \rho \pi_{13}(G_{2,-2}) \oplus \partial \pi_{14}(G_{2,-2}, G_{2,-2}^{(11)}) \cong \mathbf{Z}_3 \oplus \mathbf{Z}$ by (ii) of Lemma 3.12. Therefore, $\sigma_* = 1: \pi_{14}(G_{2,-2}, G_{2,-2}^{(11)}) \rightarrow \pi_{14}(G_{2,-2}, G_{2,-2}^{(11)})$. Since $i_*: \pi_{13}(S^3; 3) \rightarrow$

$\pi_{13}(G_{2,-2}: 3)$ is isomorphic and since $\sigma^{(3)}_* = -1: \pi_{13}(S^3) \rightarrow \pi_{13}(S^3)$, $\sigma_* = -1: \pi_{13}(G_{2,-2}) \rightarrow \pi_{13}(G_{2,-2})$. Thus we have

$$\sigma^{(11)}_* = -1 \oplus 1: \mathbf{Z}_3 \oplus \mathbf{Z} \longrightarrow \mathbf{Z}_3 \oplus \mathbf{Z}.$$

By the definition of q and (5.3), we can see that as an element of $\pi_3(G_{2,-2}^{(11)}) \oplus \pi_{11}(G_{2,-2}^{(11)})$

$$h \circ q = q \circ (\iota_3 \vee -\iota_{11}) \text{ modulo } \mathbf{Z}_2 \subset \pi_{11}(G_{2,-2}^{(11)}) \cong \mathbf{Z} \oplus \mathbf{Z}_2,$$

since $h|_{G_{2,-2}^{(6)}}$ is the inclusion. Therefore we have

$$q'^{-1}l(h) = \iota_{3(3)} \vee (-\iota_{11})_{(3)} \text{ in (5.1).}$$

It follows from this that

$$\begin{aligned} q''^{-1}l'\chi(h) &= \chi q'^{-1}l(h) = (\iota_{3(3)} \vee (-\iota_{11})_{(3)})_* = 1 \oplus -1: \\ \pi_{13}(S^3_{(3)} \vee S^1_{(3)}) &= (\pi_{13}(S^3) \otimes \mathbf{Q}_3) \oplus (\pi_{14}(S^3 \times S^{11}, S^3 \vee S^{11}) \otimes \mathbf{Q}_3) \\ &\cong \mathbf{Z}_3 \oplus \mathbf{Q}_3 \longrightarrow \mathbf{Z}_3 \oplus \mathbf{Q}_3 \cong \pi_{13}(S^3_{(3)} \vee S^1_{(3)}). \end{aligned}$$

Therefore by considering a commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_{14}(G_{2,-2(3)}, G_{2,-2(3)}^{(11)}) & \xrightarrow{\partial} & \pi_{13}(G_{2,-2(3)}^{(11)}) & \longrightarrow & \pi_{13}(G_{2,-2(3)}) \longrightarrow 0 \\ & & \uparrow \bar{q}_{(3)*} & & \uparrow q_{(3)*} & & \uparrow i_{(3)*} \\ 0 & \longrightarrow & \pi_{14}((S^3 \times S^{11})_{(3)}, (S^3 \vee S^{11})_{(3)}) & \xrightarrow{\partial} & \pi_{13}((S^3 \vee S^{11})_{(3)}) & \longrightarrow & \pi_{13}(S^3_{(3)}) \longrightarrow 0 \end{array}$$

where $\bar{q} = i + \widetilde{8}\iota_{11}: S^3 \times S^{11} \rightarrow G_{2,-2}$ and all vertical homomorphisms are isomorphic, we have $\chi(h) = 1 \oplus -1: \pi_{13}(G_{2,-2}^{(11)}) \cong \mathbf{Z}_3 \oplus \mathbf{Z} \rightarrow \mathbf{Z}_3 \oplus \mathbf{Z} \cong \pi_{13}(G_{2,-2}^{(11)})$, since $q''^{-1}l'$ is monomorphic. The other elements of $\mathcal{E}(G_{2,-2}^{(11)})$ are given by the composition of $\lambda(\xi)$, $\sigma^{(11)}$ and h by (5.2). Hence we complete the proof for the case $b = -2$.

The proof for the other $b \neq -2$ is given more easily by a similar way.

The latter half of Lemma 4.3 is obtained immediately from (ii) of Lemma 3.12. q.e.d.

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