

ON THE GROWTH OF ALGEBRAIC POLYNOMIALS IN THE WHOLE COMPLEX PLANE

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ABSTRACT. In this paper, we study the estimation for algebraic polynomials in the bounded and unbounded regions bounded by piecewise Dini smooth curve having interior and exterior zero angles.

1. Introduction and main results

Let $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ be the extended complex plane, $G \subset \mathbb{C}$ be a finite region with $0 \in G$, bounded by a Jordan curve $L := \partial G$, $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$; $\Delta := \{w : |w| > 1\}$ (with respect to $\overline{\mathbb{C}}$). Let $w = \Phi(z)$ be the univalent conformal mapping of Ω onto the Δ normalized by $\Phi(\infty) = \infty$, $\lim_{z \rightarrow \infty} \frac{\Phi(z)}{z} > 0$, and $\Psi := \Phi^{-1}$. For $R > 1$ let us set $L_R := \{z : |\Phi(z)| = R\}$, $G_R := \text{int} L_R$, $\Omega_R := \text{ext} L_R$.

Let $\{z_j\}_{j=1}^m \in L$ be a fixed system of distinct points. Consider a so-called generalized Jacobi weight function $h(z)$ being defined as follows:

$$(1.1) \quad h(z) := \prod_{j=1}^m |z - z_j|^{\gamma_j}, \quad z \in G_{R_0}, \quad R_0 > 1,$$

where $\gamma_j > -1$ for every $j = 1, 2, \dots, m$.

Denote by \mathcal{P}_n the class of all complex algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N} := \{1, 2, \dots\}$.

Let $h(z)$ be a weight function. For any $p > 0$ we introduce:

$$(1.2) \quad \|P_n\|_{A_p(h,G)} := \left(\iint_G h(z) |P_n(z)|^p d\sigma_z \right)^{1/p} < \infty,$$

where σ_z is the two-dimensional Lebesgue measure,

$$(1.3) \quad \|P_n\|_{\mathcal{L}_p(h,L)} := \left(\int_L h(z) |P_n(z)|^p |dz| \right)^{1/p} < \infty,$$

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when L is rectifiable and

$$(1.4) \quad \|P_n\|_{C(\overline{G})} := \max_{z \in \overline{G}} |P_n(z)|.$$

The classical lemma Bernstein-Walsh [23] shows that:

$$(1.5) \quad |P_n(z)| \leq |\Phi(z)|^n \|P_n\|_{C(\overline{G})}, \quad z \in \Omega.$$

If take $z \in L_R$, then from (1.5) we see that:

$$(1.6) \quad \|P_n\|_{C(\overline{G}_R)} \leq R^n \|P_n\|_{C(\overline{G})}.$$

Thus, we can estimate the growth uniform norm (C -norm) of the polynomial P_n , depending on the extension of the region G up to G_R for some $R > 1$, containing G . In particular, the C -norm $\|P_n\|_{C(\overline{G})}$ of polynomials P_n increases no more than a constant, when G expands up to $G_{1+\frac{1}{n}}$. The same effect is observed for the norm (1.3) according to the following estimate [11]:

$$(1.7) \quad \|P_n\|_{\mathcal{L}_p(L_R)} \leq R^{n+\frac{1}{p}} \|P_n\|_{\mathcal{L}_p(L)}, \quad p > 0.$$

Similar to (1.6) and (1.7) estimation for any $p > 0$ with respect to the $A_p(h, G)$ -norms was obtained in [2] for regions G with quasiconformal boundary (corresponding definition is given below) and the weight function $h(z)$ as defined in (1.1) with $\gamma_j > -2$ and for arbitrary Jordan regions G and $h(z) \equiv 1$ in [4].

N. Stylianopoulos [20] investigated analogous problem for regions with rectifiable quasiconformal boundary and obtained pointwise estimations for $|P_n(z)|$ for every points $z \in \Omega$, where he replaced the norm $\|P_n\|_{C(\overline{G})}$ with norm $\|P_n\|_{A_2(G)}$ on the right-hand side of (1.5) and found a new version of the estimation (1.5) as follows: for each $P_n \in \wp_n$

$$(1.8) \quad |P_n(z)| \leq c \frac{\sqrt{n}}{d(z, L)} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1}, \quad z \in \Omega,$$

where $d(z, L) := \inf \{|\zeta - z| : \zeta \in L\}$ and $c = c(L) > 0$ is a constant, independent from n and P_n .

To evaluate $|P_n(z)|$ on the whole complex plane with respect to $\mathcal{L}_p(h, L)$ -norm at the right-hand side, we need to get the next estimation of following type:

$$(1.9) \quad \|P_n\|_{C(\overline{G})} \leq c\mu_n(G, h, p) \|P_n\|_{\mathcal{L}_p(h, L)},$$

where $c = c(G, p) > 0$ is a constant independent from n and P_n , and $\mu_n(G, h, p) \rightarrow \infty$ as $n \rightarrow \infty$.

We note that, the first results of type (1.9), in case $h(z) \equiv 1$ for $L = \{z : |z| = 1\}$ and $0 < p < \infty$ was found by Jackson [12]. Further study estimates (1.9) were carried out in [14], [16], [18], [21], [22] and others (for more references see [18], [21]).

In this work, firstly, we study similar problems for regions with piecewise Dini-smooth boundary with exterior and interior zero angles and generalized Jacobi weight function $h(z)$ defined in (1.1), with respect to $\mathcal{L}_p(h, L)$ -norm,

$p > 1$, in the right-hand side of (1.8). Secondly, for such regions we obtain estimate for $|P_n(z)|$, $z \in \overline{G}$, of the (1.9) type. Finally, combining corresponding estimates, we obtain an estimate for $|P_n(z)|$ in whole complex plane when region G has exterior and interior cusps, depending on the geometrical properties of the region G and the weight function $h(z)$.

Let us give some definitions and notations that will be used later in the text.

Let S be a rectifiable Jordan curve or arc and $z = z(s)$, $s \in [0, |S|]$, $|S| := \text{mes}S$ (linear measure of S), denote the natural representation of S .

Definition 1.1 ([17, p. 48], see also [9, p. 32]). We say that a Jordan curve or arc S called Dini-smooth, if it has a parametrization $z = z(s)$, $0 \leq s \leq |S|$, such that $z'(s) \neq 0$, $0 \leq s \leq |S|$ and $|z'(s_2) - z'(s_1)| < g(s_2 - s_1)$, $s_1 < s_2$, where g is an increasing function for which

$$\int_0^1 \frac{g(x)}{x} dx < \infty.$$

Now, we shall define a new class of regions with piecewise Dini-smooth boundary, which have at the boundary points corners, interior and exterior cusps simultaneously.

For any $j = 1, 2, \dots$ and sufficiently small $\varepsilon_1 > 0$ we denote by $f_j : [0, \varepsilon_1] \rightarrow \mathbb{R}$ the twice differentiable functions, such that $f_j(0) = 0$, $f_j^{(k)}(x) > 0$, $x > 0$ and $k = 0, 1, 2$.

In the sequel, we denote by c, c_0, c_1, c_2, \dots are positive and $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots$ are sufficiently small positive constants (in general, different in different relations), which depend on G in general.

Definition 1.2. We say that a Jordan region $G \in PDS(\lambda_i; f_j)$, $0 < \lambda_i \leq 2$, $i = \overline{1, m_1}$, $f_j = f_j(x)$, $j = \overline{m_1 + 1, m}$, if $L = \partial G$ consists of a union of finite number of Dini-smooth arcs $\{L_j\}_{j=0}^m$, connecting at the points $\{z_j\}_{j=0}^m \in L$ such that L is locally Dini-smooth at $z_0 \in L \setminus \{z_j\}_{j=1}^m$ and:

a) for every $z_i \in L$, $i = \overline{1, m_1}$, $m_1 \leq m$, the region G has exterior (with respect to \overline{G}) angles $\lambda_i \pi$, $0 < \lambda_i \leq 2$, at the corner z_i ;

b) for every $z_j \in L$, $j = \overline{m_1 + 1, m}$, in the local co-ordinate system (x, y) with origin at z_j the following conditions are satisfied

b₁) $\{z = x + iy : |z| < \varepsilon_1, c_1 f_j(x) \leq y \leq c_2 f_j(x), 0 \leq x \leq \varepsilon_1\} \subset \overline{\Omega}$,

b₂) $\{z = x + iy : |z| < \varepsilon_1, |y| \geq \varepsilon_2 x, 0 \leq x \leq \varepsilon_1\} \subset \overline{G}$,

for some constants $-\infty < c_1 < c_2 < +\infty$, $0 < \varepsilon_i < 1$, $i = 1, 2$.

Here and in further, for any $k \geq 0$ and $m > k$, notations $j = \overline{k, m}$ denotes $j = k, k + 1, \dots, m$.

For any $z \in \mathbb{C}$ and sufficiently small $\varepsilon_1 > 0$ let

$$K(z, f_j, \varepsilon_1) := \{z = x + iy : 0 \leq x \leq \varepsilon_1, c_1 f_j(x) \leq y \leq c_2 f_j(x)\}$$

for some constants $-\infty < c_1 < c_2 < +\infty$, $0 < \varepsilon_1 < 1$. Let $f(x) := f_{j_0}(x)$, where j_0 , $m_1 + 1 \leq j_0 \leq m$, chosen such that $K(z_j, f_{j_0}, \varepsilon_1) \subseteq K(z_j, f_j, \varepsilon_1)$ for all $m_1 + 1 \leq j \leq m$ and sufficiently small $\varepsilon_1 > 0$.

As is clear from Definition 1.2, each region $G \in PDS(\lambda_i; f_j)$ may have exterior $\lambda_i\pi$, $0 < \lambda_i \leq 2$, angles (when $\lambda_i = 2$ -interior zero angles) at the points $z_i \in L$, $i = \overline{1, m_1}$, and exterior zero angles at which the boundary arcs are touching with $f_j(x)$ -speed at the points $z_j \in L$, $j = \overline{m_1 + 1, m}$. If $m_1 = m = 0$, then the region G does not have such angles, and in this case we will write: $G \in DS$; if $m_1 = m \geq 1$, then G has only $\lambda_i\pi$, $0 < \lambda_i \leq 2$, $i = \overline{1, m_1}$, exterior angles (when $\lambda_i = 2$ -interior zero angles) and in this case we will write: $G \in PDS(\lambda_i; 0)$; if $m_1 = 0$ and $m \geq 1$, then G has only exterior zero angles and in this case we will write: $G \in PDS(1; f_j)$.

Throughout this work, we will assume that the points $\{z_j\}_{j=1}^m \in L$ defined in (1.1) and Definition 1.2 are identical. Without loss of generality, we assume that these points on the curve $L = \overline{\partial G}$ are located in the positive direction such that, G has $\lambda_j\pi$, $0 < \lambda_j \leq 2$, $j = \overline{1, m_1}$, exterior angles (when $\lambda_j = 2$ -interior zero angles (interior cusps)) at the points $\{z_j\}_{j=1}^{m_1}$, $m_1 \leq m$, and has exterior zero angles (exterior cusps) on the points $\{z_j\}_{j=m_1+1}^m$ and $w_j := \Phi(z_j)$.

Before stating our main results, we introduce some notation. Let us set:

$$\Gamma := \Gamma_{1, m_1} \cup \Gamma_{2, m},$$

where

$$\Gamma_{1, k} := \{\gamma_j, j = \overline{1, k}, k \leq m_1\},$$

$$\Gamma_{2, k} := \{\gamma_j, j = \overline{m_1 + 1, m_1 + k}, 1 \leq k \leq m - m_1\};$$

$$\Gamma_{1, k}^{(1)} := \{\gamma_k \in \Gamma_{1, m_1} : \gamma_k > p - 1, k \leq m_1\} \text{ (with possible skips),}$$

$$\Gamma_{1, k}^{(2)} := \{\gamma_k \in \Gamma_{1, m_1} : \gamma_k = p - 1, k \leq m_1\},$$

$$\Gamma_{1, k}^{(3)} := \{\gamma_k \in \Gamma_{1, m_1} : \gamma_k < p - 1, k \leq m_1\};$$

$$\Gamma_{2, k}^{(1)} := \{\gamma_k \in \Gamma_{2, m} : \gamma_k > p - 1, m_1 + 1 \leq k \leq m\} \text{ (with possible skips);}$$

$$\Gamma_{2, k}^{(2)} := \{\gamma_k \in \Gamma_{2, m} : \gamma_k = p - 1, m_1 + 1 \leq k \leq m\},$$

$$\Gamma_{2, k}^{(3)} := \{\gamma_k \in \Gamma_{2, m} : \gamma_k < p - 1, m_1 + 1 \leq k \leq m\};$$

$$\gamma_k^1 = \max\{\gamma_k : \gamma_k \in \Gamma_{1, k}, k \leq m_1\}, \gamma^1 := \gamma_{m_1}^1;$$

$$\tilde{\gamma}_k^1 = \max\{0; \gamma_k : \gamma_k \in \Gamma_{1, k}, k \leq m_1\}, \tilde{\gamma}^1 := \tilde{\gamma}_{m_1}^1;$$

$$\gamma_j^2 = \max\{\gamma_j : \gamma_j \in \Gamma_{2, k}, m_1 + 1 \leq j \leq m_1 + k, 1 \leq k \leq m - m_1\},$$

$$\gamma^2 := \gamma_m^2; \tilde{\gamma}_k^2 = \max\{0; \gamma_k : \gamma_k \in \Gamma_{2, k}, k \leq m\},$$

$$\tilde{\gamma}^2 := \tilde{\gamma}_m^2; \gamma_k^* = \max\{\gamma_k : \gamma_k \in \Gamma, k \leq m\},$$

$$\gamma^* := \gamma_m^*; \tilde{\gamma}_k^* = \max\{0; \gamma_k : \gamma_k \in \Gamma, k \leq m\},$$

$$\tilde{\gamma}^* := \tilde{\gamma}_m^*,$$

$$\begin{aligned} \tilde{\gamma}_{*,k}^2 &:= \left\{ \begin{array}{ll} 0; & -1 < \gamma_k \leq 0 \\ \min \gamma_k, & \gamma_k > 0 \end{array} \right., \gamma_k \in \Gamma_{2,k}, k \leq m \}; \\ \tilde{\gamma}_*^2 &:= \tilde{\gamma}_{*,m}^2, \\ \tilde{\lambda}_k &= \max\{\lambda_j; 1 : j = \overline{1, k}, k \leq m_1\}, \\ \tilde{\lambda} &:= \tilde{\lambda}_{m_1}; \alpha_* := \min\{\alpha_j, j = \overline{m_1 + 1, m}\}, \\ \alpha^* &:= \max\{\alpha_j, j = \overline{m_1 + 1, m}\}. \end{aligned}$$

Now we can state our new results.

Theorem 1.1. *Let $p > 1$; $G \in PDS(\lambda_i; f_j)$ for some $0 < \lambda_i \leq 2$, $i = \overline{1, m_1}$ and $f_j(x) = cx^{1+\alpha_j}$, $\alpha_j > 0$, $j = \overline{m_1 + 1, m}$; $h(z)$ be defined in (1.1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, we have:*

$$(1.10) \quad |P_n(z)| \leq c_1 \frac{B_{n;1}}{d(z, L)} \|P_n\|_{\mathcal{L}_p(h, L)} \cdot \begin{cases} |\Phi(z)|^{n+1}, & z \in \Omega, \\ 1, & z \in G, \end{cases}$$

where $c_1 = c_1(G, p) > 0$ is the constant, independent from z and n , and

$$(1.11) \quad B_{n;1} := \begin{cases} \begin{aligned} &\sum_{i=1}^{m_1} n^{\frac{(\gamma_i+1-p)\tilde{\lambda}_i}{p}} && \text{if } \gamma_i \in \Gamma_{1,k}^{(1)} \\ &\sum_{j=m_1+1}^m n^{\frac{\gamma_j+1-p}{p(1+\alpha_j)}}, && \text{for all } i = \overline{1, k}, 1 \leq k \leq m_1 \\ &&& \text{and if } \gamma_j \in \Gamma_{2,k}^{(1)} \\ &&& \text{for all } j = \overline{m_1 + 1, k}, m_1 + 1 \leq k \leq m; \end{aligned} \\ (\ln n)^{1-\frac{1}{p}}, && \text{if } \gamma_i \in \Gamma_{1,m_1}^{(2)} \text{ for all } i = \overline{1, m_1} \text{ and} \\ && \text{if } \gamma_j \in \Gamma_{2,m}^{(2)} \text{ for all } j = \overline{m_1 + 1, m}; \\ 1, && \text{if } \gamma_i \in \Gamma_{1,m_1}^{(3)} \text{ for all } i = \overline{1, m_1} \text{ and} \\ && \text{if } \gamma_j \in \Gamma_{2,m}^{(3)} \text{ for all } j = \overline{m_1 + 1, m}. \end{cases}$$

Theorem 1.1 is local, that is, each term in the sum on the right side of $B_{n;1}$ shows the local growth of $|P_n(z)|$, depending on the behavior of the weight function $h(z)$ and the boundary L in the neighborhood of a each point $z_j \in L$ for any $j = \overline{1, m}$. Comparing the terms in the sum for each point $\{z_j\}$, $j = \overline{1, m}$, and using the above notations, we can obtain the following result of global character:

Theorem 1.2. *Let $p > 1$; $G \in PDS(\lambda_i; f_j)$ for some $0 < \lambda_i \leq 2$, $i = \overline{1, m_1}$ and $f_j(x) = cx^{1+\alpha_j}$, $\alpha_j > 0$, $j = \overline{m_1 + 1, m}$; $h(z)$ be defined in (1.1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, we have:*

$$(1.12) \quad |P_n(z)| \leq c_2 \frac{B_{n;2}}{d(z, L)} \|P_n\|_{\mathcal{L}_p(h, L)} \cdot \begin{cases} |\Phi(z)|^{n+1}, & z \in \Omega, \\ 1 & z \in G, \end{cases}$$

where $c_2 = c_2(G, p, m) > 0$ is the constant, independent from z and n , and

$$(1.13) \quad B_{n;2} := \begin{cases} n \frac{(\gamma^1+1-p)\tilde{\lambda}}{p} + n \frac{\gamma^2+1-p}{p(1+\alpha_2)}, & \begin{array}{l} \text{if there is at least one } \gamma_i \in \Gamma_{1,k}^{(1)} \\ \text{for some } i = \overline{1, k}, 1 \leq k \leq m_1 \\ \text{and if there is at least one } \gamma_j \in \Gamma_{2,k}^{(1)} \\ \text{for some } j = \overline{m_1+1, k}; m_1+1 \leq k \leq m; \end{array} \\ (\ln n)^{1-\frac{1}{p}}, & \begin{array}{l} \text{if there is at least one} \\ \gamma_j \in (\Gamma_{1,k}^{(2)} \cup \Gamma_{2,k}^{(2)}) \setminus (\Gamma_{1,m_1}^{(1)} \cup \Gamma_{2,m}^{(1)}) \\ \text{for some } j = \overline{1, m}, k \leq m; \end{array} \\ 1, & \begin{array}{l} \text{if } \gamma_j \in \Gamma_{1,k}^{(3)} \cup \Gamma_{2,k}^{(3)} \\ \text{for all } j = \overline{1, m}. \end{array} \end{cases}$$

In particular, in the case of L having two singular points $z_1 \in L$ and $z_2 \in L$ (i.e., $m_1 = 1, m = 2$), we obtain the following:

Corollary 1.1. *Let $p > 1; G \in PDS(\lambda_1; cx^{1+\alpha_2})$ for some $0 < \lambda_1 \leq 2, \alpha_2 > 0$; $h(z)$ be defined in (1.1) for $j = 2$. Then, for any $P_n \in \wp_n, n \in \mathbb{N}$, we have:*

$$(1.14) \quad |P_n(z)| \leq c_3 \frac{B_{n;3}}{d(z, L)} \|P_n\|_{\mathcal{L}_p(h, L)} \begin{cases} |\Phi(z)|^{n+1}, & z \in \Omega, \\ 1 & z \in G, \end{cases}$$

where $c_3 = c_3(G, p) > 0$ is the constant, independent from z and n , and

$$(1.15) \quad B_{n,3} := \begin{cases} n \frac{(\gamma_1+1-p)\tilde{\lambda}_1}{p} + n \frac{\gamma_2+1-p}{p(1+\alpha_2)}, & \text{if } \gamma_1, \gamma_2 > p-1, \\ (\ln n)^{1-\frac{1}{p}}, & \begin{array}{l} \text{if } \gamma_1 = p-1, -1 < \gamma_2 \leq p-1 \\ \text{or } -1 < \gamma_1 \leq p-1, \gamma_2 = p-1, \end{array} \\ 1, & \text{if } -1 < \gamma_1, \gamma_2 < p-1. \end{cases}$$

We can take individual cases when the region G has only one singular point on the boundary L : exterior non zero (interior zero) angle or exterior zero angle. In this case, from Corollary 1.1, we obtain the following:

Corollary 1.2. *Under the conditions of Corollary 1.1, we have:*

$$(1.16) \quad |P_n(z)| \leq c_4 \frac{B_{n;4}}{d(z, L)} \|P_n\|_{\mathcal{L}_p(h, L)} \begin{cases} |\Phi(z)|^{n+1}, & z \in \Omega, \\ 1 & z \in G, \end{cases}$$

where $c_4 = c_4(G, p) > 0$ is the constant, independent from z and n , and

$$(1.17) \quad B_{n,4} := \begin{cases} n^{\frac{(\gamma_1+1-p)\tilde{\lambda}_1}{p}}, & \text{if } \alpha_2 = 0, \gamma_1 > p - 1, \\ n^{\frac{\gamma_2+1-p}{p(1+\alpha_2)}}, & \text{if } \lambda_1 = 1, \alpha_2 > 0, \gamma_2 > p - 1; \\ (\ln n)^{1-\frac{1}{p}}, & \text{if } \alpha_2 = 0, \gamma_1 = p - 1, \\ & \text{or } \lambda_1 = 1, \gamma_2 = p - 1; \\ 1, & \text{if } \alpha_2 = 0, -1 < \gamma_1 < p - 1, \\ & \text{or } \lambda_1 = 1, -1 < \gamma_2 < p - 1. \end{cases}$$

As we can see from the above results, it remains to find an estimate on the $|P_n(z)|$ for the boundary points. In this case the following is true:

Theorem 1.3. *Let $p > 1$; $G \in PDS(\lambda_i; f_j)$ for some $0 < \lambda_i < 2$, $i = \overline{1, m_1}$ and $f_j(x) = cx^{1+\alpha_j}$, $\alpha_j > 0$, $j = \overline{m_1 + 1, m}$; $h(z)$ be defined as in (1.1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, we have:*

$$(1.18) \quad \|P_n\|_{C(\overline{G})} \leq c_5 \left(\sum_{i=1}^{m_1} n^{\frac{(\tilde{\gamma}_i+1)\tilde{\lambda}_i}{p}} + \sum_{i=m_1+1}^m D_{n,m}^i \right) \|P_n\|_{\mathcal{L}_p(h,L)},$$

where $c_5 = c_5(G, p) > 0$ is the constant, independent from z and n ,

$$D_{n,m}^i := \begin{cases} n^{\frac{\tilde{\gamma}_i+1}{p(1+\alpha_i)} + \frac{\alpha_i}{p(1+\alpha_i)}}, & 1 < p < 2 + \frac{\tilde{\gamma}_i}{1+\alpha_i}, \\ (n \ln)^{1-\frac{1}{p}}, & p = 2 + \frac{\tilde{\gamma}_i}{1+\alpha_i}, \\ n^{1-\frac{1}{p}}, & p > 2 + \frac{\tilde{\gamma}_i}{1+\alpha_i}, \end{cases} \quad \text{for any } i = \overline{m_1 + 1, m}.$$

Comparing the terms in the sum for each point $\{z_j\}$, $j = \overline{1, m}$, we obtain result of global character as follows:

Corollary 1.3. *Under the conditions of Theorem 1.3, we have*

$$(1.19) \quad \|P_n\|_{C(\overline{G})} \leq c_6 (D_{n,m_1} + D_{n,m}) \|P_n\|_{\mathcal{L}_p(h,L)},$$

$$D_{n,m_1} := n^{\frac{(\tilde{\gamma}^1+1)\tilde{\lambda}}{p}}; \quad D_{n,m} := \begin{cases} n^{\frac{\tilde{\gamma}^2+1}{p(1+\alpha_*)} + \frac{\alpha_*}{p(1+\alpha_*)}}, & 1 < p < 2 + \frac{\tilde{\gamma}^2}{1+\alpha_*}, \\ (n \ln)^{1-\frac{1}{p}}, & p = 2 + \frac{\tilde{\gamma}^2}{1+\alpha_*}, \\ n^{1-\frac{1}{p}}, & p > 2 + \frac{\tilde{\gamma}^2}{1+\alpha_*}, \end{cases}$$

where $c_6 = c_6(G, p) > 0$ is the constant, independent from z and n .

In particular, when the region has one exterior angle $\lambda_1\pi$, $0 < \lambda_1 \leq 2$, at the boundary point z_1 and one exterior zero angle $x^{1+\alpha_2}$, $\alpha_2 > 0$, at the boundary point z_2 , then from Theorem 1.3 we obtain the following:

Corollary 1.4. *Let $p > 1$; $G \in PDS(\lambda_1; f_2)$ for some $0 < \lambda_1 \leq 2$ and $f_2(x) = cx^{1+\alpha_2}$, $\alpha_2 > 0$; $h(z)$ be defined as in (1.1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, we have:*

$$(1.20) \quad \|P_n\|_{C(\overline{G})} \leq c_7 (D_{n,1} + D_{n,2}) \|P_n\|_{\mathcal{L}_p(h, L)},$$

where $c_7 = c_7(G, p, \lambda_1, \alpha_2) > 0$ is the constant, independent from z and n , and

$$D_{n,1} := n^{\frac{(\tilde{\gamma}_1+1)\tilde{\lambda}}{p}}; D_{n,2} := \begin{cases} n^{\frac{\tilde{\gamma}_2+1}{p(1+\alpha_2)} + \frac{\alpha_2}{p(1+\alpha_2)}}, & 1 < p < 2 + \frac{\tilde{\gamma}_2}{1+\alpha_2}, \\ (n \ln)^{1-\frac{1}{p}}, & p = 2 + \frac{\tilde{\gamma}_2}{1+\alpha_2}, \\ n^{1-\frac{1}{p}}, & p > 2 + \frac{\tilde{\gamma}_2}{1+\alpha_2}. \end{cases}$$

Combining Corollary 1.4 with Corollary 1.2, we obtain an estimate at the growth of $|P_n(z)|$ on the whole complex plane (for simplicity, we assume $m_1 = 1, m = 2$):

Corollary 1.5. *Let $p > 1; G \in PDS(\lambda_1; f_2)$ for some $0 < \lambda_1 \leq 2$ and $f_2(x) = cx^{1+\alpha_2}, \alpha_2 > 0; h(z)$ be defined as in (1.1). Then, for any $P_n \in \wp_n, n \in \mathbb{N}$, we have:*

$$(1.21) \quad |P_n(z)| \leq c_8 \|P_n\|_{\mathcal{L}_p(h,L)} \begin{cases} \frac{B_{n,4}}{d(z,L)} |\Phi(z)|^{n+1}, & z \in \Omega, \\ (D_{n,1} + D_{n,2}), & z \in \overline{G}, \end{cases}$$

where $c_8 = c_8(G, p) > 0$ is the constant, independent from z and n ; $B_{n,4}$ and $D_{n,1}, D_{n,2}$ are defined as in (1.17) and (1.20), respectively.

1.1. Sharpness of estimates

The sharpness of the estimations (1.10)-(1.21) for some special cases can be discussed by comparing them with the following results:

Remark 1.1. For any $n \in \mathbb{N}$ and $i = 1, 2$ there exist polynomials $P_{i,n} \in \wp_n$, regions $G^i \subset \mathbb{C}$ and constants $c_9 = c_9(G^1) > 0, c_{10} = c_{10}(G^2) > 0$ such that

$$(1.22) \quad \|P_{1,n}\|_{C(\overline{G^1})} \geq c_9 n^{\frac{1}{p}} \|P_{1,n}\|_{\mathcal{L}_p(\partial G^1)},$$

and

$$(1.23) \quad |P_{2,n}(z)| \geq c_{10} |\Phi(z)|^n \|P_{2,n}\|_{\mathcal{L}_2(\partial G^2)}, \forall z \in F,$$

where F is a closed subset in $\overline{\Omega} \setminus \overline{G^2}$.

2. Some auxiliary results

For the nonnegative functions $a > 0$ and $b > 0$, we shall use the notations “ $a \leq b$ ” (order inequality), if $a \leq cb$ and “ $a \asymp b$ ” are equivalent to $c_1a \leq b \leq c_2a$ for some constants c, c_1, c_2 (independent of a and b) respectively.

We can find a well known definition of a K -quasiconformal curve in [5], [13, p. 97], [17, p. 286] and [19] as follows:

Definition 2.1 ([13, p. 97], [19]). The Jordan curve (or arc) L is called K -quasiconformal ($K \geq 1$), if there is a K -quasiconformal mapping f of the region $D \supset L$ such that $f(L)$ is a circle (or line segment).

Let $F(L)$ denote the set of all sense preserving plane homeomorphisms f of the region $D \supset L$ such that $f(L)$ is a line segment (or circle) and let

$$K_L := \inf \{K(f) : f \in F(L)\},$$

where $K(f)$ is the maximal dilatation of a such mapping f . Then L is a quasiconformal curve if $K_L < \infty$, and L is a K -quasiconformal curve if $K_L \leq K$.

We well know that there exist quasiconformal curves which are not rectifiable [13, p. 104].

According to the “three-point” criterion [5, p. 100], every piecewise Dini-smooth curve (without any cusps) is quasiconformal.

Lemma 2.1 ([1], [3]). *Let L be a K -quasiconformal curve, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \leq d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $(z_2, z_3 \in G \cap \{z : |z - z_1| \leq d(z_1, L_{R_0})\}$; $w_j = \varphi(z_j)$), $j = 1, 2, 3$. Then*

- a) *The statements $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$ are equivalent. So are $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$;*
- b) *If $|z_1 - z_2| \leq |z_1 - z_3|$, then*

$$\left| \frac{w_1 - w_3}{w_1 - w_2} \right|^{K^2} \asymp \left| \frac{z_1 - z_3}{z_1 - z_2} \right| \asymp \left| \frac{w_1 - w_3}{w_1 - w_2} \right|^{K^{-2}},$$

where $0 < r_0 < 1$, $R_0 := r_0^{-1}$ are constants, depending on G .

Recall that for $0 < \delta_j < \delta_0 := \frac{1}{4} \min \{|z_i - z_j| : i, j = 1, 2, \dots, m, i \neq j\}$, we put $\Omega(z_j, \delta_j) := \Omega \cap \{z : |z - z_j| \leq \delta_j\}$; $\delta := \min_{1 \leq j \leq m} \delta_j$, $\Omega(\delta) := \bigcup_{j=1}^m \Omega(z_j, \delta)$, $\widehat{\Omega} := \Omega \setminus \Omega(\delta)$. Additionally, let $\Delta_j := \Phi(\Omega(z_j, \delta))$, $\Delta(\delta) := \bigcup_{j=1}^m \Phi(\Omega(z_j, \delta))$, $\widehat{\Delta}(\delta) := \Delta \setminus \Delta(\delta)$. Let $w_j := \Phi(z_j)$ and for $\varphi_j := \arg w_j$, $j = 1, 2, \dots, m$, we put $\Delta'_j := \left\{ t = Re^{i\theta} : R > 1, \frac{\varphi_{j-1} + \varphi_j}{2} \leq \theta < \frac{\varphi_j + \varphi_{j+1}}{2} \right\}$, where $\varphi_0 \equiv \varphi_m$, $\varphi_1 \equiv \varphi_{m+1}$; $\Omega_j := \Psi(\Delta'_j)$, $L^j := L \cap \overline{\Omega}_j$, $i = 1, 2, \dots, m$. Clearly, $\Omega = \bigcup_{j=1}^m \Omega_j$. $L^j_R := L_R \cap \overline{\Omega}_j$, $F^i := \Phi(L^i) = \overline{\Delta}_i \cap \{\tau : |\tau| = 1\}$, $F^i_R := \Phi(L^i_R) = \overline{\Delta}_i \cap \{\tau : |\tau| = R\}$, $i = \overline{1, m}$.

The following lemma is a consequence of the results given in [17, pp. 41–58], [9, pp. 32–36], and estimation for the $|\Psi'|$ (see, for example, [8, Th. 2.8]):

$$(2.1) \quad |\Psi'(\tau)| \asymp \frac{d(\Psi(\tau), L)}{|\tau| - 1}.$$

Lemma 2.2. *Let a Jordan region $G \in PDS(\lambda_j; 0)$, $0 < \lambda_j \leq 2$, $j = \overline{1, m_1}$. Then,*

- i) *for any $w \in \Delta_j$, $|\Psi(w) - \Psi(w_j)| \asymp |w - w_j|^{\lambda_j}$, $|\Psi'(w)| \asymp |w - w_j|^{\lambda_j - 1}$;*
- ii) *for any $w \in \overline{\Delta} \setminus \Delta_j$, $|\Psi(w) - \Psi(w_j)| \asymp |w - w_j|$, $|\Psi'(w)| \asymp 1$.*

Let L be a quasiconformal curve; $y(\zeta)$ denotes the regular quasiconformal reflection across L (see, for example, [1], [8], [13]), and for any $R > 1$, let $L^* := y(L_R)$, $G^* := \text{int} L^*$, $\Omega^* := \text{ext} L^*$; $w = \Phi_R(z)$ be the conformal mapping of Ω^* onto the Δ normalized by $\Phi_R(\infty) = \infty$, $\lim_{z \rightarrow \infty} \frac{\Phi_R(z)}{z} > 0$, and $\Psi_R := \Phi_R^{-1}$.

For $t > 1$, let us set $L_t^* := \{z : |\Phi_R(z)| = t\}$. According to [7], for all $z \in L^*$ and $t \in L$ such that $|z - t| = d(z, L)$ we have

$$(2.2) \quad d(z, L) \asymp d(t, L_R) \asymp d(z, L_R^*).$$

Lemma 2.3 ([1]). *Let L be a quasiconformal curve. Then, for any polynomial $P_n(z) \in \wp_n$, $n \in \mathbb{N}$ and $R > 1$, we have*

$$(2.3) \quad \|P_n\|_{\mathcal{C}(\overline{G})} \leq \|P_n\|_{\mathcal{C}(\overline{G}^*)}.$$

Let $\{z_j\}_{j=1}^m$ be a fixed system of the points on L and the weight function $h(z)$ defined as (1.1):

Lemma 2.4. *Let L be a rectifiable Jordan curve $h(z)$ as defined in (1.1). Then, for arbitrary $P_n(z) \in \wp_n$, any $R > 1$ and $n \in \mathbb{N}$, we have*

$$(2.4) \quad \|P_n\|_{\mathcal{L}_p(h, L_R)} \leq R^{n + \frac{1 + \gamma^*}{p}} \|P_n\|_{\mathcal{L}_p(h, L)}, \quad p > 0,$$

where $\gamma^* = \max\{\gamma_k : \gamma_k \in \Gamma, k \leq m\}$.

Proof. Let us set:

$$f_n(t) := \prod_{j=1}^m \left[t \left(\Psi\left(\frac{1}{t}\right) - \Psi(w_j) \right) \right]^{\frac{\gamma_j}{p}} t^n P_n \left(\Psi\left(\frac{1}{t}\right) \right) \cdot \left(\Psi'\left(\frac{1}{t}\right) \right)^{\frac{1}{p}}.$$

Since f_n is analytic in B and L is rectifiable, f_n belongs to Hardy class H_p , and then, by Hardy’s convexity theorem, we have

$$\int_{|t|=\frac{1}{R}} |f_n(t)|^p \frac{|dt|}{|t|} \leq \int_{|t|=1} |f_n(t)|^p |dt|,$$

which implies that

$$\begin{aligned} & \int_{L_R} h(z) |P_n(z)|^p |dz| \\ &= \int_{|w|=R} \prod_{j=1}^m |\Psi(w) - \Psi(w_j)|^{\gamma_j} \left| P_n(\Psi(w)) \left(\Psi'(w) \right)^{\frac{1}{p}} \right|^p |dw| \\ &= \int_{|t|=\frac{1}{R}} \prod_{j=1}^m \left| \Psi\left(\frac{1}{t}\right) - \Psi(w_j) \right|^{\gamma_j} \left| P_n\left(\Psi\left(\frac{1}{t}\right)\right) \left(\Psi'\left(\frac{1}{t}\right) \right)^{\frac{1}{p}} \right|^p |dt| \\ &\leq R^{np+1+\gamma_j} \int_{|t|=1} \prod_{j=1}^m \left| \Psi\left(\frac{1}{t}\right) - \Psi(w_j) \right|^{\gamma_j} \left| P_n\left(\Psi\left(\frac{1}{t}\right)\right) \left(\Psi'\left(\frac{1}{t}\right) \right)^{\frac{1}{p}} \right|^p |dt| \\ &= R^{np+1+\gamma_j} \int_L h(z) |P_n(z)|^p |dz|. \end{aligned}$$

We completed the proof of (2.4). □

Remark 2.1. In case of $h(z) \equiv 1$, the estimate (2.4) has been proved in [11].

3. Proofs

3.1. Proof of Theorem 1.1

Proof. Suppose that $G \in PDS(\lambda_i; f_j)$ for some $0 < \lambda_i \leq 2$, $i = \overline{1, m_1}$ and $f_j(x) = cx^{1+\alpha_j}$, $\alpha_j > 0$, $j = \overline{m_1 + 1, m}$; $h(z)$ be defined in (1.1). Let's denote

$$(3.1) \quad T_n(z) := \frac{P_n(z)}{\Phi^{n+1}(z)},$$

and, Cauchy integral representation for T_n :

$$(3.2) \quad T_n(z) = -\frac{1}{2\pi i} \int_L T_n(\zeta) \frac{d\zeta}{\zeta - z}, \quad z \in \Omega.$$

Hence

$$|T_n(z)| \leq \frac{1}{2\pi} \int_L |T_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|} \leq \frac{1}{2\pi d(z, L)} \int_L |P_n(\zeta)| |d\zeta|,$$

since $|\Phi(\zeta)| = 1$, and

$$(3.3) \quad |P_n(z)| \leq \frac{|\Phi(z)|^{n+1}}{2\pi d(z, L)} \int_L |P_n(\zeta)| |d\zeta|.$$

Let

$$(3.4) \quad A_n := \int_L |P_n(\zeta)| |d\zeta| = \sum_{i=1}^m \int_{L^i} |P_n(\zeta)| |d\zeta|.$$

Multiplying the numerator and denominator of the integrand by $\prod_{j=1}^m |\zeta - z_j|^{\frac{\gamma_j}{p}}$, after then applying Hölder inequality, we obtain:

$$(3.5) \quad A_n \leq \sum_{i=1}^m \left(\int_{L^i} \prod_{j=1}^m |\zeta - z_j|^{\gamma_j} |P_n(\zeta)|^p |d\zeta| \right)^{\frac{1}{p}} \times \left(\int_{L^i} \frac{|d\zeta|}{\prod_{j=1}^m |\zeta - z_j|^{\frac{q\gamma_j}{p}}} \right)^{\frac{1}{q}} =: \sum_{i=1}^m A_n^i,$$

where

$$A_n^i := \left(\int_{L^i} h(\zeta) |P_n(\zeta)|^p |d\zeta| \right)^{\frac{1}{p}} \left(\int_{L^i} \frac{|d\zeta|}{\prod_{j=1}^m |\zeta - z_j|^{\frac{q\gamma_j}{p}}} \right)^{\frac{1}{q}} =: J_{n,1}^i \cdot J_{n,2}^i, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

For the $J_{n,1}^i$ we have:

$$(3.6) \quad J_{n,1}^i \leq \|P_n\|_{\mathcal{L}_p(h,L)}, \quad i = \overline{1, m}.$$

Then, from (3.5) and (3.6) we obtain:

$$A_n \leq \|P_n\|_{\mathcal{L}_p(h,L)} \sum_{i=1}^m (J_{n,2}^i)^{\frac{1}{q}}.$$

For the integral $J_{n,2}^i$ we get:

$$(3.7) \quad J_{n,2}^i := \int_{L^i} \frac{|d\zeta|}{\prod_{j=1}^m |\zeta - z_j|^{\frac{q\gamma_j}{p}}} \asymp \int_{L^i} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i(q-1)}},$$

since the points $\{z_j\}_{j=1}^m$ are distinct on L . For simplicity calculations, we assume that $m_1 = 1, m = 2, z_1 = -1, z_2 = 1; (-1, 1) \subset G$ and let local coordinate axis in Definition 1.2 be parallel to OX and OY in the coordinate system; $L = L^+ \cup L^-$, where $L^+ := \{z \in L : \text{Im } z \geq 0\}, L^- := \{z \in L : \text{Im } z < 0\}$; Let $w^\pm := \left\{ w = e^{i\theta} : \theta = \frac{\varphi_1 \pm \varphi_2}{2} \right\}, z^\pm \in \Psi(w^\pm), l_i^\pm(z_i, z^\pm)$ -denoted the arcs, connected the points z_i which z^\pm , respectively; $|l_i^\pm| := \text{mes } l_i^\pm(z_i, z^\pm), i = 1, 2$. Let z_0 be taken as an arbitrary point on L^+ (or on L^- subject to the chosen direction). Then, from (3.7), we have:

$$(3.8) \quad A_n \leq \|P_n\|_{\mathcal{L}_p(h,L)} \sum_{i=1}^2 (J_{n,2}^i)^{\frac{1}{q}},$$

where

$$(3.9) \quad J_{n,2}^1 = \int_{L^1} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1(q-1)}}; \quad J_{n,2}^2 = \int_{L^2} \frac{|d\zeta|}{|\zeta - z_2|^{\gamma_2(q-1)}}.$$

For estimation the integral, we give some notations:

$$\begin{aligned} R &= 1 + \frac{1}{n}; \quad d_{i,R} := d(z_i, L_R); \\ E_1^{1,\pm} &:= \{ \zeta \in L^1 : |\zeta - z_1| < c_1 d_{1,R} \}, \\ E_2^{1,\pm} &:= \{ \zeta \in L^1 : c_1 d_{1,R} \leq |\zeta - z_1| \leq |l_1^\pm| \}, \\ E_1^{2,\pm} &:= \{ \zeta \in L^2 : |\zeta - z_2| < c_2 d_{2,R} \}, \\ E_2^{2,\pm} &:= \{ \zeta \in L^2 : c_2 d_{2,R} \leq |\zeta - z_2| \leq |l_2^\pm| \}; \\ I_{n,k}^{i,\pm} &:= I_{n,k}^i(E_k^{i,\pm}) := \int_{E_k^{i,\pm}} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i(q-1)}}; \quad i, k = 1, 2. \end{aligned}$$

Taking into consideration these notations, (3.8) can be written as:

$$(3.10) \quad \begin{aligned} A_n &\leq \|P_n\|_{\mathcal{L}_p(h,L)} \sum_{i=1}^2 (J_{n,2}^i)^{\frac{1}{q}} \\ &=: \|P_n\|_{\mathcal{L}_p(h,L)} \sum_{i=1}^2 \left[I_{n,1}^i(E_1^{i,\pm}) + I_{n,2}^i(E_2^{i,\pm}) \right]^{\frac{1}{q}} \end{aligned}$$

$$=: \|P_n\|_{\mathcal{L}_p(h,L)} \sum_{i=1}^2 \left[I_{n,1}^{i,\pm} + I_{n,2}^{i,\pm} \right]^{\frac{1}{q}}, \quad i = 1, 2.$$

According to (3.3) and (3.4), it suffices to estimate the integrals $I_{n,k}^{i,\pm}$ for each $i = 1, 2$ and $k = 1, 2, 3$.

Let's start with the evaluation of the follows integral $J_{n,2}^1$ from (3.9) and (3.10):

$$\begin{aligned} (3.11) \quad J_{n,2}^1 &= \int_{L^1} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1(q-1)}} \\ &:= \left(\sum_{k=1}^2 \int_{E_k^{1,\pm}} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_1(q-1)}} \right)^{\frac{1}{q}} =: \left[I_{n,1}^{1,\pm} + I_{n,2}^{1,\pm} \right]^{\frac{1}{q}}. \end{aligned}$$

Given the possible values of γ_i ($-1 < \gamma_i < 0$, $\gamma_i \geq 0$, $i = 1, 2$), we will consider the estimates for the $J_{n,2}^1$ separately.

Let $\gamma_1 \geq 0$ and $\gamma_2 \geq 0$. In this case for the integral $J_{n,2}^1$, we get:

$$\begin{aligned} (3.12) \quad I_{n,1}^{1,\pm} &\preceq \int_{E_1^{1,\pm}} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1(q-1)}} \\ &\preceq \int_0^{c_1 d_{1,R}} \frac{ds}{s^{\gamma_1(q-1)}} \preceq \begin{cases} d_{1,R}^{1-\gamma_1(q-1)}, & \gamma_1 > p-1, \\ 1, & \gamma_1 \leq p-1; \end{cases} \\ I_{n,2}^{1,\pm} &\preceq \int_{E_2^{1,\pm}} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1(q-1)}} \preceq \int_{c_1 d_{1,R}}^{|i_1^\pm|} \frac{ds}{s^{\gamma_1(q-1)}} \preceq \begin{cases} d_{1,R}^{1-\gamma_1(q-1)}, & \gamma_1 > p-1, \\ \ln \frac{1}{d_{1,R}}, & \gamma_1 = p-1, \\ 1, & \gamma_1 < p-1; \end{cases} \end{aligned}$$

Similar estimates for the integral J_n^2 given:

$$\begin{aligned} (3.13) \quad I_{n,1}^2 &\preceq \int_{E_1^2} \frac{|d\zeta|}{|\zeta - z_2|^{\gamma_2(q-1)}} \\ &\preceq \int_0^{c_2 d_{2,R}} \frac{ds}{s^{\gamma_2(q-1)}} \preceq \begin{cases} d_{2,R}^{1-\gamma_2(q-1)}, & \gamma_2 > p-1, \\ 1, & \gamma_2 \leq p-1; \end{cases} \\ I_{n,2}^2 &\preceq \int_{E_2^{2,\pm}} \frac{|d\zeta|}{|\zeta - z_2|^{\gamma_2(q-1)}} \preceq \int_{c_2 d_{2,R}}^{|i_2^\pm|} \frac{ds}{s^{\gamma_2(q-1)}} \preceq \begin{cases} d_{2,R}^{1-\gamma_2(q-1)}, & \gamma_2 > p-1, \\ \ln \frac{1}{d_{2,R}}, & \gamma_2 = p-1, \\ 1, & \gamma_2 < p-1; \end{cases} \end{aligned}$$

Let $\gamma_1 < 0$ and $\gamma_2 < 0$. Then, analogously to the (3.12) and (3.13)

$$(3.14) \quad \begin{aligned} I_{n,1}^1 &\preceq \int_{E_1^1} |\zeta - z_1|^{(-\gamma_1)(q-1)} |d\zeta| \preceq d_{1,n}^{(-\gamma_1)(q-1)} \text{mes} E_1^1 \preceq 1, \\ I_{n,2}^1 &\preceq \int_{E_2^1} |\zeta - z_1|^{(-\gamma_1)(q-1)} |d\zeta| \preceq |l_1^\pm|^{(-\gamma_1)(q-1)+1} \preceq 1, \end{aligned}$$

and

$$(3.15) \quad \begin{aligned} I_{n,1}^{2,\pm} &\preceq \int_{E_1^{2,\pm}} |\zeta - z_2|^{(-\gamma_2)(q-1)} |d\zeta| \preceq d_{2,R}^{(-\gamma_2)(q-1)} \text{mes} E_1^{2,\pm} \preceq 1, \\ I_{n,2}^{2,\pm} &\preceq \int_{E_2^{2,\pm}} |\zeta - z_2|^{(-\gamma_2)(q-1)} |d\zeta| \preceq |l_2^\pm|^{(-\gamma_2)(q-1)+1} \preceq 1. \end{aligned}$$

Therefore, in this case, from (3.10)-(3.15), we obtain:

$$(3.16) \quad A_n \preceq \|P_n\|_{\mathcal{L}_p(h,L)} \begin{cases} d_{1,R}^{\frac{1-\gamma_1(q-1)}{q}} + d_{2,R}^{\frac{1-\gamma_2(q-1)}{q}}, & \gamma_1, \gamma_2 > p-1, \\ \left(\ln \frac{1}{d_{1,R}}\right)^{\frac{1}{q}} + \left(\ln \frac{1}{d_{2,R}}\right)^{\frac{1}{q}}, & \gamma_1 = \gamma_2 = p-1, \\ 1, & \gamma_1, \gamma_2 < p-1. \end{cases}$$

Comparing (3.3), (3.4) and (3.16), we have:

$$(3.17) \quad |P_n(z)| \leq c \frac{B_{n,1}^0}{d(z,L)} \|P_n\|_{\mathcal{L}_p(h,L)} |\Phi(z)|^{n+1},$$

where $c = c(G, p, \gamma_i) > 0$, $i = \overline{1, p}$, constant independent from n and z , and

$$(3.18) \quad B_{n,1}^0 := \begin{cases} d_{1,R}^{1-\gamma_1(q-1)} + d_{2,R}^{1-\gamma_2(q-1)}, & \gamma_1, \gamma_2 > p-1, \\ \ln \frac{1}{d_{1,R}} + \ln \frac{1}{d_{2,R}}, & \gamma_1 = \gamma_2 = p-1, \\ 1, & \gamma_1, \gamma_2 < p-1. \end{cases}$$

According to Lemma 2.2, for the point z_1 we get:

$$(3.19) \quad d_{1,R} \succeq n^{-\tilde{\lambda}_1}.$$

For the estimate $d_{2,R}$, let's set: $z_R \in L_R$ such that $d_{2,R} = |z_2 - z_R|$, $\zeta^\pm \in L^\pm$ such that $d(z_R, L^2 \cap L^\pm) := d(z_R, L^+)$; $z_2^\pm := \zeta \in L^2 : |\zeta - z_2| = c_2 d_{2,R}$. Under this notations, from Lemma 2.1, we obtain:

$$(3.20) \quad d_R^\pm := d(z_R, L^2 \cap L^\pm) \asymp |z_R - z_2^\pm| \asymp d_{2,R}^{1+\alpha_2}.$$

In this, $d_{2,R} = (d_R^\pm)^{\frac{1}{1+\alpha_2}}$. On the other hand, according to Lemma 2.2 and [6, Corollary 2], we get: $d_R^\pm \succeq n^{-1}$. Therefore,

$$(3.21) \quad d_{2,R} \succeq n^{-\frac{1}{1+\alpha_2}}.$$

Comparing (3.17)-(3.21), we get:

$$|P_n(z)| \preceq \frac{B_{n,1}^0}{d(z,L)} \|P_n\|_{\mathcal{L}_p(h,L)} |\Phi(z)|^{n+1},$$

where

$$(3.22) \quad B_{n,1}^0 \preceq \begin{cases} n^{\frac{(\gamma_1(q-1)-1)\tilde{\lambda}_1}{q}} + n^{\frac{\gamma_2(q-1)-1}{q(1+\alpha_2)}}, & \gamma_1, \gamma_2 > p-1, \\ (\ln n)^{\frac{1}{q}}, & \gamma_1 = \gamma_2 = p-1, \\ 1, & \gamma_1, \gamma_2 < p-1. \end{cases}$$

Combining the corresponding estimates for each point $\{z_j\}$, $j = \overline{1, m}$, and taking into account the above notation, we complete the proof for the points $z \in \Omega$.

Let $z \in G$ be an arbitrary fixed point. Cauchy integral representation for the G gives

$$(3.23) \quad P_n(z) = \frac{1}{2\pi i} \int_L P_n(\zeta) \frac{d\zeta}{\zeta - z}, \quad z \in G.$$

In this, we have

$$(3.24) \quad \begin{aligned} |P_n(z)| &= \frac{1}{2\pi} \int_L |P_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|} \\ &\leq \frac{1}{2\pi d(z,L)} \int_L |P_n(\zeta)| |d\zeta| =: \frac{1}{2\pi d(z,L)} A_n, \end{aligned}$$

where A_n defined as in (3.4). Combining relations (3.16), (3.18) and (3.24), we complete the proof. \square

3.2. Proof of Theorem 1.3

Proof. Let $G \in PDS(\lambda_i; f_j)$ for some $0 < \lambda_i \leq 2$, $i = \overline{1, m_1}$ and $f_j(x) = cx^{1+\alpha_j}$, $\alpha_j > 0$, $j = \overline{m_1+1, m}$, is given. If $G \in PDS(\lambda_i; 0)$ for some $0 < \lambda_i < 2$, $i = \overline{1, m_1}$, i.e., $\alpha_j = 0$ for all $j = \overline{m_1+1, m}$, the proof is just. Really, for any $z \in G$, according to Theorem 1.1, (3.23) and (3.24), we have

$$(3.25) \quad |P_n(z)| \preceq \frac{B_{n,1}''}{d(z,L)} \|P_n\|_{\mathcal{L}_p(h,L)}, \quad z \in G,$$

where

$$(3.26) \quad B''_{n;1} := \begin{cases} n^{\frac{(1-p+\tilde{\gamma}^1)\tilde{\lambda}}{p}}, & \text{if there is at least one } \gamma_i \in \Gamma_{1,k}^{(1)} \\ & \text{for some } i = \overline{1, k}, k \leq m_1; \\ (\ln n)^{1-\frac{1}{p}}, & \text{if there is at least one} \\ & \gamma_j \in \Gamma_{1,k}^{(2)} \setminus \Gamma_{1,m_1}^{(1)} \\ & \text{for some } j = \overline{1, k}, k \leq m_1; \\ 1, & \text{if } \gamma_j \in \Gamma_{1,k}^{(3)} \\ & \text{for all } j = \overline{1, m}. \end{cases}$$

On the other hand, according to the “three-point” criterion [5, p. 100], we see that $L := \partial G$ is Q -quasiconformal for some $Q > 1$. Let $z \in L^*$ be arbitrary fixed and $R = 1 + \frac{1}{n}$. According to Lemma 2.2 and (2.2), we obtain:

$$(3.27) \quad \begin{aligned} d(z, L) &\asymp d(t, L_R) \geq n^{-\tilde{\lambda}}, \\ |P_n(z)| &\leq n^{\tilde{\lambda}} B''_{n,1} \|P_n\|_{\mathcal{L}_p(h,L)}, \quad z \in \overline{G^*}. \end{aligned}$$

Combining (2.2), (3.25) and (3.27), from Lemma 2.3, we obtain the proof.

Now, we consider the general case. Let $\lambda_i = 2$ for some $i = \overline{1, m_1}$ and $\alpha_j > 0$ for some $j = \overline{m_1 + 1, m}$. Cauchy integral representation for a region G_R gives:

$$P_n(z) = \frac{1}{2\pi i} \int_{L_R} P_n(\zeta) \frac{d\zeta}{\zeta - z}, \quad z \in G_R.$$

Replacing the variable $\tau = \Phi(\zeta)$, multiplying the numerator and denominator of the integrand by $\prod_{i=1}^m |\Psi(\tau) - \Psi(w_i)|^{\frac{\gamma_i}{p}} |\Psi'(\tau)|^{\frac{1}{p}}$, according to the Hölder inequality, we get:

(3.28)

$$\begin{aligned} |P_n(z)| &\leq \frac{1}{2\pi} \int_{L_R} |P_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|} \\ &= \frac{1}{2\pi} \int_{|\tau|=R} \frac{|P_n(\Psi(\tau))| |\Psi'(\tau)|}{|\Psi(\tau) - \Psi(w)|} |d\tau| \\ &\leq \frac{1}{2\pi} \left(\int_{|\tau|=R} \prod_{j=1}^m |\Psi(\tau) - \Psi(w_j)|^{\gamma_j} |P_n(\Psi(\tau))|^p |\Psi'(\tau)| |d\tau| \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_{|\tau|=R} \frac{|\Psi'(\tau)|^{(1-\frac{1}{p})q}}{\prod_{j=1}^m |\Psi(\tau) - \Psi(w_j)|^{\frac{q\gamma_j}{p}} |\Psi(\tau) - \Psi(w')|^q} |d\tau| \right)^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2\pi} \left(\int_{L_R} h(\zeta) |P_n(\zeta)|^p |d\zeta| \right)^{\frac{1}{p}} \\
 &\quad \times \left(\int_{|\tau|=R} \frac{|\Psi'(\tau)|}{\prod_{j=1}^m |\Psi(\tau) - \Psi(w_j)|^{\gamma_j(q-1)} |\Psi(\tau) - \Psi(w')|^q} |d\tau| \right)^{\frac{1}{q}} \\
 &= \frac{1}{2\pi} J_{n,1} \times J_{n,2},
 \end{aligned}$$

where

$$\begin{aligned}
 J_{n,1} &:= \|P_n\|_{\mathcal{L}_p(h, L_R)}, \\
 J_{n,2} &:= \left(\int_{|\tau|=R} \frac{|\Psi'(\tau)|}{\prod_{j=1}^m |\Psi(\tau) - \Psi(w_j)|^{\gamma_j(q-1)} |\Psi(\tau) - \Psi(w)|^q} |d\tau| \right)^{\frac{1}{q}}.
 \end{aligned}$$

Then, from Lemma 2.4, we have:

$$(3.29) \quad |P_n(z)| \leq J_{n,1} \cdot J_{n,2} \leq \|P_n\|_{\mathcal{L}_p(h, L)} \cdot J_{n,2}, \quad z \in L, \quad w = \Phi(z).$$

In order to estimate the integral $J_{n,2}$, taking into account the estimation for the $|\Psi'|$ (2.1), we get

$$\begin{aligned}
 (3.30) \quad J_{n,2} &= \left(\sum_{i=1}^m \int_{F_R^i} \frac{|\Psi'(\tau)| |d\tau|}{\prod_{j=1}^m |\Psi(\tau) - \Psi(w_j)|^{\gamma_j(q-1)} |\Psi(\tau) - \Psi(w)|^q} \right)^{\frac{1}{q}} \\
 &=: \left(\sum_{i=1}^m J_{n,2}^i \right)^{\frac{1}{q}} \leq \sum_{i=1}^m (J_{n,2}^i)^{\frac{1}{q}},
 \end{aligned}$$

where

$$\begin{aligned}
 (3.31) \quad J_{n,2}^i &:= \int_{F_R^i} \frac{|\Psi'(\tau)| |d\tau|}{\prod_{j=1}^m |\Psi(\tau) - \Psi(w_j)|^{\gamma_j(q-1)} |\Psi(\tau) - \Psi(w)|^q}, \quad i = \overline{1, m}. \\
 &\asymp \int_{F_R^i} \frac{|\Psi'(\tau)| |d\tau|}{|\Psi(\tau) - \Psi(w_i)|^{\gamma_i(q-1)} |\Psi(\tau) - \Psi(w)|^q} =: J(F_R^i),
 \end{aligned}$$

since the points $w_i := \Phi(z_i)$ are distinct.

It remains to estimate the integrals $J(F_R^i)$ for each $i = \overline{1, m}$.

For simplicity of the our next calculations, we assume that:

$$(3.32) \quad i = 1, 2; \quad m_1 = 1, \quad m = 2; \quad z_1 = -1, \quad z_2 = 1; \quad R = 1 + \frac{1}{n}.$$

In addition to the previously notations, we introduce the following: $L_R = L_R^+ \cup L_R^-$, where $L_R^+ := \{z \in L_R : \text{Im } z \geq 0\}$, $L_R^- := \{z \in L_R : \text{Im } z < 0\}$; Let $w_R^\pm := \left\{ w = Re^{i\theta} : \theta = \frac{\varphi_1 \pm \varphi_2}{2} \right\}$, $z_R^\pm \in \Psi(w_R^\pm)$, let's set: $z_{i,R} \in L_R$ such that $d_{i,n} = |z_i - z_{i,R}|$ and $\zeta^\pm \in L^\pm$ such that $d(z_{2,R}, L^2 \cap L^\pm) := d(z_{2,R}, L^\pm)$; $z_i^\pm := \{ \zeta \in L^i : |\zeta - z_i| = c_i d(z_i, L_R) \}$, $z_{i,R}^\pm := \{ \zeta \in L_R^i : |\zeta - z_{i,R}| = c_i d(z_{i,R}, L_R) \}$, $w_{i,R}^\pm = \Phi(z_{i,R}^\pm) \cdot l_{i,R}^\pm(z_{i,R}^\pm, z_R^\pm)$ -denoted the arcs, connected the points $z_{i,R}^\pm$ with z_R^\pm , respectively and $|l_{i,R}^\pm| := \text{mes} l_{i,R}^\pm(z_{i,R}^\pm, z_R^\pm)$, $i = 1, 2$. Let z_0 be taken as an arbitrary fixed point on L^+ distinct from z_1 and z_2 (or on L^- subject to the chosen direction). For simplicity, without loss of generality, we assume that $z_0 = z_R^+$ ($z_0 = z_R^-$). We denote: $E_{1,R}^{1,\pm} := \{ \zeta \in L_R : |\zeta - z_1| < c_1 d_{1,R} \}$, $E_{2,R}^{1,\pm} := \left\{ \zeta \in L_R : c_1 d_{1,R} \leq |\zeta - z_1| \leq |l_{1,R}^\pm| \right\}$, $E_{1,R}^{2,\pm} := \{ \zeta \in L_R^2 : |\zeta - z_2| < c_2 d_{2,R} \}$, $E_{2,R}^{2,\pm} := \left\{ \zeta \in L_R^2 : c_2 d_{2,R} \leq |\zeta - z_2| \leq |l_{2,R}^\pm| \right\}$, and $F_{j,R}^{i,\pm} := \Phi(E_{j,R}^{i,\pm})$, $i, j = 1, 2$.

Taking into consideration these designations, (3.31) can be written as:

$$(3.33) \quad \begin{aligned} J_{n,2}^i &\asymp \sum_{i,j=1}^2 \int_{F_{j,R}^{i,\pm}} \frac{|\Psi'(\tau)| |d\tau|}{|\Psi(\tau) - \Psi(w_i)|^{\gamma_i(q-1)} |\Psi(\tau) - \Psi(w)|^q} \\ &=: \sum_{i,j=1}^2 J(F_{j,R}^{i,\pm}). \end{aligned}$$

So, we need to evaluate the integrals $J(F_{j,R}^{i,\pm})$ for each $i, j = 1, 2$.

Let be

$$(3.34) \quad \|P_n\|_{C(\overline{G})} =: |P_n(z')|, \quad z' \in L = L^1 \cup L^2,$$

and let $w' = \Phi(z')$. There are two possible cases: point z' may lie or L^1 , or L^2 .

1) Suppose first that $z' \in L^1$. In this case, from Lemma 2.2 and Lemma 2.1, we have:

1.1) If $z' \in E_1^{1,\pm}$, then

$$(3.35) \quad \begin{aligned} &J(F_{1,R}^{1,+}) + J(F_{1,R}^{1,-}) \\ &= \int_{F_{1,R}^{1,+} \cup F_{1,R}^{1,-}} \frac{|\Psi'(\tau)| |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma_1(q-1)} |\Psi(\tau) - \Psi(w')|^q} \\ &\asymp \int_{F_{1,R}^{1,+} \cup F_{1,R}^{1,-}} \frac{|\tau - w_1|^{\lambda_1 - 1} |d\tau|}{|\tau - w_1|^{\gamma_1(q-1)\lambda_1} |\tau - w'|^{q\lambda_1}} \\ &\asymp \int_{F_{1,R}^{1,+} \cup F_{1,R}^{1,-}} \frac{|d\tau|}{[\min \{ |\tau - w_1|; |\tau - w'| \}]^{(\gamma_1(q-1)+q)\lambda_1 - \lambda_1 + 1}} \\ &\asymp n^{(\gamma_1+1)(q-1)\lambda_1} \end{aligned}$$

for $\gamma_1 > 0$, and

$$\begin{aligned}
 (3.36) \quad & J(F_{1,R}^{1,+}) + J(F_{1,R}^{1,-}) \\
 &= \int_{F_{1,R}^{1,+} \cup F_{1,R}^{1,-}} \frac{|\Psi(\tau) - \Psi(w_1)|^{(-\gamma_1)(q-1)} |\Psi'(\tau)| |d\tau|}{|\Psi(\tau) - \Psi(w')|^q} \\
 &\preceq \int_{F_{1,R}^{1,+} \cup F_{1,R}^{1,-}} \frac{|\tau - w_1|^{(-\gamma_1)(q-1)\lambda_1 + \lambda_1 - 1} |d\tau|}{|\tau - w'|^{q\lambda_1}} \\
 &\preceq n^{(\gamma_1+1)(q-1)\lambda_1}
 \end{aligned}$$

for $-1 < \gamma_1 \leq 0$;

1.2) If $z' \in E_2^{1,\pm}$, then

$$\begin{aligned}
 (3.37) \quad & J(F_{1,R}^{1,+}) + J(F_{1,R}^{1,-}) \\
 &= \int_{F_{1,R}^{1,+} \cup F_{1,R}^{1,-}} \frac{|\Psi'(\tau)| |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma_1(q-1)} |\Psi(\tau) - \Psi(w')|^q} \\
 &\preceq \int_{F_{1,R}^{1,+} \cup F_{1,R}^{1,-}} \frac{|\tau - w_1|^{\lambda_1 - 1} |d\tau|}{|\tau - w_1|^{\gamma_1(q-1)\lambda_1 + q\lambda_1}} \\
 &\preceq \int_{F_{1,R}^{1,+} \cup F_{1,R}^{1,-}} \frac{|d\tau|}{|\tau - w_1|^{\gamma_1(q-1)\lambda_1 + q\lambda_1 - \lambda_1 + 1}} \\
 &\preceq n^{(\gamma_1+1)(q-1)\lambda_1}
 \end{aligned}$$

for all $\gamma_1 > -1$;

1.3) If $z' \in E_1^{1,\pm}$, then

$$\begin{aligned}
 (3.38) \quad & J(F_{2,R}^{1,+}) + J(F_{2,R}^{1,-}) \\
 &= \int_{F_{2,R}^{1,+} \cup F_{2,R}^{1,-}} \frac{|\Psi'(\tau)| |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma_1(q-1)} |\Psi(\tau) - \Psi(w')|^q} \\
 &\preceq \int_{F_{2,R}^{1,+} \cup F_{2,R}^{1,-}} \frac{|\Psi'(\tau)| |d\tau|}{\min\{|\Psi(\tau) - \Psi(w_1)|; |\Psi(\tau) - \Psi(w')|\}^{\gamma_1(q-1)+q}} \\
 &\preceq \int_{F_{2,R}^{1,+} \cup F_{2,R}^{1,-}} \frac{|d\tau|}{\min\{|\tau - w_1|; |\tau - w'|\}^{[\gamma_1(q-1)+q]\lambda_1 - \lambda_1 + 1}} \\
 &\preceq n^{(\gamma_1+1)(q-1)\lambda_1}
 \end{aligned}$$

for $\gamma_1 > 0$ and

$$\begin{aligned}
 (3.39) \quad & J(F_{2,R}^{1,+}) + J(F_{2,R}^{1,-}) \\
 &= \int_{F_{2,R}^{1,+} \cup F_{2,R}^{1,-}} \frac{|\Psi(\tau) - \Psi(w_1)|^{(-\gamma_1)(q-1)} |\Psi'(\tau)| |d\tau|}{|\Psi(\tau) - \Psi(w')|^q}
 \end{aligned}$$

$$\begin{aligned}
&\leq \int_{F_{2,R}^{1,+} \cup F_{2,R}^{1,-}} \frac{|\tau - w_1|^{(-\gamma_1)(q-1) + \lambda_1 - 1} |d\tau|}{|\tau - w'|^{q\lambda_1}} \\
&\leq \int_{F_{2,R}^{1,+} \cup F_{2,R}^{1,-}} \frac{|d\tau|}{\min\{|\tau - w_1|; |\tau - w'|\}^{[\gamma_1(q-1) + q]\lambda_1 - \lambda_1 + 1}} \\
&\leq n^{(\gamma_1 + 1)(q-1)\lambda_1}
\end{aligned}$$

for $-1 < \gamma_1 \leq 0$;

1.4) If $z' \in E_2^{1,\pm}$, then

$$\begin{aligned}
(3.40) \quad &J(F_{2,R}^{1,+}) + J(F_{2,R}^{1,-}) \\
&= \int_{F_{2,R}^{1,+} \cup F_{2,R}^{1,-}} \frac{|\Psi'(\tau)| |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma_1(q-1)} |\Psi(\tau) - \Psi(w')|^q} \\
&\leq \int_{F_{2,R}^{1,+} \cup F_{2,R}^{1,-}} \frac{|\Psi'(\tau)| |d\tau|}{[\min\{|\Psi(\tau) - \Psi(w_1)|; |\Psi(\tau) - \Psi(w')|\}]^{\gamma_1(q-1) + q - 1}} \\
&\leq \int_{F_{2,R}^{1,+} \cup F_{2,R}^{1,-}} \frac{|\tau - w_1|^{\lambda_1 - 1} |d\tau|}{[\min\{|\tau - w_1|; |\tau - w'|\}]^{\gamma_1(q-1) + q - 1}} \\
&\leq n^{(\gamma_1 + 1)(q-1)\lambda_1}
\end{aligned}$$

for $\gamma_1 > 0$, and

$$\begin{aligned}
(3.41) \quad &J(F_{2,R}^{1,+}) + J(F_{2,R}^{1,-}) \\
&= \int_{F_{2,R}^{1,+} \cup F_{2,R}^{1,-}} \frac{|\Psi(\tau) - \Psi(w_1)|^{(-\gamma_1)(q-1)} |\Psi'(\tau)| |d\tau|}{|\Psi(\tau) - \Psi(w')|^q} \\
&\leq \int_{F_{2,R}^{1,+} \cup F_{2,R}^{1,-}} \frac{|d\tau|}{\min\{|\tau - w_1|; |\tau - w'|\}^{[\gamma_1(q-1) + q]\lambda_1 - \lambda_1 + 1}} \\
&\leq n^{(\gamma_1 + 1)(q-1)\lambda_1}
\end{aligned}$$

for $-1 < \gamma_1 \leq 0$. Combining the relations (3.35)-(3.41), we obtain:

$$(3.42) \quad J(F_{1,R}^{1,+}) + J(F_{1,R}^{1,-}) \leq n^{(\tilde{\gamma}_1 + 1)(q-1)\tilde{\lambda}_1}.$$

2) Now, suppose that $z' \in L^2$. In this case, according to (2.1):

2.1) If $z' \in E_1^{2,\pm}$, then

$$\begin{aligned}
&J(F_{1,R}^{2,+}) + J(F_{1,R}^{2,-}) \\
&= \int_{F_{1,R}^{2,+} \cup F_{1,R}^{2,-}} \frac{|\Psi'(\tau)| |d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2(q-1)} |\Psi(\tau) - \Psi(w')|^q} \\
&\asymp \int_{F_{1,R}^{2,+} \cup F_{1,R}^{2,-}} \frac{d(\Psi(\tau), L) |d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2(q-1)} |\Psi(\tau) - \Psi(w')|^q (|\tau| - 1)} \\
&\leq \int_{F_{1,R}^{2,+}} \frac{|d\tau|}{d_{2,R}^{\gamma_2(q-1) - 1} |\Psi(\tau) - \Psi(w')|^q (|\tau| - 1)}
\end{aligned}$$

$$+ \int_{F_{1,R}^{2,-}} \frac{|d\tau|}{d_{2,R}^{\gamma_2(q-1)-1} |\Psi(\tau) - \Psi(w')|^q (|\tau| - 1)}$$

for all $\gamma_2 > -1$. The last two integrals are evaluated identically, therefore, we evaluate one of them, say the first. When $\tau \in F_{1,R}^{2,+}$ for the $|\Psi(\tau) - \Psi(w')|$ we obtain:

$$\begin{aligned} |\Psi(\tau) - \Psi(w')| &\succeq \max \{ |\Psi(\tau) - \Psi(w_2)|; |\Psi(\tau) - z_2^+| \} \\ &= |\Psi(\tau) - \Psi(w_2)| \succeq |\Psi(\tau) - z_2^+|^{\frac{1}{1+\beta_2}}. \end{aligned}$$

Then,

$$\begin{aligned} J(F_{1,R}^{2,+}) &\leq n \int_{F_{1,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau) - z_2^+|^{\frac{\gamma_2(q-1)+q-1}{1+\alpha_2}}} \\ &\leq n \int_{F_{1,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau) - z_2^+|^{\frac{(\gamma_2+1)(q-1)}{1+\alpha_2}}} \\ &\leq n \int_{F_{1,R}^{2,+}} \frac{|d\tau|}{|\tau - w_2^+|^{\frac{(\gamma_2+1)(q-1)}{1+\alpha_2}}} \\ &\leq \begin{cases} n^{\frac{(\gamma_2+1)(q-1)}{1+\alpha_2}}, & \frac{(\gamma_2+1)}{1+\alpha_2}(q-1) > 1, \\ n \ln n, & \frac{(\gamma_2+1)}{1+\alpha_2}(q-1) = 1, \\ n, & \frac{(\gamma_2+1)}{1+\alpha_2}(q-1) < 1, \end{cases} \end{aligned}$$

if $\gamma_2 > 0$, and

$$\begin{aligned} J(F_{1,R}^{2,+}) &\leq n \int_{F_{1,R}^{2,+}} \frac{|\Psi(\tau) - \Psi(w_2)|^{(-\gamma_2)(q-1)} |d\tau|}{|\Psi(\tau) - z_2^+|^{\frac{q-1}{1+\alpha_2}}} \\ &\leq n \int_{F_{1,R}^{2,+}} \frac{|d\tau|}{|\tau - w_2^+|^{\frac{q-1}{1+\alpha_2}}} \\ &\leq \begin{cases} n^{\frac{q-1}{1+\alpha_2}}, & \frac{q-1}{1+\alpha_2} > 1, \\ n \ln n, & \frac{q-1}{1+\alpha_2} = 1, \\ n, & \frac{q-1}{1+\alpha_2} < 1, \end{cases} \end{aligned}$$

if $-1 < \gamma_2 \leq 0$, and so, in this case we get:

$$(3.43) \quad J(F_{1,R}^{2,+}) + J(F_{1,R}^{2,-}) \leq \begin{cases} n^{\frac{(\tilde{\gamma}_2+1)(q-1)}{1+\alpha_2}}, & \frac{(\tilde{\gamma}_2+1)}{1+\alpha_2}(q-1) > 1, \\ n \ln n, & \frac{(\tilde{\gamma}_2+1)}{1+\alpha_2}(q-1) = 1, \\ n, & \frac{(\tilde{\gamma}_2+1)}{1+\alpha_2}(q-1) < 1. \end{cases}$$

2.2) If $z' \in E_2^{\pm}$, then

$$J(F_{1,R}^{2,+}) + J(F_{1,R}^{2,-})$$

$$\begin{aligned}
 &= \int_{F_{1,R}^{2,+} \cup F_{1,R}^{2,-}} \frac{d(\Psi(\tau), L) |d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2(q-1)} |\Psi(\tau) - \Psi(w')|^q (|\tau| - 1)} \\
 &\leq n \int_{F_{1,R}^{2,+} \cup F_{1,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2(q-1)} |\Psi(\tau) - \Psi(w')|^{q-1}}
 \end{aligned}$$

for all $\gamma_2 > -1$. When $\tau \in F_{1,R}^{2,+}$ for the $|\Psi(\tau) - \Psi(w')|$ we obtain: $|\Psi(\tau) - \Psi(w')| \geq |\Psi(\tau) - z_2^+|$, and analogous to previous case, we get:

$$\begin{aligned}
 J(F_{1,R}^{2,+}) &\leq n \int_{F_{1,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2(q-1)} |\Psi(\tau) - z_2^+|^{q-1}} \\
 &\leq n \int_{F_{1,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau) - z_2^+|^{\frac{\gamma_2(q-1)}{1+\alpha_2} + q-1}} \\
 &\leq n \int_{F_{1,R}^{2,+}} \frac{|d\tau|}{|\tau - w_2^+|^{\frac{\gamma_2(q-1)}{1+\alpha_2} + q-1}} \\
 &\leq \begin{cases} n \frac{\gamma_2(q-1)}{1+\alpha_2} + q-1, & \frac{\gamma_2(q-1)}{1+\alpha_2} + q-1 > 1, \\ n \ln n, & \frac{\gamma_2(q-1)}{1+\alpha_2} + q-1 = 1, \\ n, & \frac{\gamma_2(q-1)}{1+\alpha_2} + q-1 < 1, \end{cases}
 \end{aligned}$$

if $\gamma_2 > 0$, and

$$\begin{aligned}
 J(F_{1,R}^{2,+}) &\leq n \int_{F_{1,R}^{2,+}} \frac{|\Psi(\tau) - \Psi(w_2)|^{(-\gamma_2)(q-1)} |d\tau|}{|\Psi(\tau) - z_2^+|^{q-1}} \\
 &\leq n \int_{F_{1,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau) - z_2^+|^{q-1}} \\
 &\leq n \int_{F_{1,R}^{2,+}} \frac{|d\tau|}{|\tau - w_2^+|^{q-1}} \\
 &\leq \begin{cases} n^{q-1}, & q > 2, \\ n \ln n, & q = 2, \\ n, & q < 2, \end{cases}
 \end{aligned}$$

if $-1 < \gamma_2 \leq 0$. So, in this case we have:

$$(3.44) \quad J(F_{1,R}^{2,+}) + J(F_{1,R}^{2,-}) \leq \begin{cases} n^{\frac{\tilde{\gamma}_2(q-1)}{1+\alpha_2} + q-1}, & \frac{\tilde{\gamma}_2(q-1)}{1+\alpha_2} + q > 2, \\ n \ln n, & \frac{\tilde{\gamma}_2(q-1)}{1+\alpha_2} + q = 2, \\ n, & \frac{\tilde{\gamma}_2(q-1)}{1+\alpha_2} + q < 2. \end{cases}$$

2.3) If $z' \in E_1^{2,\pm}$, then

$$\begin{aligned}
 (3.45) \quad &J(F_{2,R}^{2,+}) + J(F_{2,R}^{2,-}) \\
 &= \int_{F_{2,R}^{2,+} \cup F_{2,R}^{2,-}} \frac{|\Psi'(\tau)| |d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2(q-1)} |\Psi(\tau) - \Psi(w')|^q}
 \end{aligned}$$

$$\begin{aligned} &\preceq n \int_{F_{2,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2(q-1)} |\Psi(\tau) - \Psi(w')|^{q-1}} \\ &\quad + n \int_{F_{2,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2(q-1)} |\Psi(\tau) - \Psi(w')|^{q-1}} \end{aligned}$$

for $\gamma_2 > 0$. The last two integrals are evaluated identically. Therefore, we evaluate one of them, say the first. For $\tau \in F_{2,R}^{2,+}$ and $z' \in E_1^{2,\pm}$, we have:

$$\begin{aligned} |\Psi(\tau) - \Psi(w')| &\succeq |\Psi(\tau) - z_2^+|; \\ |\Psi(\tau) - \Psi(w_2)| &\succeq d_{2,R} \succeq |z_{2,R} - z_2^+|^{\frac{1}{1+\alpha_2}} \succeq \left(\frac{1}{n}\right)^{\frac{1}{1+\alpha_2}}. \end{aligned}$$

Then

$$\begin{aligned} J(F_{2,R}^{2,+}) &\preceq n \int_{F_{2,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau) - z_2^+|^{\gamma_2(q-1)+q-1}} \\ &\preceq n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+1} \int_{F_{2,R}^{2,+}} \frac{|d\tau|}{|\tau - w_2^+|^{q-1}} \preceq \begin{cases} n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+q-1}, & q > 2, \\ n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+1} \ln n, & q = 2, \\ n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+1}, & q < 2, \end{cases} \end{aligned}$$

and so, for $\gamma_2 > 0$ we obtain:

$$J(F_{2,R}^{2,+}) + J(F_{2,R}^{2,-}) \preceq \begin{cases} n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+q-1}, & q > 2, \\ n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+1} \ln n, & q = 2, \\ n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+1}, & q < 2. \end{cases}$$

For $-1 < \gamma_2 \leq 0$ we get:

$$\begin{aligned} &J(F_{2,R}^{2,+}) + J(F_{2,R}^{2,-}) \\ &= \int_{F_{2,R}^{2,+} \cup F_{2,R}^{2,-}} \frac{|\Psi(\tau) - \Psi(w_2)|^{(-\gamma_2)(q-1)} |\Psi'(\tau)| |d\tau|}{|\Psi(\tau) - \Psi(w')|^q} \\ &\preceq n \int_{F_{2,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w')|^{q-1}} \preceq n \int_{F_{2,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau) - z_2^+|^{q-1}} \\ &\preceq n \int_{F_{2,R}^{2,+}} \frac{|d\tau|}{|\tau - w_2^+|^{q-1}} \preceq \begin{cases} n^q, & q > 2, \\ n \ln n, & q = 2, \\ n, & q < 2. \end{cases} \end{aligned}$$

Then, in this case we have:

$$(3.46) \quad J(F_{2,R}^{2,+}) + J(F_{2,R}^{2,-}) \preceq \begin{cases} n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+q-1}, & q > 2, \\ n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+1} \ln n, & q = 2, \\ n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+1}, & q < 2. \end{cases}$$

2.4) If $z' \in E_2^{2,+}$, then for $\gamma_2 > 0$

$$\begin{aligned}
 (3.47) \quad & J(F_{2,R}^{2,+}) \\
 &= \int_{F_{2,R}^{2,+}} \frac{|\Psi'(\tau)| |d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2(q-1)} |\Psi(\tau) - \Psi(w')|^q} \\
 &\preceq \frac{n}{d_{2,R}^{\gamma_2(q-1)}} \int_{F_{2,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w')|^{q-1}} \\
 &\preceq n^{1+\frac{\gamma_2(q-1)}{1+\alpha_2}} \int_{F_{2,R}^{2,+}} \frac{|d\tau|}{|\tau - w'|^{q-1}} \preceq \begin{cases} n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+q-1}, & q > 2, \\ n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+1} \ln n, & q = 2, \\ n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+1}, & q < 2 \end{cases}
 \end{aligned}$$

and

$$\begin{aligned}
 (3.48) \quad & J(F_{2,R}^{2,-}) \\
 &= \int_{F_{2,R}^{2,-}} \frac{|\Psi'(\tau)| |d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2(q-1)} |\Psi(\tau) - \Psi(w')|^q} \\
 &\preceq \frac{n}{d_{2,R}^{\gamma_2(q-1)}} \int_{F_{2,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w')|^{q-1}} \\
 &\preceq n^{1+\frac{\gamma_2(q-1)}{1+\alpha_2}} \int_{F_{2,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w')|^{q-1}} \\
 &\preceq n^{1+\frac{\gamma_2(q-1)}{1+\alpha_2}} \int_{F_{2,R}^{2,-}} \frac{|d\tau|}{|\tau - w'|^{q-1}} \preceq \begin{cases} n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+q-1}, & q > 2, \\ n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+1} \ln n, & q = 2, \\ n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+1}, & q < 2. \end{cases}
 \end{aligned}$$

Case of $z' \in E_2^{2,-}$ is absolute identically to the case $z' \in E_2^{2,-}$.

If $-1 < \gamma_2 \leq 0$, then

$$\begin{aligned}
 (3.49) \quad & J(F_{2,R}^{2,+}) = \int_{F_{2,R}^{2,+}} \frac{|\Psi(\tau) - \Psi(w_2)|^{(-\gamma_2)(q-1)} |\Psi'(\tau)| |d\tau|}{|\Psi(\tau) - \Psi(w')|^q} \\
 &\preceq \int_{F_{2,R}^{2,+}} \frac{|d\tau|}{(|\tau| - 1)^q} \preceq \int_{F_{2,R}^{2,+}} \frac{|d\tau|}{(|\tau| - 1)^q} \preceq n^{q-1}
 \end{aligned}$$

and

$$\begin{aligned}
 (3.50) \quad & J(F_{2,R}^{2,-}) = \int_{F_{2,R}^{2,-}} \frac{|\Psi(\tau) - \Psi(w_2)|^{(-\gamma_2)(q-1)} |\Psi'(\tau)| |d\tau|}{|\Psi(\tau) - \Psi(w')|^q} \\
 &\preceq n \int_{F_{2,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w')|^{q-1}} \preceq \begin{cases} n^{q-1}, & q > 2, \\ n \ln n, & q = 2, \\ n, & q < 2. \end{cases}
 \end{aligned}$$

Combining the relations (3.43)-(3.50), we obtain:

$$(3.51) \quad (J_{n,2}^2)^{\frac{1}{q}} \preceq \begin{cases} n^{\frac{1}{p}}, & p < 2, \\ n^{1-\frac{1}{p}} \ln n & p = 2 \\ n^{1-\frac{1}{p}}, & p > 2, \end{cases}$$

for each $-1 < \gamma_2 \leq 0$,

$$(3.52) \quad (J_{n,2}^2)^{\frac{1}{q}} \preceq \begin{cases} n^{\frac{\gamma_2}{p(1+\alpha_2)} + \frac{1}{p}}, & 1 < p < 2 + \frac{\gamma_2}{1+\alpha_2}, \\ n^{1-\frac{1}{p}} \ln n, & p = 2 + \frac{\gamma_2}{1+\alpha_2}, \\ n^{1-\frac{1}{p}}, & p > 2 + \frac{\gamma_2}{1+\alpha_2}, \end{cases}$$

for each $\gamma_2 > 0$.

Combining (3.33), (3.42), (3.51) and (3.52), from (3.33), for $i = 1, 2; m_1 = 1, m_2 = 1$, we get:

$$J_{n,2} \preceq n^{\frac{(\tilde{\gamma}_1+1)\tilde{\lambda}_1}{p}} + \begin{cases} n^{\frac{\gamma_2}{p(1+\alpha_2)} + \frac{1}{p}}, & 1 < p < 2 + \frac{\gamma_2}{1+\alpha_2}, \\ n^{1-\frac{1}{p}} \ln n, & p = 2 + \frac{\gamma_2}{1+\alpha_2}, \\ n^{1-\frac{1}{p}}, & p > 2 + \frac{\gamma_2}{1+\alpha_2}. \end{cases}$$

From (3.29), (3.30) and (3.31) according to (3.32) and (3.34), we have

$$\begin{aligned} & \|P_n\|_{C(\overline{G})} \\ & \preceq \|P_n\|_{\mathcal{L}_p(h,L)} \cdot J_{n,2} \\ & \preceq \|P_n\|_{\mathcal{L}_p(h,L)} \sum_{i=1}^m (J_{n,2}^i)^{\frac{1}{q}} \\ & \preceq \|P_n\|_{\mathcal{L}_p(h,L)} \sum_{i=1}^m \sum_{j=1}^2 \cdot \left(J(F_{j,R}^{i,\pm}) \right)^{\frac{1}{q}} \\ & \preceq \|P_n\|_{\mathcal{L}_p(h,L)} \left(\sum_{i=1}^{m_1} n^{\frac{(\tilde{\gamma}_i+1)(q-1)\tilde{\lambda}_i}{q}} + \sum_{i=m_1+1}^m n^{\frac{\tilde{\gamma}_i(q-1)-1}{q(1+\alpha_i)}+1} \right) \\ & \preceq \|P_n\|_{\mathcal{L}_p(h,L)} \cdot \left(\sum_{i=1}^{m_1} n^{\frac{(\tilde{\gamma}_i+1)\tilde{\lambda}_i}{p}} + \begin{cases} \sum_{i=m_1+1}^m n^{\frac{\tilde{\gamma}_i+1}{p(1+\alpha_i)} + \frac{\alpha_i}{p(1+\alpha_i)}}, & 1 < p < 2 + \frac{\tilde{\gamma}_i}{1+\alpha_i}, \\ (n \ln n)^{1-\frac{1}{p}}, & p = 2 + \frac{\tilde{\gamma}_i}{1+\alpha_i}, \\ n^{1-\frac{1}{p}}, & p > 2 + \frac{\tilde{\gamma}_i}{1+\alpha_i}, \end{cases} \right). \end{aligned}$$

and the proof is completed. □

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