On the Growth of Entire Functions of Bounded Index

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1. Introduction. Let $f(z)$ be a transcendental entire function. If there exists an integer N, independent of z, such that

$$\frac{|f^{(n)}(z)|}{n!} \leqq \max_{0 \leqq k \leqq N} \left\{ \frac{|f^{(k)}(z)|}{k!} \right\}$$

for all n and all z then $f(z)$ is said to be of bounded index. The least such integer N is called the index of $f(z)$. It is known [3] that in this case the growth of $f(z)$ is at most exponential type $(1, N + 1)$. If $f(z)$ satisfies a linear differential equation with polynomial coefficients

$$(1.1) \qquad P_0(z)f^{(k)}(z) + P_1(z)f^{(k-1)}(z) + \cdots + P_k(z)f(z) = Q(z)$$

where $P_j(z)$, $j = 0, 1, \cdots, k$, $Q(z)$ are polynomials and $P_0(z)$ $(\not\equiv 0)$ is of degree not less than that of any $P_j(z)$, then $f(z)$ is of bounded index [4]. Pugh and Shah [2] have shown that if

$$f(z) = \Pi_1^\infty \left(1 - \frac{z}{z_j}\right), \ |z_1| \geqq 5, \ |z_{j+1}| \geqq 5^j \ |z_j| \qquad (j = 1, 2, \cdots),$$

then $f(z)$ is of bounded index. If the zeros a_n of the canonical product $f(z)$ are real and positive and $a_{n+1}/a_n \geqq \gamma > 1$, then $f(z)$ is also of bounded index [1]. We extend this result and prove

Theorem 1. *Let*

$$f(z) = e^{\alpha z + \beta} \Pi_1^\infty \left(1 - \frac{z}{a_n}\right)$$

where α and β are any complex numbers, and

$$(1.2) \qquad a_{n+1}/a_n \geqq \gamma > 1, \ a_1 > 0.$$

Then each $f^{(k)}(z)$, $k = 0, 1, \cdots,$ is of bounded index.

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The functions $f(z)$ satisfying (1.1) are necessarily of positive rational order and of perfectly regular growth. We now show that there exist entire functions f of bounded index and arbitrarily slow growth.

Theorem 2. *Given an indefinitely increasing sequence* $\{\psi_n\}_1^\infty$ *of positive numbers and a transcendental entire function* $g(z)$, $g(0) \neq 0$, *there exists an entire function* $f(z)$ *of bounded index such that, for all* z *and* $k = 0, 1, \cdots$, *we have*

$$M(|z|, g) > M(|z|, f^{(k)}) > \psi_{k+1} M(|z|, f^{(k+1)}). \tag{1.3}$$

2. Proof of Theorem 1. We require the following lemmas.

Lemma 1. *Let* $f(z)$ *be an entire function and* R *and* M *be given positive numbers. Then there exists an integer* p *such that for* $|z| \leqq R$,

$$\max_{0 \leq k \leq p} \left\{ \frac{|f^{(k)}(z)|}{k!} \right\} \geqq M \frac{|f^{(j)}(z)|}{j!}, \qquad j = p+1, \; p+2, \cdots .$$

The proof is similar to that of (2.7) of [4] and is omitted.

Lemma 2. ([1]) *Let*

$$P(z) = \Pi_1^\infty \left(1 - \frac{z}{a_n}\right) \tag{2.1}$$

where the zeros a_n *satisfy the condition* (1.2).

Then given any $M > 0$, *there exists a number* $R = R(M, P)$ *such that for every* z, $|z| > R$,

$$\max \{ |P(z)|, |P'(z)| \} > M \frac{|P^{(j)}(z)|}{j!}, \qquad j = 2, 3, \cdots .$$

Let $\mathcal{E}$ denote the class of entire functions f such that given any $M > 0$, there exists $R = R(M, f)$ with the property that for every z, $|z| > R$,

$$\max \{ |f(z)|, |f'(z)| \} > M \frac{|f^{(j)}(z)|}{j!}, \qquad j \geqq 2.$$

Lemma 2 shows that the class $\mathcal{E}$ is not empty.

Lemma 3. *If* $Q(z) \in \mathcal{E}$, $Q'(z) \in \mathcal{E}$ *then* $Q'(z) + \lambda Q(z) \in \mathcal{E}$ *where* λ *is any complex number.*

Proof. We may assume $\lambda \neq 0$. Let $M > 2$ be given. Denote

$$H(z) = Q'(z) + \lambda Q(z),$$

and

$$M_0 = \left(1 + \frac{1}{|\lambda|}\right)^2 M + \frac{2}{|\lambda|^2}.$$

Since $Q \in \mathcal{E}$ there exists $R_0 = R_0(M_0)$ such that for every z, $|z| > R_0$,

$$\max\{|Q(z)|, |Q'(z)|\} > M_0 \frac{|Q^{(j)}(z)|}{j!}, \qquad j \geqq 2. \tag{2.2}$$

We consider two cases. Let z_0 be a given point such that $|z_0| > R_0$ and suppose first that $|Q'(z_0)| \geqq |Q(z_0)|$. Then

$$\begin{aligned} |H'(z_0)| &\geqq |\lambda|\, |Q'(z_0)| - |Q''(z_0)| \\ &\geqq |\lambda|\, |Q'(z_0)| - \frac{2}{M_0} |Q'(z_0)| \\ &= \left(|\lambda| - \frac{2}{M_0}\right) \max\{|Q(z_0)|, |Q'(z_0)|\}. \end{aligned}$$

If $|Q'(z_0)| < |Q(z_0)|$, then

$$\lambda H(z_0) - H'(z_0) = \lambda^2 Q(z_0) - Q''(z_0).$$

Hence

$$\begin{aligned} (|\lambda| + 1) \max\{|H(z_0)|, |H'(z_0)|\} &> |\lambda H(z_0) - H'(z_0)| \\ &= |\lambda^2 Q(z_0)| - |Q''(z_0)| \\ &> |\lambda^2 Q(z_0)| - \frac{2}{M_0} \max\{|Q(z_0)|, |Q'(z_0)|\} \\ &= \left(|\lambda|^2 - \frac{2}{M_0}\right) \max\{|Q(z_0)|, |Q'(z_0)|\}. \end{aligned}$$

Consequently

$$\max\{|H(z_0)|, |H'(z_0)|\} > \left[\frac{|\lambda|^2 - \frac{2}{M_0}}{|\lambda| + 1}\right] \max\{|Q(z_0)|, |Q'(z_0)|\}.$$

Since

$$|\lambda| - \frac{2}{M_0} > \frac{|\lambda|^2 - \frac{2}{M_0}}{|\lambda| + 1},$$

we have in both cases

$$\max\{|H(z)|, |H'(z)|\} > \left[\frac{|\lambda|^2 - \frac{2}{M_0}}{|\lambda| + 1}\right] \max\{|Q(z)|, |Q'(z)|\}. \tag{2.3}$$

Now $H^{(j)}(z) = Q^{(j+1)}(z) + \lambda Q^{(j)}(z)$ and hence for every z,

$$M_0 \frac{|H^{(j)}(z)|}{j!} \leqq (|\lambda| + 1) \max\left\{M_0 \frac{|Q^{(j+1)}(z)|}{j!}, M_0 \frac{|Q^{(j)}(z)|}{j!}\right\}, \qquad j \geqq 1. \tag{2.4}$$

Since $Q'(z) \varepsilon \mathcal{E}$ we have for $|z| > R_1 = R_1(M_0)$

$$\max\{|Q'(z)|, |Q''(z)|\} > M_0 \frac{|Q^{(j+1)}(z)|}{j!}, \qquad j \geqq 2. \tag{2.5}$$

Let $R = \max(R_0, R_1)$. Then (2.2) and (2.5) imply, for $|z| > R$,

$$\max\{|Q(z)|, |Q'(z)|\} > \frac{M_0}{2}|Q''(z)| > |Q''(z)|,$$

and hence

$$\max\{|Q(z)|, |Q'(z)|\} \geqq M_0 \max\left\{\frac{|Q^{(j)}(z)|}{j!}, \frac{|Q^{(j+1)}(z)|}{j!}\right\}, \qquad j \geqq 2. \tag{2.6}$$

For $|z| > R$ and $j \geqq 2$, we have from (2.3) (2.6) and (2.4),

$$\max\{|H(z)|, |H'(z)|\} > M\frac{|H^{(j)}(z)|}{j!}.$$

This proves the lemma.

Lemma 4. *If $Q(z) \in \mathcal{E}$ and λ is any complex number then the entire function $e^{\lambda z}Q(z)$ is of bounded index.*

Proof. Write $f(z) = e^{\lambda z}Q(z)$ and choose $M \geqq 2(1 + |\lambda|)e^{|\lambda|}$. Then

$$\frac{|f^{(k)}(z)|}{k!} \leqq |e^{\lambda z}| \sum_{j=0}^{k} \frac{k!}{j!\,(k-j)!} \frac{|\lambda|^j\,|Q^{(k-j)}(z)|}{k!}.$$

Hence, by Lemma 2, we have for $|z| > R$

$$\frac{|f^{(k)}(z)|}{k!} \leqq |e^{\lambda z}| \left(\frac{e^{|\lambda|}}{M} + \frac{|\lambda|^{(k-1)}}{(k-1)!} + \frac{|\lambda|^k}{k!}\right) \max\{|Q(z)|, |Q'(z)|\}$$

$$< \left(\frac{e^{|\lambda|}}{M} + \frac{|\lambda|^{(k-1)}}{(k-1)!} + \frac{|\lambda|^k}{k!}\right)(1 + |\lambda|) \max\{|f(z)|, |f'(z)|\}.$$

Choose L such that, for $k \geqq L$,

$$\left\{\frac{|\lambda|^{k-1}}{(k-1)!} + \frac{|\lambda|^k}{k!}\right\}(1 + |\lambda|) \leqq 1/2.$$

Then for $k \geqq N = \max(M, L)$ and $|z| > R$.

$$\max\{|f(z)|, |f'(z)|\} \geqq \frac{|f^{(k)}(z)|}{k!}.$$

This inequality and Lemma 1 complete the proof.

Proof of Theorem 1.

Write

$$Q_0(z) = P(z),$$

$$Q_n(z) = Q'_{n-1}(z) + \alpha Q_{n-1}(z), \qquad n = 1, 2, \cdots.$$

We may suppose that $\beta = 0$. Then Lemmas 2 and 4 prove that $f(z)$ is of bounded index. Furthermore by Laguerre's Theorem we see that the zeros of $P'(z)$ satisfy a condition of the type (1.2) with γ replaced by another number $\gamma' > 1$ (*cf*: (2.3) of [1]). Hence $P'(z) \varepsilon \mathcal{E}$ and so Lemmas 3 and 4 imply that $Q_1(z) \varepsilon \mathcal{E}$ and $e^{\alpha z}Q_1(z) = f'(z)$ is of bounded index.

Similarly $P^{(k)}(z) \varepsilon \mathcal{E}$ for every $k \geqq 1$ and so $Q_1^{(k)}(z) = \alpha P^{(k)}(z) + P^{(k+1)}(z) \varepsilon \mathcal{E}$. An induction argument on n shows that $Q_n^{(k)}(z) \varepsilon \mathcal{E}$. Hence $f^{(n)}(z) = e^{\alpha z}Q_n(z)$ $(n \geqq 1)$ is of bounded index.

3. Proof of Theorem 2. We require a lemma.

Lemma 5. ([1]) *Suppose* $P(z)$ *is defined by* (2.1) *where* $a_{n+1}/a_n \geqq n + 1$, $n = 2, 3, \cdots, a_2 > a_1 > 0$. *Write* $B = \max(A, 2A^2)$ *where* $A = (7.5)/(a_2 - a_1)$. *Then for every* z

$$B \max\{|P(z)|, |P'(z)|\} > |P''(z)|.$$

We now prove Theorem 2. We may assume that $1 < \psi_1 < \psi_2 < \cdots$. Since $g(z)$ is transcendental, $\log M(r, g)/\log r$ is a strictly increasing function of r for $r \geqq r_0(g)$.

We take $r_0 > 1$ and let

$$h(r) = \begin{cases} \dfrac{\log M(r, g)}{2 \log r}, & r \geqq r_0 \\ h(r_0), & r < r_0 . \end{cases} \tag{3.1}$$

Then $h(r) \to \infty$ as $r \to \infty$. Choose $a_1 \geqq \max(8, r_0, 2\psi_1)$ such that $h(a_1) \geqq 1$. Having chosen $a_1, a_2, \cdots a_n$ choose

$$a_{n+1} \geqq \max\{(n + 2)a_n, 2(1 + n)(2 + n)\psi_{n+1}\}, n \geqq 1 \tag{3.2}$$

such that $h(a_{n+1}) \geqq n + 1$. Let

$$F(z) = \Pi_1^\infty\left(1 - \frac{z}{a_n}\right). \tag{3.3}$$

Then $\log M(r, F) \sim N(r, 1/F) \leqq n(r, 1/F) \log r$

$$\leqq h(r) \log r$$

$$\leqq \frac{\log M(r, g)}{2}.$$

Hence $M(r, F) \leqq M(r, g), r \geqq r_1 > 0$. Write

$$c = |g(0)|, k = \max(1, c)$$

and

$$f(z) = \frac{c}{k} \frac{F(z)}{F(-r_1)} \tag{3.4}$$

$$= f(0)\Pi_1^\infty\left(1 - \frac{z}{a_n}\right).$$

Then $f(z)$ is an entire function and

$$M(r, f) < \frac{c}{k} M(r, g) \quad \text{for} \quad r \leqq r_1 . \tag{3.5}$$

When $r > r_1$,

$$M(r, f) = \frac{c}{k} \frac{1}{F(-r_1)} M(r, F) < M(r, g). \tag{3.5$'$}$$

Denote the zeros of $f^{(k)}(z)$ by $\{a_n^{(k)}\}_{n=1}^{\infty}$. Then we have the following

Lemma 6. (i) *If* $a_{n+1}/a_n \geqq n + 2$ *then for every* $k \geqq 1$, $a_{n+1}^{(k)}/a_n^{(k)} \geqq n + 2$.
(ii) *For every* $k \geqq 0$, $n \geqq 1$

$$a_n^{(k)} > \frac{n(n+1)}{(n+k)(n+k+1)} a_{n+k} .$$

We omit the proof since these results are similar to those in Lemma 4 (ii) and (5.7) of [1].

Now for $r \geqq 0$,

$$\begin{aligned} M(r, f') &= |f(0)| \left(\sum_1^{\infty} \frac{1}{a_n}\right) \Pi_1^{\infty}\left(1 + \frac{r}{a_n^{(1)}}\right) \\ &\leqq \left(\sum_1^{\infty} \frac{1}{a_n}\right) M(r, f) < \frac{2}{a_1} M(r, f) \leqq \frac{1}{\psi} M(r, f). \end{aligned}$$

By Lemma 5 and Lemma 6 we have for every z

$$\max\{|F(z)|, |F'(z)|\} > \frac{a_2}{12} |F''(z)| \geqq \psi_2 |F''(z)|;$$

$$\begin{aligned} \max\{|F^{(k)}(z)|, |F^{(k+1)}(z)|\} &> \frac{a_2^{(k)}}{12} |F^{(k+2)}(z)| \\ &> |F^{(k+2)}(z)| \frac{a_{2+k}}{2(2+k)(3+k)} \\ &> |F^{(k+2)}(z)| \psi_{k+2} . \end{aligned}$$

This proves that, for every z and $k \geqq 0$,

$$\max\{|f^{(k)}(z)|, |f^{(k+1)}(z)|\} > |f^{(k+2)}(z)| \psi_{k+2} . \tag{3.6}$$

Now

$$f^{(k+1)}(z) = f^{(k+1)}(0) \Pi_1^{\infty}\left(1 - \frac{z}{a_n^{(k+1)}}\right) ,$$

and so by Lemma 6,

$$M(r, f^{(k+1)}) = |f^{(k)}(0)| \left(\sum_{n=1}^{\infty} \frac{1}{a_n^{(k)}}\right) \Pi_1^{\infty}\left(1 + \frac{r}{a_n^{(k+1)}}\right)$$

$$< \frac{2}{a_1^{(k)}} M(r, f^{(k)})$$

$$< \frac{(k+1)(k+2)}{a_{k+1}} M(r, f^{(k)}) < \frac{M(r, f^{(k)})}{\psi_{k+1}};$$

that is,

$$M(r, f^{(k)}) > \psi_{k+1} M(r, f^{(k+1)}). \tag{3.7}$$

The inequality (3.6) shows that each $f^{(k)}(z)$ is of bounded index $N_k = 1$. The inequality (1.3) follows from (3.5), (3.5)′ and (3.7).

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