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ON THE H^1–L^1 BOUNDEDNESS OF OPERATORS

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ABSTRACT. We prove that if q is in $(1,\infty)$, Y is a Banach space, and T is a linear operator defined on the space of finite linear combinations of $(1,q)$-atoms in $\mathbb{R}^n$ with the property that

$$\sup\{\|Ta\|Y : a \text{ is a } (1,q)\text{-atom}\} < \infty,$$

then T admits a (unique) continuous extension to a bounded linear operator from $H^1(\mathbb{R}^n)$ to Y. We show that the same is true if we replace $(1,q)$-atoms by *continuous* $(1,\infty)$-atoms. This is known to be false for $(1,\infty)$-atoms.

1. Introduction

In a recent paper, M. Bownik [3] showed that there exists a linear functional F defined on finite linear combinations of $(1,\infty)$-atoms in $\mathbb{R}^n$ with the property that

$$\sup\{|F(a)| : a \text{ is a } (1,\infty)\text{-atom}\} < \infty,$$

but which does not admit a continuous extension to $H^1(\mathbb{R}^n)$. If v is a fixed function in $L^1(\mathbb{R}^n)\setminus\{0\}$, then the operator B, defined on finite linear combinations of $(1,\infty)$-atoms by $Bf = F(f)\,v$, satisfies

$$\sup\{\|Ba\|_{L^1(\mathbb{R}^n)} : a \text{ is a } (1,\infty)\text{-atom}\} < \infty$$

but does not admit an extension to a bounded operator from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. This shows that the argument "the operator T maps $(1,\infty)$-atoms uniformly into $L^1(\mathbb{R}^n)$, and hence it extends to a bounded operator from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$" is fallacious.

Fortunately, if T is a Calderón–Zygmund operator, then the uniform boundedness of T on $(1,\infty)$-atoms implies the boundedness from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ (see, for instance, [11, Ch. 7.3, Lemma 1], [2, Ch. 1.9], [7, Ch. III.7] and [8, Thm 6.7.1]).

The purpose of this paper is to show that the operator B constructed above is, to a certain extent, pathological. Indeed, we prove that if q is in $(1,\infty)$, Y is a Banach space, and T is a linear operator defined on finite linear combinations of $(1,q)$-atoms in $\mathbb{R}^n$ with the property that

$$\sup\{\|Ta\|_Y : a \text{ is a } (1,q)\text{-atom}\} < \infty, \tag{1.1}$$

then T admits a unique continuous extension to a bounded linear operator from $H^1(\mathbb{R}^n)$ to Y. The same conclusion holds if we assume that T is a linear operator

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on finite linear combinations of *continuous* $(1,\infty)$-atoms in $\mathbb{R}^n$ with the property that

$$\sup\{\|Ta\|_Y : a \text{ is a continuous } (1,\infty)\text{-atom}\} < \infty. \tag{1.2}$$

Note that this does not contradict Bownik's example. Indeed, the restriction of the operator B to continuous $(1,\infty)$-atoms extends to a bounded operator $\widetilde{B}$ from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. However, B and $\widetilde{B}$ will agree on continuous $(1,\infty)$-atoms but not on all $(1,\infty)$-atoms.

To explain the idea of the proofs of these results, we need more notation. Suppose that q is in $(1,\infty]$, and denote by $H^{1,q}_{\mathrm{fin}}(\mathbb{R}^n)$ the vector space of all finite linear combinations of $(1,q)$-atoms. Notice that $H^{1,q}_{\mathrm{fin}}(\mathbb{R}^n)$ consists of all $L^q(\mathbb{R}^n)$ functions with compact support and integral 0. Clearly, $H^{1,q}_{\mathrm{fin}}(\mathbb{R}^n)$ is a dense subspace of $H^1(\mathbb{R}^n)$. We may define a norm on $H^{1,q}_{\mathrm{fin}}(\mathbb{R}^n)$ as follows:

$$\|f\|_{H^{1,q}_{\mathrm{fin}}(\mathbb{R}^n)} = \inf\Big\{\sum_{j=1}^N |\lambda_j| : f = \sum_{j=1}^N \lambda_j\, a_j,\ a_j \text{ is a } (1,q)\text{-atom},\ N \in \mathbb{N}\Big\}.$$

Obviously $\|f\|_{H^1(\mathbb{R}^n)} \leq \|f\|_{H^{1,q}_{\mathrm{fin}}(\mathbb{R}^n)}$ for every f in $H^{1,q}_{\mathrm{fin}}(\mathbb{R}^n)$. An example due to Y. Meyer (see [12, p. 513], Bownik's paper [3] or [7, p. 370]) shows that $\|\cdot\|_{H^1(\mathbb{R}^n)}$ and $\|\cdot\|_{H^{1,\infty}_{\mathrm{fin}}(\mathbb{R}^n)}$ are inequivalent norms on $H^{1,\infty}_{\mathrm{fin}}(\mathbb{R}^n)$. This is the starting point of Bownik's construction.

We prove that Meyer's example itself is somewhat exceptional. Indeed, by using the maximal characterisation of $H^1(\mathbb{R}^n)$, we show that if $q < \infty$, then $\|\cdot\|_{H^1(\mathbb{R}^n)}$ and $\|\cdot\|_{H^{1,q}_{\mathrm{fin}}(\mathbb{R}^n)}$ are equivalent norms on $H^{1,q}_{\mathrm{fin}}(\mathbb{R}^n)$ (see Section 3). Similarly, we prove that $\|\cdot\|_{H^1(\mathbb{R}^n)}$ and $\|\cdot\|_{H^{1,\infty}_{\mathrm{fin}}(\mathbb{R}^n)}$ are equivalent norms on $H^{1,\infty}_{\mathrm{fin}}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. This immediately implies that operators defined on $H^{1,q}_{\mathrm{fin}}(\mathbb{R}^n)$ which have either property (1.1) or property (1.2) automatically extend to bounded operators from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.

As discussed briefly in Section 3, this equivalence of norms remains true for $H^p(\mathbb{R}^n)$ with $0 < p < 1$ and (p,q)-atoms.

The extension property for operators was also proved, by different methods, for $0 < p \leq 1$ and $(p,2)$-atoms and operators taking values in quasi-Banach spaces, by D. Yang and Y. Zhou [17].

A theory of Hardy spaces has been developed in spaces of homogeneous type; see R.R. Coifman and G. Weiss [5]. It is, however, not evident whether our results extend to this case in general. Nevertheless, let M be such a space. By a simple functional analysis argument, we show that if q is in $(1,\infty)$ and T is an operator defined on $H^{1,q}_{\mathrm{fin}}(M)$ satisfying the analogue of (1.1), then T automatically extends to a bounded operator from $H^1(M)$ to $L^1(M)$ (see Section 4). It may be worth noticing that the proof of this result also applies to certain metric measured spaces (M,ρ,μ) where μ is only "locally doubling" [10], [4], and [16].

For so-called RD-spaces, which are spaces of homogeneous type having "dimension n" in a certain sense, our complete results were recently extended in the paper [9] by L. Grafakos, L. Liu and Yang. These authors consider $n/(n+1) < p \leq 1$ and quasi-Banach-valued operators.

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2. Notation and terminology

Suppose that (M, ρ, μ) is a space of homogeneous type in the sense of Coifman and Weiss [5] and that μ is a σ-finite measure. For the sake of simplicity, we shall assume that $\mu(M)$ is infinite.

Suppose that q is in $(1, \infty]$. For each closed ball B in M, we denote by $L^q_0(B)$ the space of all functions in $L^q(M)$ which are supported in B and have integral 0. Clearly $L^q_0(B)$ is a closed subspace of $L^q(M)$. The union of all spaces $L^q_0(B)$ as B varies over all balls coincides with the space $L^q_{c,0}(M)$ of all functions in $L^q(M)$ with compact support and integral 0. Fix a reference point o in M and for each positive integer k denote by B_k the ball centred at o with radius k. A convenient way of topologising $L^q_{c,0}(M)$ is to interpret $L^q_{c,0}(M)$ as the strict inductive limit of the spaces $L^q_{c,0}(B_k)$ (see [1, II, p. 33] for the definition of the strict inductive limit topology). We denote by X^q the space $L^q_{c,0}(M)$ with this topology, and write X^q_k for $L^q_{c,0}(B_k)$.

We recall the basic definitions and results concerning the atomic Hardy space $H^1(M)$. The reader is referred to [5] and the references therein for this and more on Hardy spaces defined on spaces of homogeneous type. Suppose that q is in $(1, \infty]$. A $(1, q)$-atom is a function a in $L^q(M)$ supported in a ball B, with mean value 0 and such that

$$\Big(\frac{1}{\mu(B)} \int_B |a|^q \,\mathrm{d}\mu\Big)^{1/q} \leq \mu(B)^{-1}$$

if q is finite, and $\|a\|_\infty \leq \mu(B)^{-1}$ if $q = \infty$. We denote by $H^{1,q}(M)$ the space of all functions g in $L^1(M)$ which admit a decomposition of the form $g = \sum_j \lambda_j \, a_j$, where the a_j are $(1, q)$-atoms and the λ_j are complex numbers such that $\sum_j |\lambda_j| < \infty$. The norm $\|g\|_{H^{1,q}}$ of g in $H^{1,q}(M)$ is the infimum of $\sum_j |\lambda_j|$ over all such decompositions. It is well known that all the spaces $H^{1,q}(M)$ with $q \in (1, \infty)$ coincide with $H^{1,\infty}(M)$, and we denote them all by $H^1(M)$. Clearly, the vector space $H^{1,q}_{\mathrm{fin}}(M)$ of all finite linear combinations of $(1, q)$-atoms is dense in $H^1(M)$ with respect to the norm of $H^1(M)$, for q in $(1, \infty]$. Observe also that $H^{1,q}_{\mathrm{fin}}(M)$ and $L^q_{c,0}(M)$ agree as vector spaces, and so do the space of finite linear combinations of continuous $(1, \infty)$-atoms and $H^{1,\infty}_{\mathrm{fin}}(M) \cap C(\mathbb{R}^n)$.

For each ball B and each locally integrable function f, we denote by f_B the average of f on B. Recall that BMO is the Banach space of all locally integrable functions f, defined modulo constants, such that

$$\|f\|_{BMO} = \sup_B \frac{1}{\mu(B)} \int_B \big|f - f_B\big| \,\mathrm{d}\mu < \infty.$$

The dual of $H^1(M)$ may be identified with BMO.

There are several characterisations of the space $H^1(\mathbb{R}^n)$. We shall make use of the so-called maximal characterisation, which we briefly recall. Suppose that m is an integer with $m > n$, and denote by $\mathcal{A}_m$ the set of all functions φ in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ such that

$$\sup_{|\beta| \leq m} \sup_{x \in \mathbb{R}^n} (1 + |x|)^m \,\big|D^\beta \varphi(x)\big| \leq 1,$$

where $|\beta|$ denotes the length of the multi-index β. For φ in $\mathcal{S}(\mathbb{R}^n)$ denote by φ_t the function $t^{-n} \varphi(\cdot/t)$. Given f in $L^1(\mathbb{R}^n)$, define the "grand maximal function"

$\mathcal{M}_m f$ by

$$\mathcal{M}_m f = \sup_{\varphi\in\mathcal{A}_m} \sup_{t>0} |\varphi_t * f| .$$

The following result is classical [6], [13], [7], and [15].

Theorem 2.1. *Suppose that f is in $L^1(\mathbb{R}^n)$. The following are equivalent:*

(i) *f is in $H^1(\mathbb{R}^n)$;*

(ii) *the grand maximal function $\mathcal{M}_m f$ is in $L^1(\mathbb{R}^n)$.*

Furthermore, $f \mapsto \|\mathcal{M}_m f\|_{L^1(\mathbb{R}^n)}$ is an equivalent norm on $H^1(\mathbb{R}^n)$.

The letter C will denote a positive constant, which need not be the same at different occurrences. Given two positive quantities A and B, we shall mean by $A \sim B$ that there exists a constant C such that $1/C \leq A/B \leq C$.

3. The Euclidean case

In this section we work in the classical setting of $\mathbb{R}^n$.

Theorem 3.1. *The following hold:*

(i) *if $q < \infty$, then $\|\cdot\|_{H^{1,q}_{\mathrm{fin}}(\mathbb{R}^n)}$ and $\|\cdot\|_{H^1(\mathbb{R}^n)}$ are equivalent norms on $H^{1,q}_{\mathrm{fin}}(\mathbb{R}^n)$;*

(ii) *the two norms $\|\cdot\|_{H^{1,\infty}_{\mathrm{fin}}(\mathbb{R}^n)}$ and $\|\cdot\|_{H^1(\mathbb{R}^n)}$ are equivalent on $H^{1,\infty}_{\mathrm{fin}}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$.*

Proof. Clearly, $\|f\|_{H^1(\mathbb{R}^n)} \leq \|f\|_{H^{1,q}_{\mathrm{fin}}(\mathbb{R}^n)}$ for f in $H^{1,q}_{\mathrm{fin}}(\mathbb{R}^n)$ and for q in $(1,\infty]$. Thus, we have to show that for every q in $(1,\infty)$ there exists a constant C such that

$$\|f\|_{H^{1,q}_{\mathrm{fin}}(\mathbb{R}^n)} \leq C\,\|f\|_{H^1(\mathbb{R}^n)} \qquad \forall f \in H^{1,q}_{\mathrm{fin}}(\mathbb{R}^n),$$

and that a similar estimate holds for $q = \infty$ and all f in $H^{1,\infty}_{\mathrm{fin}}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$.

Suppose that q is in $(1,\infty]$ and that f is in $H^{1,q}_{\mathrm{fin}}(\mathbb{R}^n)$ with $\|f\|_{H^1(\mathbb{R}^n)} = 1$. By the translation invariance of Lebesgue measure, we may assume that the support of f is contained in the closed ball $B = B(0,R)$ centred at 0 with radius R. For each k in $\mathbb{Z}$, denote by Ω_k the level set $\{x \in \mathbb{R}^n : \mathcal{M}_m f(x) > 2^k\}$ of the grand maximal function $\mathcal{M}_m f$ of f. We choose Whitney cubes Q_i^k, $i \in \mathbb{N}$, with disjoint interiors satisfying $\Omega_k = \bigcup_i Q_i^k$ and

$$\mathrm{diam}(Q_i^k) \leq \eta\,\mathrm{dist}(Q_i^k, \Omega_k^c) \leq 4\,\mathrm{diam}(Q_i^k), \tag{3.1}$$

where η is a suitable constant in $(0,1)$. Except for the factor η, this is Theorem VI.1 of [14, p. 167]. The only modification needed in the proof of [14] concerns the choice of the constant denoted c.

By following closely the proof of [15, Theorem III.2, p. 107] or [13, Theorem 3.5, pp. 12-18], we produce an atomic decomposition of f of the form

$$f = \sum_{i,k} \lambda_i^k\, a_i^k, \tag{3.2}$$

such that the following hold:

(a) $\left|\lambda_i^k\, a_i^k\right| \leq C\,2^k$ for every k in $\mathbb{Z}$;

(b) for each k in $\mathbb{Z}$, the atoms a_i^k are supported in balls B_i^k concentric with the Q_i^k and contained in Ω_k. By choosing the constant η in (3.1) small enough, depending on the dimension, we can also ensure that the family $\{B_i^k\}_i$ has the bounded overlap property, uniformly with respect to k;

(c) there exists a constant C independent of f such that

$$\sum_{i,k} \left|\lambda_i^k\right| \leq C \, \|f\|_{H^1(\mathbb{R}^n)} = C.$$

We write $2B$ for the closed ball concentric with B whose radius is twice as large. For φ in $\mathcal{A}_m$ and x in $\mathbb{R}^n \setminus (2B)$ one then has

$$\begin{aligned} |\varphi_t * f(x)| &\leq t^{-n} \sup_{y \in B^c} |\varphi(y/t)| \; \|f\|_{L^1(\mathbb{R}^n)} \\ &\leq t^{-n} \, (1 + R/t)^{-m} \, \|f\|_{L^1(\mathbb{R}^n)} \qquad \forall t \in \mathbb{R}^+, \end{aligned}$$

so that

$$\mathcal{M}_m f(x) = \sup_{\varphi \in \mathcal{A}_m} \sup_{t > R} |\varphi_t * f(x)| \leq R^{-n},$$

since $m > n$. Now, if x is in $\Omega_k \setminus (2B)$, the above inequality and the definition of Ω_k force $2^k < R^{-n}$; denote by k' the largest integer k such that $2^k < R^{-n}$. Then $\overline{\Omega_k}$ is contained in $2B$ for $k > k'$.

Next we define the functions h and ℓ by

$$(3.3) \qquad h = \sum_{k \leq k'} \sum_i \lambda_i^k \, a_i^k \qquad \text{and} \qquad \ell = \sum_{k > k'} \sum_i \lambda_i^k \, a_i^k.$$

Observe that both these series converge in $L^1(\mathbb{R}^n)$, simply because $\sum_{i,k} \left|\lambda_i^k\right| < \infty$, so that h and ℓ have integral 0. Clearly, $f = h + \ell$. Furthermore, the support of ℓ is contained in $2B$, because it is contained in $\overline{\Omega}_k$ by (b) above, and $\overline{\Omega}_k$ is contained in $2B$ for all $k > k'$. Therefore $h = f = 0$ in $(2B)^c$.

To estimate the size of h in $2B$, we use (a) above and the bounded overlap property of (b), getting

$$|h| \leq C \sum_{k \leq k'} 2^k \; \leq \; C 2^{k'} \; \leq \; C \, |2B|^{-1}.$$

This proves that h/C is a $(1, \infty)$-atom, where C is independent of f.

Now we assume that $q < \infty$ and conclude the proof of (i). Observe that ℓ is in $L^q(\mathbb{R}^n)$, because $\ell = f - h$, and both f and h are in $L^q(\mathbb{R}^n)$.

We claim that the series $\sum_{k > k'} \sum_i \lambda_i^k \, a_i^k$ converges to ℓ in $L^q(\mathbb{R}^n)$.

Fixing s in $\mathbb{Z}$, we shall estimate $\sum_{k > k'} \sum_i \left|\lambda_i^k \, a_i^k\right|$ in $\Omega_s \setminus \Omega_{s+1}$. First observe that all terms with $k > s$ vanish outside Ω_{s+1}. Then apply (a) and (b) to get the pointwise bound

$$\sum_{k > k'} \sum_i \left|\lambda_i^k \, a_i^k\right| \; \leq \; C \sum_{k \leq s} 2^k \; \leq \; C 2^s \; \leq \; C \mathcal{M}_m f.$$

The constants C above are independent of f and s, so that

$$\sum_{k > k'} \sum_i \left|\lambda_i^k \, a_i^k\right| \leq C \, \mathcal{M}_m f$$

in all of $\mathbb{R}^n$, with C independent of f. Note that $\mathcal{M}_m f$ is in $L^q(\mathbb{R}^n)$, since f is. This implies that the series defining ℓ converges almost everywhere and the limit must coincide with the L^1 limit ℓ. The Lebesgue dominated convergence theorem now implies that $\sum_{k > k'} \sum_i \lambda_i^k \, a_i^k$ converges to ℓ in $L^q(\mathbb{R}^n)$, and the claim is proved.

Finally, for each positive integer N we denote by F_N the finite set of all pairs of integers (i, k) such that $k > k'$ and $|i| + |k| \leq N$, and by ℓ_N the function $\sum_{(i,k) \in F_N} \lambda_i^k \, a_i^k$. The function ℓ_N is in $H^{1,q}_{\text{fin}}(\mathbb{R}^n)$, and $f = h + \ell_N + (\ell - \ell_N)$.

Observe that $\ell - \ell_N$ will be a small multiple of a $(1,q)$-atom for large N. Indeed, by taking N large enough, we can make the corresponding coefficient less than any given ε in $\mathbb{R}^+$. Then

$$\|f\|_{H^{1,q}_{\mathrm{fin}}(\mathbb{R}^n)} \le C + \sum_{(i,k)\in F_N} \left|\lambda_i^k\right| + \varepsilon,$$

so that

$$\|f\|_{H^{1,q}_{\mathrm{fin}}(\mathbb{R}^n)} \le C + \sum_{(i,k)\in F_N} \left|\lambda_i^k\right| \le C,$$

by property (c) above, as required to conclude the proof of (i).

Now we finish the proof of (ii). Assume that f is a continuous function in $H^{1,\infty}_{\mathrm{fin}}(\mathbb{R}^n)$. A careful examination of the proof of [15, Theorem III.2, pp. 107-8] or [13, Theorem 3.5, pp. 12-18] shows that the atoms a_i^k that appear in the decomposition (3.2) are then continuous. Furthermore, we see that for each k and i the function $\lambda_i^k\, a_i^k$ depends only on the restriction of f to a ball $\tilde{B}_i^k$ which is a concentric enlargement of the ball B_i^k from (b) above, by a fixed scaling factor. It is straightforward to check that if f is constant in $\tilde{B}_i^k$, then $\lambda_i^k\, a_i^k = 0$ and that there exists an absolute constant C such that if $|f| < \varepsilon$ in $\tilde{B}_i^k$, then $\left|\lambda_i^k\, a_i^k\right| < C\,\varepsilon$.

Since trivially $\mathcal{M}_m f \le C_n \|f\|_\infty$, where the constant C_n depends only on n, the level set Ω_k is empty for all k such that $2^k \ge C_n\|f\|_\infty$. We denote by k'' the largest integer for which the last inequality does not hold. Then the index k in the sum defining ℓ in (3.3) will run only over $k' < k \le k''$.

Let ε be positive. Since f is uniformly continuous, there exists a positive δ such that $|x-y| < \delta$ implies

$$\left|f(x) - f(y)\right| < \varepsilon.$$

Write $\ell = \ell_1^\varepsilon + \ell_2^\varepsilon$ with

$$\ell_1^\varepsilon = \sum_{(i,k)\in F_1} \lambda_i^k\, a_i^k \qquad \text{and} \qquad \ell_2^\varepsilon = \sum_{(i,k)\in F_2} \lambda_i^k\, a_i^k,$$

where $F_1 = \{(i,k) : \mathrm{diam}(\tilde{B}_i^k) \ge \delta,\ k' < k \le k''\}$ and $F_2 = \{(i,k) : \mathrm{diam}(\tilde{B}_i^k) < \delta,\ k' < k \le k''\}$. Since F_1 is a finite set, ℓ_1^ε is continuous.

To estimate ℓ_2^ε, we denote by x_i^k the centre of the ball B_i^k and write for (i,k) in F_2

$$f(x) = f(x_i^k) + f(x) - f(x_i^k).$$

Then $\left|\lambda_i^k\, a_i^k\right| < C\,\varepsilon$, because $\left|f(x) - f(x_i^k)\right| < \varepsilon$ for x in $\tilde{B}_i^k$. For fixed k the balls $\{B_i^k\}_i$ have uniformly bounded overlap, so there exists an absolute constant C such that

$$|\ell_2^\varepsilon| \le C \sum_{k'<k\le k''} \varepsilon \le C\,(k''-k')\,\varepsilon.$$

Since ε is arbitrary, we can thus split ℓ into a continuous part and a part that is uniformly arbitarily small. It follows that ℓ is continuous. But then $h = f - \ell$ is also continuous, so that h is a continuous $(1,\infty)$-atom, multiplied by a factor C.

To find a finite atomic decomposition of ℓ, we again use the splitting $\ell = \ell_1^\varepsilon + \ell_2^\varepsilon$. Clearly ℓ_1^ε is for each ε a finite linear combination of continuous $(1,\infty)$-atoms, and the ℓ^1 norm of the coefficients is controlled by $\|f\|_{H^1}$, in view of (c). Observe that $\ell_2^\varepsilon = \ell - \ell_1^\varepsilon$ is continuous. Further, ℓ_2^ε is supported in $2B$, has integral 0 and satisfies $|\ell_2^\varepsilon| \le C(k''-k')\varepsilon$. Choosing ε, we can thus make ℓ_2^ε into an arbitrarily small multiple of a continuous $(1,\infty)$-atom.

To sum up, $f = h + \ell_1^\varepsilon + \ell_2^\varepsilon$ gives the desired finite atomic decomposition of f, with coefficients controlled by $\|f\|_{H^1}$.

We have completed the proof of (ii) and that of the theorem. □

Remark 3.2. Theorem 3.1 (ii) implies that any function f in $H^{1,\infty}_{\text{fin}}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ admits a finite decomposition in $(1,\infty)$-atoms such that the sum of the corresponding coefficients is $\leq C\,\|f\|_{H^1(\mathbb{R}^n)}$. Actually, the proof of Theorem 3.1 (ii) shows that we can construct this finite decomposition in such a way that it involves only continuous $(1,\infty)$-atoms.

Remark 3.3. Theorem 3.1 extends to $H^p(\mathbb{R}^n)$ with $0 < p < 1$ and (p,q)-atoms, where one can now have $1 \leq q \leq \infty$. The proof is rather similar to the one given above, so we only briefly describe the modifications needed for part (i). Thus let $1 \leq q < \infty$. Given $f \in H^{p,q}_{\text{fin}}(\mathbb{R}^n)$ supported in a ball B_R, the first step is the inequality $\mathcal{M}_m f \leq CR^{-n/p}\|f\|_{H^p(\mathbb{R}^n)}$, valid outside a larger ball B_{CR}. One proves this by comparing the values of $\mathcal{M}_m f$ at different points and using the fact that $\|\mathcal{M}_m f\|_{L^p(\mathbb{R}^n)} \sim \|f\|_{H^p(\mathbb{R}^n)}$. Then the Ω_k and the decompositions $f = \sum \lambda_i^k a_i^k = h + \ell$ are introduced as above. The sum ℓ now converges in $\mathcal{S}'$ and is dominated by $\mathcal{M}_m f$. If $q > 1$, we have $\mathcal{M}_m f \in L^q(\mathbb{R}^n)$ and conclude as before that ℓ converges in $L^q(\mathbb{R}^n)$. For $q = 1$, the tail sum $S_\kappa = \sum_{k \geq \kappa} \sum_i \lambda_i^k a_i^k$ tends to 0 in $L^1(\mathbb{R}^n)$ as $\kappa \to +\infty$, because S_κ is nonzero only in Ω_κ and not larger than $|f| + C2^\kappa$ there, and $|\Omega_\kappa| = o(2^{-\kappa})$ as $\kappa \to +\infty$. The rest of the proof proceeds as before. See also [9, Theorem 5.6].

Corollary 3.4. *Suppose that Y is a Banach space and that one of the following holds:*

(i) *q is in $(1,\infty)$ and $T : H^{1,q}_{\text{fin}}(\mathbb{R}^n) \to Y$ is a linear operator such that*
$$A := \sup\{\|Ta\|_Y : a \text{ is a } (1,q)\text{-atom}\} < \infty;$$

(ii) *T is a Y-valued linear operator defined on continuous $(1,\infty)$-atoms such that*
$$A := \sup\{\|Ta\|_Y : a \text{ is a continuous } (1,\infty)\text{-atom}\} < \infty.$$

Then there exists a unique bounded linear operator $\widetilde{T}$ from $H^1(\mathbb{R}^n)$ to Y which extends T.

Proof. We consider the case (i). Suppose that f is in $H^{1,q}_{\text{fin}}(\mathbb{R}^n)$, $f = \sum_{j=1}^N \lambda_j\, a_j$ say, where a_j are $(1,q)$-atoms. Then the assumption and the triangle inequality give
$$\|Tf\|_Y \leq A \sum_{j=1}^N |\lambda_j|\,.$$
By taking the infimum of the right-hand side with respect to all decompositions of f as a finite sum of $(1,q)$-atoms, we obtain
$$\|Tf\|_Y \leq A\,\|f\|_{H^{1,q}_{\text{fin}}(\mathbb{R}^n)}.$$
Now, Theorem 3.1 (i) implies that the right-hand side is dominated by $CA\|f\|_{H^1(\mathbb{R}^n)}$, where C does not depend on f, and a density argument completes the proof of the corollary.

The case (ii) is similar. □

Remark 3.5. The statement of Corollary 3.4 (i) becomes false if we replace q by ∞. A counterexample is given by the operator B defined in the Introduction. Note also that Corollary 3.4 applies to linear functionals.

4. Results on spaces of homogeneous type

In this section, we work in a space of homogeneous type (M, ρ, μ). Recall that we assume that μ is σ-finite and that $\mu(M)$ is infinite.

Theorem 4.1. *Suppose that q is in $(1,\infty)$ and that T is a linear operator defined on $H^{1,q}_{\mathrm{fin}}(M)$ with the property that*

$$A := \sup\{\|Ta\|_{L^1(M)} : a \text{ is a } (1,q)\text{-atom}\} < \infty.$$

Then there exists a unique bounded linear operator $\widetilde{T}$ from $H^1(M)$ to $L^1(M)$ which extends T.

Proof. We prove the result in the case where $q = 2$. The proof in the other cases is similar.

Suppose that B is a ball. For each f in $L^2_0(B)$ such that $\|f\|_{L^2(M)} = 1$, the function $\mu(B)^{-1/2} f$ is a $(1,2)$-atom, so that

$$\|Tf\|_{L^1(M)} \leq A\,\mu(B)^{1/2} \qquad \forall f \in L^2_0(B)$$

by the assumption. In particular, the restriction of T to X^2_k is bounded from X^2_k to $L^1(M)$ for each k. Thus, T is bounded from X^2 to $L^1(M)$. It follows that T^* is bounded from $L^\infty(M)$ to the dual of X^2. But the dual of X^2 is the quotient space $L^2_{\mathrm{loc}}(M)/\mathbb{C}$, since that of $L^2_{c,0}(B_k)$ is $L^2(B_k)/\mathbb{C}$. Now, for every f in $L^\infty(M)$ and for every $(1,2)$-atom a,

$$\langle Ta, f\rangle = \langle a, T^*f\rangle = \int_M a\, T^*f \,\mathrm{d}\mu,$$

so that

$$\Big|\int_M a\, T^*f\,\mathrm{d}\mu\Big| = \big|\langle Ta, f\rangle\big| \leq A\,\|f\|_\infty.$$

A standard argument then shows that T^*f belongs to $BMO(M)$ and that

$$\|T^*f\|_{BMO(M)} \leq 2A\,\|f\|_\infty \qquad \forall f \in L^\infty(M). \tag{4.1}$$

We give the details for the reader's convenience. Suppose that B is a ball and observe that

$$\Big[\int_B |T^*f - (T^*f)_B|^2 \,\mathrm{d}\mu\Big]^{1/2} = \sup_{\|\varphi\|_{L^2(B)}=1} \Big|\int_B \varphi\,\big(T^*f - (T^*f)_B\big)\,\mathrm{d}\mu\Big|.$$

But

$$\begin{aligned}\int_B \varphi\,\big(T^*f - (T^*f)_B\big)\,\mathrm{d}\mu &= \int_B \big(\varphi - \varphi_B\big)\,\big(T^*f - (T^*f)_B\big)\,\mathrm{d}\mu \\ &= \int_B \big(\varphi - \varphi_B\big)\,T^*f\,\mathrm{d}\mu,\end{aligned}$$

and since $\|\varphi\|_{L^2(B)} = 1$,

$$|\varphi_B| \leq \Big[\frac{1}{\mu(B)}\int_B |\varphi|^2\,\mathrm{d}\mu\Big]^{1/2} \leq \mu(B)^{-1/2}.$$

Write ψ instead of $\varphi - \varphi_B$. Then

$$\|\psi\|_{L^2(B)} \le \|\varphi\|_{L^2(B)} + |\varphi_B|\ \mu(B)^{1/2} \le 2,$$

so that $\psi/(2\,\mu(B)^{1/2})$ is a $(1,2)$-atom. Therefore

$$\Big|\int_B \psi\ T^* f \,\mathrm{d}\mu\Big| \le 2A\,\mu(B)^{1/2}\,\|f\|_\infty.$$

Combining the above, we conclude that for every ball B

$$\Big[\frac{1}{\mu(B)}\int_B |T^*f - (T^*f)_B|^2\ \mathrm{d}\mu\Big]^{1/2} \le 2A\,\|f\|_\infty,$$

and (4.1) follows.

Now we show that T extends to a bounded operator from $H^1(M)$ to $L^1(M)$ with norm at most $2A$. Observe that X^2 and $H^{1,2}_{\mathrm{fin}}(M)$ coincide as vector spaces. For every g in $H^{1,2}_{\mathrm{fin}}(M)$ and for every f in $L^\infty(M)$

$$\begin{aligned}\big|\langle Tg, f\rangle\big| &= \big|\langle g, T^*f\rangle\big| \\ &\le \|g\|_{H^1(M)}\,\|T^*f\|_{BMO(M)} \\ &\le 2A\,\|g\|_{H^1(M)}\,\|f\|_{L^\infty(M)}.\end{aligned}$$

By taking the supremum of both sides over all functions f in $L^\infty(M)$ with $\|f\|_{L^\infty(M)} = 1$, we obtain that

$$\|Tg\|_{L^1(M)} \le 2A\,\|g\|_{H^1(M)} \qquad \forall g \in H^{1,2}_{\mathrm{fin}}(M).$$

Finally we observe that $H^{1,2}_{\mathrm{fin}}(M)$ is dense in $H^1(M)$ (with respect to the norm of $H^1(M)$), and the required conclusion follows by a density argument. □

Quite often one encounters the following situation. Suppose that T is a bounded linear operator on $L^2(M)$. Then T is automatically defined on $H^{1,2}_{\mathrm{fin}}(M)$. Assume that

$$A := \sup\{\|Ta\|_{L^1(M)} : a \text{ is a } (1,2)\text{-atom}\} < \infty.$$

By the previous result, the restriction of T to $H^{1,2}_{\mathrm{fin}}(M)$ has a unique extension to a bounded linear operator $\widetilde{T}$ from $H^1(M)$ to $L^1(M)$. The question is whether the operators T and $\widetilde{T}$ are consistent, i.e., whether they coincide on the intersection $H^1(M)\cap L^2(M)$ of their domains. The answer to this question is in the affirmative, as the following proposition shows.

Proposition 4.2. *Suppose that T is bounded on $L^2(M)$ and that*

$$A := \sup\{\|Ta\|_{L^1(M)} : a \textit{ is a } (1,2)\textit{-atom}\} < \infty.$$

Denote by $\widetilde{T}$ the unique continuous linear extension of the restriction of T to $H^{1,2}_{\mathrm{fin}}(M)$ to an operator from $H^1(M)$ to $L^1(M)$. Then the operators T and $\widetilde{T}$ agree on $H^1(M)\cap L^2(M)$.

Proof. Suppose that f is in $L^2(M)\cap L^\infty(M)$ and that g is in $L^2_{c,0}(M)$. Denote by T^* the transpose operator of T (as an operator on $L^2(M)$). Then

$$\int_M g\ T^*f\,\mathrm{d}\mu = \int_M Tg\ f\,\mathrm{d}\mu. \tag{4.2}$$

Since g is in $H^{1,2}_{\mathrm{fin}}(M)$ and the operators T and $\widetilde{T}$ agree on $H^{1,2}_{\mathrm{fin}}(M)$, we see that

$$\begin{aligned}\int_M Tg\; f\,\mathrm{d}\mu &= \int_M \widetilde{T}g\; f\,\mathrm{d}\mu \\ &= \Big\langle g, (\widetilde{T})^* f\Big\rangle,\end{aligned} \tag{4.3}$$

where $(\widetilde{T})^*$ denotes the transpose of the operator $\widetilde{T}$ from $H^1(M)$ to $L^1(M)$. Note that $(\widetilde{T})^* f$ is in $BMO(M)$ and g is a multiple of an atom. Thus the above scalar product $\Big\langle g, (\widetilde{T})^* f\Big\rangle$ (with respect to the duality between $H^1(M)$ and $BMO(M)$) may be written as $\int_M g\; (\widetilde{T})^* f\,\mathrm{d}\mu$. Therefore, (4.2) and (4.3) imply that

$$\int_M g\,\left[T^* f - (\widetilde{T})^* f\right]\mathrm{d}\mu = 0 \qquad \forall g \in L^2_{c,0}(M),$$

i.e., for all g in X^2. Therefore $T^* f - (\widetilde{T})^* f = 0$ in the dual space of X^2, i.e., in $L^2_{\mathrm{loc}}(M)/\mathbb{C}$. This implies that $T^* f - (\widetilde{T})^* f$ is constant.

Now, suppose that g is in $H^1(M) \cap L^2(M)$ and that f is in $L^2(M) \cap L^\infty(M)$. Then

$$\begin{aligned}\int_M Tg\; f\,\mathrm{d}\mu &= \int_M g\; T^* f\,\mathrm{d}\mu \\ &= \int_M g\; (\widetilde{T})^* f\,\mathrm{d}\mu \\ &= \int_M \widetilde{T}g\; f\,\mathrm{d}\mu.\end{aligned} \tag{4.4}$$

Since f is an arbitrary function in $L^2(M)\cap L^\infty(M)$, $Tg - \widetilde{T}g = 0$ almost everywhere, as required. □

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