ON THE HADAMARD PRODUCTS OF SCHLICHT FUNCTIONS AND APPLICATIONS

H.S. AL-AMIRI

Department of Mathematics
Bowling Green State University
Bowling Green, Ohio 43403

ABSTRACT. We show that each of the schlicht classes of starlike, convex, close-to-convex and strongly starlike with respect to symmetric points is invariant under the Hadamard product with the class of convex functions. The influence of certain operators over these classes is also investigated.

KEY WORDS AND PHRASES. Hadamard products, schlicht functions, and starlike, convex, close-to convex and strongly starlike with respect to symmetric points.
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1. INTRODUCTION.

Let A be the class of analytic functions on the unit disk $U = \{z: |z| < 1\}$.
By S we denote those functions in A which are univalent and normalized by $f(0) = f'(0) - 1 = 0$. Let C, S* and K be the usual subclasses of S which are consisting of the convex, starlike and close-to-convex functions, respectively.

The Hadamard product (convolution) of two power series $f(z) = \sum_0^\infty a_n z^n$,
$g(z) = \sum_0^\infty b_n z^n$ is defined as the power series $(f * g)(z) = \sum_0^\infty a_n b_n z^n$. Polya and Schoenberg [9] have conjectured that f * g is in C whenever both f and g are in C. This conjecture was finally shown to be true by Ruscheweyh and Sheil-Small [12].
THEOREM A (Ruscheweyh and Sheil-Small): Let h be in C. If f is in C,S* or K
then h * f is in C, S* or K respectively.

The proof of Theorem A requires a key lemma that turned out to be beautiful in its simplicity and powerful in its applicability to various problems. We state this key lemma of Ruscheweyh and Sheil-Small in the form stated and proved by Barnard and Kellogg in [1].
LEMMA A. Let ϕ and g be analytic in $|z| < 1$ with $\phi(0) = g(0) = 0$ and
$\phi'(0)g'(0) \neq 0$. Suppose that for each $\alpha(|\alpha| = 1)$ and $\sigma(|\sigma| = 1)$ we have

$$[\phi * (\frac{1 + \alpha\sigma z}{1 - \sigma z})g](z) \neq 0 \tag{1.1}$$

on $0 < |z| < r < 1$. Then for each F in A the image of $|z| < r$ under

$(\phi * Fg)/(\phi * g)$ is a subset of the convex hull of $F(U)$.

REMARK 1. In [12], it was shown that condition [1] is satisfied for all z in U whenever ϕ is in C and g is in $S*$.

A function $f \cdot A$ is said to be starlike with respect to symmetric points in U, denoted $f \in S_s*$, if and only if

$$\text{Re } \frac{zf'(z)}{f(z) - f(-z)} > 0, \quad \text{for all } z \text{ in } U. \tag{1.2}$$

This class was introduced by Sakaguchi [13].

REMARK 2. Sakaguchi has shown that $C \cdot S_s* \subset K$. Further, if $f \in S_s*$ then $(f(z) - f(-z))/2$ is in $S*$.

We say that f is strongly starlike of order λ, with respect to symmetric points in U, denoted $f \cdot S_s*(\lambda)$, if and only if

$$\left| \arg\left\{ \frac{z f'(z)}{f(z) - f(-z)} \right\} \right| < \frac{\lambda \pi}{2}, \quad 0 < \lambda < 1, \quad z \in U. \tag{1.3}$$

This class was introduced by Padmanabhan and Thangamant [8].

Das and Singh [4] introduced the classes of function f which are convex with respect to symmetric points in U, denoted $f \in C_s$, and those which are close-to-convex with respect to symmetric points, denoted $f \in K_s$ as follows: $f \in C_s$ if and only if

$$\text{Re } \frac{z(zf'(z))'}{zf'(z) + zf'(-z)} > 0, \quad z \in U, \tag{1.4}$$

and $f \in K_s$ if and only if there exists $K \in S_s*$ such that

$$\text{Re } \frac{zf'(z)}{K(z) - K(-z)} > 0, \quad z \in U. \tag{1.5}$$

REMARK 3. In [4], it was shown that both C_s and K_s are subsets of S and furthermore, $f \in C_s$ if and only if $zf' \in S_s*$.

The purpose of this note is to investigate these subclasses of S which are symmetric with respect to points in U via the Hadamard products. In section 2, we state our main theorem which shows that these classes are invariant under the Hadamard product with convex functions. In section 3, we use the results of section 2 and an observation made in [1] to obtain results concerning the radii of these classes under the influence of certain well known operators.

2. MAIN RESULTS

The elegant technique and results of Ruscheweyh and Sheil-Small allow us to produce a theorem similar to Theorem A for the classes S_s*, C_s, K_s and $S_s*(\lambda)$, $0 < \lambda < 1$.

THEOREM 1. Let $h \in C$. If f is in S_s*, C_s, K_s or $S_s*(\lambda)$, $0 < \lambda \leq 1$, then $h * f$ is in S_s*, C_s, K_s or $S_s*(\lambda)$ respectively.

PROOF. The proof of the four cases requires an application of Lemma A. In order to apply this lemma, we need to identify the convex function ϕ and the starlike function g for the different cases.

Case 1. Let $f \in S_s*$ and $h \in C$. Then $g(z) = \dfrac{f(z) - f(-z)}{2} \in S*$ by Remark 2. Thus

condition (1.1) is satisfied according to Remark 1, i.e.,

$$[h * (\frac{1 + \alpha\sigma z}{1 - \sigma z})g](z) \neq 0$$

for all z in U. Now let

$$F(z) = \frac{z\ f'(z)}{f(z) - f(-z)} .$$ (1.6)

Applying Lemma A with F as given in (2.1) we obtain

$$(\frac{h * Fg}{h * g})(z) = \frac{h(z) * \dfrac{z\ f'(z)}{(f(z) - f(-z))} \cdot \dfrac{(f(z) - f(-z))}{2}}{h(z) * (\dfrac{f(z) - f(-z)}{2})}$$

$$= \frac{h(z) * z\ f'(z)}{(h * f)(z) - (h * f)(-z)} = \frac{z(h * f)'(z)}{(h * f)(z) - (h * f)(-z)}$$

which is contained in the convex hull of F in U. This implies

$$Re\ \frac{z(h * f)'(z)}{(h * f)(z) - (h * f)(-z)} > 0,\ z \in U,$$

or $h * f \in S_s^*$ according to (1.2).

CASE 2. Let $f \in C_s$ and $h \in C$. This case follows immediately from Remark 3 and Case 1 above.

CASE 3. Let $f \in K_s$ and $h \in C$. Let $K(z)$ be the associated function appearing in (1.5). Then according to Remark 2 $g(z) = \frac{K(z) - K(-z)}{2} \in S*$. Thus condition (1.1) is satisfied for h and g of this case. Applying Lemma A with

$$F(z) = \frac{z\ f'(z)}{K(z) - K(-z)}$$

we easily get after some simplification as in Case 1 that

$$\frac{z(h * f)'(z)}{(h * K)(z) - (h * K)(-z)}$$

is contained in the convex hull of F. That is

$$Re\ \frac{z(h * f)'(z)}{(h * K)(z) - (h * K)(-z)} > 0,\ \ z \in U.$$

Thus $h * f \in K_s$ provided $h * K \in S_s*$. However, the latter is true as seen by Case 1.

CASE 4. Let $f \in S_s*(\lambda)$, $0 < \lambda \leq 1$, and $h \in C$. It is clear from (1.3) and (1.2) that $S_s*(\lambda) \subset S_s*$ for all $0 < \lambda < 1$. Let g and F as defined in Case 1, then, as in the proof of Case 1,

$$\frac{z(h * f)'(z)}{(h * f)(z) - (h * f)(-z)}$$

lies in the convex hull of F. This means

$$\left| arg\ \frac{z(h * f)'(z)}{(h * f)(z) - (h * f)(-z)} \right| < \lambda\pi/2,\ 0 < \lambda < 1.$$

Thus $h * f \in S_s*(\lambda)$ by (1.3). This completes the proof of Theorem 1.

3. APPLICATIONS

For $X \subset A$, let $r_c(X)$ denote the largest positive number so that every $f \in X$

is convex in $|z| < r_c(X)$. Consequently, the conclusion of Theorem 1 remains valid
for $h * f$ in the disk $|z| < r_c(X)$ for each $h \in X$ and every f in the given four
classes. This observation was first used effectively by Barnard and Kellogg in [1]
to unify and reduce the amount of work usually associated with such problems. Let

$$h_\gamma(z) = \sum_{n=1}^\infty \frac{\gamma + n}{\gamma + 1} z^n = \frac{\gamma}{1 + \gamma} \frac{z}{1 - z} + \frac{1}{1 + \gamma} \frac{z}{(1 - z)^2} ,$$

$$H(z) = \sum_{n=1}^\infty \frac{\gamma + 1}{\gamma + n} z^n , \text{ and}$$

$$P(z) = \sum_{n=1}^\infty \frac{1 - x^n}{(1 - x)n} z^n = \frac{1}{1 - x} \log \frac{1 - xz}{1 - z} ,$$

where $\operatorname{Re}\gamma \geq 0$, $|x| \leq 1$, $x \neq 1$ and for all z in U. The operators B_j, $j = 1,2,3$,
defined below have been studied by many authors. Let

$$B_1 f(z) = \frac{\gamma f(z) + z f'(z)}{1 + \gamma} = (h_\gamma * f)(z), \operatorname{Re}\gamma \geq 0,$$

$$B_2 f(z) = \frac{1 + \gamma}{z^\gamma} \int_0^z t^{\gamma-1} f(t) dt = (H_\gamma * f)(z), \operatorname{Re}\gamma \geq 0,$$

$$B_3 f(z) = \int_0^z \frac{f(t) - f(xt)}{t - xt} dt = (P * f)(z), |x| \leq 1, x \neq 1.$$

For example, the operator B_1 is due to Livingston [7] when $\gamma = 1$, B_2 is due to
Libera [5] when $\gamma = 1$ while B_3 was first used by Pommerenke in [10], see also [2],
[3], [6], [11]. Consequently, the operators B_j, $j = 1,2,3$, preserve the classes
S_s^*, C_s, K_s or $S_s^*(\lambda)$ up to $r_c(h_\gamma)$, $r_c(H_\gamma)$ or $r_c(P)$ as mentioned at the start of
this section. However, $r_c(H_\gamma) = 1$, $\operatorname{Re}\gamma \geq 0$ was shown by Ruscheweyh [11], and
$r_c(P) = 1$ was shown by Pommerenke [10]. We need only to calculate $r_c(h_\gamma)$. From the
definition of h_γ we get

$$h_\gamma'(z) = \frac{1 + \frac{1 - \gamma}{1 + \gamma} z}{(1 - z)^3} = \frac{1 + cz}{(1 - z)^3}$$

where $c = \frac{1 - \gamma}{1 + \gamma}$. Thus

$$1 + \frac{z h_\gamma''(z)}{h_\gamma'(z)} = \frac{2 + z}{1 - z} - \frac{1}{1 + cz} .$$

This implies

$$\operatorname{Re}(1 + \frac{z h_\gamma''(z)}{h_\gamma'(z)}) \geq \frac{2 - r}{1 + r} - \frac{1}{1 - |c|r} , |z| = r.$$

This shows that $r_c(h_\gamma)$ is the smallest positive number r satisfying

$$|c|r^2 - 2(1 + |c|)r + 1 = 0$$

or

$$r_c(h_\gamma) = \frac{1}{1 + |c| + \sqrt{1 + |c| + |c|^2}} .$$

Finally, we have the following theorem.

THEOREM 2. If f is in S_s^*, C_s, K_s or $S_s^*(\lambda)$, $0 < \lambda \leq 1$, then

   i) $B_j f$ is in S_s^*, C_s, K_s or $S_s^*(\lambda)$, respectively for each $j = 2, 3$, and

   ii) $B_1 f$ is in S_s^*, C_s, K_s or $S_s^*(\lambda)$ respectively for
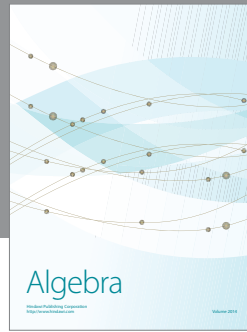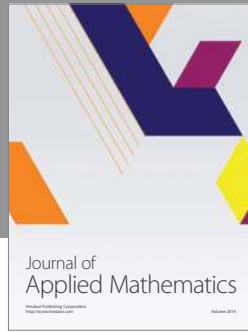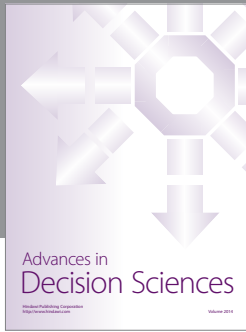
   $$|z| < r_c(h_\gamma) = 1/(1 + |c| + \sqrt{1 + |c| + |c|^2}), \quad c = \frac{1 - \gamma}{1 + \gamma},$$

   $\beta, \gamma \geq 0$.

REMARK 4. Theorem 2 part (i) was shown in [8] for the class $S_s^*(\lambda)$ only. Part (i) with $\gamma = 1$ and $x = -1$ was shown in [4] for the classes S_s^*, C_s and K_s.

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