

Research Article

On The Hadamard's Inequality for Log-Convex Functions on the Coordinates

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Inequalities of the Hadamard and Jensen types for coordinated log-convex functions defined in a rectangle from the plane and other related results are given.

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1. Introduction

Let $f : I \subseteq \mathbf{R} \rightarrow \mathbf{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2} \quad (1.1)$$

holds, this inequality is known as the Hermite-Hadamard inequality. For refinements, counterparts, generalizations and new Hadamard-type inequalities, see [1–8].

A positive function f is called log-convex on a real interval $I = [a, b]$, if for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq f^\lambda(x)f^{1-\lambda}(y). \quad (1.2)$$

If f is a positive log-concave function, then the inequality is reversed. Equivalently, a function f is log-convex on I if f is positive and $\log f$ is convex on I . Also, if $f > 0$ and f'' exists on I , then f is log-convex if and only if $f \cdot f'' - (f')^2 \geq 0$.

The logarithmic mean of the positive real numbers $a, b, a \neq b$, is defined as

$$L(a, b) = \frac{b - a}{\log b - \log a}. \quad (1.3)$$

A version of Hadamard's inequality for log-convex (concave) functions was given in [9], as follows.

Theorem 1.1. *Suppose that f is a positive log-convex function on $[a, b]$, then*

$$\frac{1}{b - a} \int_a^b f(x) dx \leq L(f(a), f(b)). \quad (1.4)$$

If f is a positive log-concave function, then the inequality is reversed.

For refinements, counterparts and generalizations of log-convexity see [9–13].

A convex function on the coordinates was introduced by Dragomir in [8]. A function $f : \Delta \rightarrow \mathbf{R}$ which is convex in Δ is called coordinated convex on Δ if the partial mapping $f_y : [a, b] \rightarrow \mathbf{R}, f_y(u) = f(u, y)$ and $f_x : [c, d] \rightarrow \mathbf{R}, f_x(v) = f(x, v)$, are convex for all $y \in [c, d]$ and $x \in [a, b]$.

An inequality of Hadamard's type for coordinated convex mapping on a rectangle from the plane \mathbf{R}^2 established by Dragomir in [8], is as follows.

Theorem 1.2. *Suppose that $f : \Delta \rightarrow \mathbf{R}$ is coordinated convex on Δ , then*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\ &\leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}. \end{aligned} \quad (1.5)$$

The above inequalities are sharp.

The maximum modulus principle in complex analysis states that if f is a holomorphic function, then the modulus $|f|$ cannot exhibit a true local maximum that is properly within the domain of f . Characterizations of the maximum principle for sub(super)harmonic functions are considered in [14], as follows.

Theorem 1.3. *Let $G \subseteq \mathbf{R}^2$ be a region and let $f : G \rightarrow \mathbf{R}$ be a sub(super)harmonic function. If there is a point $\lambda \in G$ with $f(\lambda) \geq f(x)$, for all $x \in G$ then $f(x)$ is a constant function.*

Theorem 1.4. *Let $G \subseteq \mathbf{R}^2$ be a region and let f and g be bounded real-valued functions defined on G such that f is subharmonic and g is superharmonic. If for each point $a \in \partial_\infty G$*

$$\limsup_{x \rightarrow a} f(x) \leq \liminf_{x \rightarrow a} g(x), \quad (1.6)$$

then $f(x) < g(x)$ for all $x \in G$ or $f = g$ and f is harmonic.

In this paper, a new version of the maximum (minimum) principle in terms of convexity, and some inequalities of the Hadamard type are obtained.

2. On Coordinated Convexity and Sub(Super)Harmonic Functions

Consider the 2-dimensional interval $\Delta := [a, b] \times [c, d]$ in \mathbf{R}^2 . A function $f : \Delta \rightarrow \mathbf{R}$ is called convex in Δ if

$$f(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{y}) \quad (2.1)$$

holds for all $\mathbf{x}, \mathbf{y} \in \Delta$ and $\lambda \in [0, 1]$.

As in [8], we define a log-convex function on the coordinates as follows: a function $f : \Delta \rightarrow \mathbf{R}_+$ will be called *coordinated log-convex* on Δ if the partial mappings $f_y : [a, b] \rightarrow \mathbf{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \rightarrow \mathbf{R}$, $f_x(v) = f(x, v)$, are log-convex for all $y \in [c, d]$ and $x \in [a, b]$. A formal definition of a coordinated log-convex function may be stated as follows.

Definition 2.1. A function $f : \Delta \rightarrow \mathbf{R}_+$ will be called *coordinated log-convex* on Δ , for all $t, s \in [0, 1]$ and $(x, y), (u, v) \in \Delta$, if the following inequality holds,

$$\begin{aligned} & f(tx + (1 - t)y, su + (1 - s)w) \\ & \leq f^{ts}(x, u) f^{s(1-t)}(y, u) f^{t(1-s)}(x, w) f^{(1-t)(1-s)}(y, w). \end{aligned} \quad (2.2)$$

Equivalently, we can determine whether or not the function f is coordinated log-convex by using the following lemma.

Lemma 2.2. *Let $f : \Delta \rightarrow \mathbf{R}_+$. If f is twice differentiable then f is coordinated log-convex on Δ if and only if for the functions $f_y : [a, b] \rightarrow \mathbf{R}$, defined by $f_y(u) = f(u, y)$ and $f_x : [c, d] \rightarrow \mathbf{R}$, defined by $f_x(v) = f(x, v)$, we have*

$$f_x \cdot f_x'' - (f_x')^2 \geq 0, \quad f_y \cdot f_y'' - (f_y')^2 \geq 0. \quad (2.3)$$

Proof. The proof is straight forward using the elementary properties of log-convexity in one variable. \square

Proposition 2.3. *Suppose that $g : [a, b] \rightarrow \mathbf{R}_+$ is twice differentiable on (a, b) and log-convex on $[a, b]$ and $h : [c, d] \rightarrow \mathbf{R}_+$ is twice differentiable on (c, d) and log-convex on $[c, d]$. Let $f : \Delta = [a, b] \times [c, d] \rightarrow \mathbf{R}_+$ be a twice differentiable function defined by $f(x, y) = g(x)h(y)$, then f is coordinated log-convex on Δ .*

Proof. This follows directly using Lemma 2.2. \square

The following result holds.

Proposition 2.4. *Every log-convex function $f : \Delta = [a, b] \times [c, d] \rightarrow \mathbf{R}_+$ is log-convex on the coordinates, but the converse is not generally true.*

Proof. Suppose that $f : \Delta \rightarrow \mathbf{R}$ is convex in Δ . Consider the function $f_x : [c, d] \rightarrow \mathbf{R}_+$, $f_x(v) = f(x, v)$, then for $\lambda \in [0, 1]$, and $v, w \in [c, d]$, we have

$$\begin{aligned} f_x(\lambda v + (1 - \lambda)w) &= f(x, \lambda v + (1 - \lambda)w) \\ &= f(\lambda x + (1 - \lambda)x, \lambda v + (1 - \lambda)w) \\ &\leq f^\lambda(x, v)f^{1-\lambda}(x, w) \\ &= f_x^\lambda(v)f_x^{1-\lambda}(w), \end{aligned} \tag{2.4}$$

which shows the log-convexity of f_x . The proof that $f_y : [a, b] \rightarrow \mathbf{R}_+$, $f_y(u) = f(u, y)$, is also log-convex on $[a, b]$ for all $y \in [c, d]$ follows likewise. Now, consider the mapping $f_0 : [0, 1]^2 \rightarrow \mathbf{R}_+$ given by $f_0(x, y) = e^{xy}$. It is obvious that f_0 is log-convex on the coordinates but not log-convex on $[0, 1]^2$. Indeed, if $(u, 0), (0, w) \in [0, 1]^2$ and $\lambda \in [0, 1]$, we have:

$$\begin{aligned} \log f_0(\lambda(u, 0) + (1 - \lambda)(0, w)) &= \log f_0(\lambda u, (1 - \lambda)w) = \lambda(1 - \lambda)uw, \\ \lambda \log f_0(u, 0) + (1 - \lambda) \log f_0(0, w) &= 0. \end{aligned} \tag{2.5}$$

Thus, for all $\lambda \in (0, 1)$ and $u, w \in (0, 1)$, we have

$$\log f_0(\lambda(u, 0) + (1 - \lambda)(0, w)) > \lambda \log f_0(u, 0) + (1 - \lambda) \log f_0(0, w) \tag{2.6}$$

which shows that f_0 is not log-convex on $[0, 1]^2$. \square

In the following, a Jensen-type inequality for coordinated log-convex functions is considered.

Proposition 2.5. *Let f be a positive coordinated log-convex function on the open set $(a, b) \times (c, d)$ and let $x_i \in (a, b)$, $y_j \in (c, d)$. If $\alpha_i, \beta_j > 0$ and $\sum_{i=1}^n \alpha_i = 1$, $\sum_{j=1}^m \beta_j = 1$, then*

$$\log f\left(\sum_{i=1}^n \alpha_i x_i, \sum_{j=1}^m \beta_j y_j\right) \leq \sum_{i=1}^n \sum_{j=1}^m \alpha_i \beta_j \log f(x_i, y_j). \tag{2.7}$$

Proof. Let $x_i \in (a, b)$, $\alpha_i > 0$ be such that $\sum_{i=1}^n \alpha_i = 1$, and let $y_j \in (c, d)$, $\beta_j > 0$ be such that $\sum_{j=1}^m \beta_j = 1$, then we have,

$$f\left(\sum_{i=1}^n \alpha_i x_i, \sum_{j=1}^m \beta_j y_j\right) \leq \prod_{i=1}^n f^{\alpha_i}\left(x_i, \sum_{j=1}^m \beta_j y_j\right) \leq \prod_{i=1}^n \prod_{j=1}^m f^{\alpha_i \beta_j}(x_i, y_j), \tag{2.8}$$

and, since f is positive,

$$\log f\left(\sum_{i=1}^n \alpha_i x_i, \sum_{j=1}^m \beta_j y_j\right) \leq \sum_{i=1}^n \sum_{j=1}^m \alpha_i \beta_j \log f(x_i, y_j), \tag{2.9}$$

which is as required. \square

Remark 2.6. Let $f(x, y) = xy$, then the following inequality holds:

$$\log \left[\left(\sum_{i=1}^n \alpha_i x_i \right) \left(\sum_{j=1}^m \beta_j y_j \right) \right] \leq \sum_{i=1}^n \sum_{j=1}^m \alpha_i \beta_j \log x_i y_j. \tag{2.10}$$

The above result may be generalized to the integral form as follows.

Proposition 2.7. Let f be a positive coordinated log-convex function on the $\overset{\circ}{\Delta} := (a, b) \times (c, d)$, and let $x(t) : [r_1, r_2] \rightarrow \mathbf{R}$ be integrable with $a < x(t) < b$, and let $y(t) : [s_1, s_2] \rightarrow \mathbf{R}$ be integrable with $c < y(t) < d$. If $\alpha : [r_1, r_2] \rightarrow \mathbf{R}$ is positive, $\int_{r_1}^{r_2} \alpha(t) dt = 1$, and $(\alpha x)(t)$ is integrable on $[r_1, r_2]$ and $\beta : [s_1, s_2] \rightarrow \mathbf{R}$ is positive, $\int_{s_1}^{s_2} \beta(t) dt = 1$, and $(\beta y)(t)$ is integrable on $[s_1, s_2]$, then

$$\begin{aligned} \log f \left(\int_{r_1}^{r_2} \alpha(t) x(t) dt, \int_{s_1}^{s_2} \beta(u) y(u) du \right) \\ \leq \int_{r_1}^{r_2} \int_{s_1}^{s_2} \alpha(t) \beta(u) \log f(x(t), y(u)) du dt. \end{aligned} \tag{2.11}$$

Proof. Applying Jensen’s integral inequality in one variable on the x -coordinate and on the y -coordinate we get the required result. The details are omitted. \square

Theorem 2.8. Let $f : \Delta \rightarrow \mathbf{R}_+$ be a positive coordinated log-convex function in Δ , then for all distinct $x_1, x_2, x_3 \in [a, b]$, such that $x_1 < x_2 < x_3$ and distinct $y_1, y_2, y_3 \in [c, d]$ such that $y_1 < y_2 < y_3$, the following inequality holds:

$$\begin{aligned} & f^{x_2 y_2 + y_3 x_3}(x_1, y_1) \cdot f^{y_1 x_2 + y_2 x_3}(x_1, y_3) \cdot f^{x_1 y_2 + x_2 y_3}(x_3, y_1) \\ & \cdot f^{x_1 y_1 + y_2 x_2}(x_3, y_3) \cdot f^{x_1 y_3 + x_3 y_1}(x_2, y_2) \\ & \geq f^{x_2 y_3 + y_2 x_3}(x_1, y_1) \cdot f^{y_1 x_3 + x_2 y_2}(x_1, y_3) \cdot f^{x_1 y_3 + x_2 y_2}(x_3, y_1) \\ & \cdot f^{x_1 y_2 + y_1 x_2}(x_3, y_3) \cdot f^{x_1 y_1 + x_3 y_3}(x_2, y_2). \end{aligned} \tag{2.12}$$

Proof. Let x_1, x_2, x_3 be distinct points in $[a, b]$ and let y_1, y_2, y_3 be distinct points in $[c, d]$. Setting $\alpha = (x_3 - x_2)/(x_3 - x_1)$, $x_2 = \alpha x_1 + (1 - \alpha)x_3$ and let $\beta = (y_3 - y_2)/(y_3 - y_1)$, $y_2 = \beta y_1 + (1 - \beta)y_3$, we have

$$\begin{aligned} \log f(x_2, y_2) &= \log f(\alpha x_1 + (1 - \alpha)x_3, \beta y_1 + (1 - \beta)y_3) \\ &\leq \alpha \beta \log f(x_1, y_1) + \alpha(1 - \beta) \log f(x_1, y_3) \\ &\quad + \beta(1 - \alpha) \log f(x_3, y_1) + (1 - \alpha)(1 - \beta) \log f(x_3, y_3) \\ &= \frac{x_3 - x_2}{x_3 - x_1} \frac{y_3 - y_2}{y_3 - y_1} \log f(x_1, y_1) + \frac{x_3 - x_2}{x_3 - x_1} \frac{y_2 - y_1}{y_3 - y_1} \log f(x_1, y_3) \\ &\quad + \frac{x_2 - x_1}{x_3 - x_1} \frac{y_3 - y_2}{y_3 - y_1} \log f(x_3, y_1) + \frac{x_2 - x_1}{x_3 - x_1} \frac{y_2 - y_1}{y_3 - y_1} \log f(x_3, y_3), \end{aligned} \tag{2.13}$$

and we can write

$$\log \frac{f^{x_2 y_2 + y_3 x_3}(x_1, y_1) f^{y_1 x_2 + y_2 x_3}(x_1, y_3) f^{x_1 y_2 + x_2 y_3}(x_3, y_1) f^{x_1 y_1 + y_2 x_2}(x_3, y_3) f^{x_1 y_3 + x_3 y_1}(x_2, y_2)}{f^{x_2 y_3 + y_2 x_3}(x_1, y_1) f^{y_1 x_3 + x_2 y_2}(x_1, y_3) f^{x_1 y_3 + x_2 y_2}(x_3, y_1) f^{x_1 y_2 + y_1 x_2}(x_3, y_3) f^{x_1 y_1 + x_3 y_3}(x_2, y_2)} \geq 0. \quad (2.14)$$

From this inequality it is easy to deduce the required result (2.12). \square

The subharmonic functions exhibit many properties of convex functions. Next, we give some results for the coordinated convexity and sub(super)harmonic functions.

Proposition 2.9. *Let $f : \Delta \subseteq \mathbf{R}^2 \rightarrow \mathbf{R}$ be coordinated convex (concave) on Δ . If f is a twice differentiable on Δ° , then f is sub(super)harmonic on Δ° .*

Proof. Since f is coordinated convex on Δ then the partial mappings $f_y : [a, b] \rightarrow \mathbf{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \rightarrow \mathbf{R}$, $f_x(v) = f(x, v)$, are convex for all $y \in [c, d]$ and $x \in [a, b]$. Equivalently, since f is differentiable we can write

$$0 \leq f''_x = \frac{\partial^2 f}{\partial^2 y} \quad (2.15)$$

for all $y \in (c, d)$, and

$$0 \leq f''_y = \frac{\partial^2 f}{\partial^2 x} \quad (2.16)$$

for all $x \in (a, b)$, which imply that

$$f''_x + f''_y = \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y} \geq 0 \quad (2.17)$$

which shows that f is subharmonic. If f is coordinated concave on Δ , replace “ \leq ” by “ \geq ” above, we get that f is superharmonic on Δ° . \square

We now give two version(s) of the Maximum (Minimum) Principle theorem using convexity on the coordinates.

Theorem 2.10. *Let $f : \Delta \subseteq \mathbf{R}^2 \rightarrow \mathbf{R}$ be a coordinated convex (concave) function on Δ . If f is twice differentiable in Δ° and there is a point $a = (a_1, a_2) \in \Delta^\circ$ with $f(a_1, a_2) \geq (\leq) f(x, y)$, for all $(x, y) \in \Delta$ then f is a constant function.*

Proof. By Proposition 2.9, we get that f is sub(super)harmonic. Therefore, by Theorem 1.3 and the maximum principal the required result holds (see [14, page 264]). \square

Theorem 2.11. Let f and g be two twice differentiable functions in Δ° . Assume that f and g are bounded real-valued functions defined on Δ such that f is coordinated convex and g is coordinated concave. If for each point $a = (a_1, a_2) \in \partial_\infty \Delta$

$$\lim_{(x,y) \rightarrow (a_1, a_2)} \sup f(x, y) \leq \lim_{(x,y) \rightarrow (a_1, a_2)} \inf g(x, y), \quad (2.18)$$

then $f(x, y) < g(x, y)$ for all $(x, y) \in \Delta$ or $f = g$ and f is harmonic.

Proof. By Proposition 2.9, we get that f is subharmonic and g is superharmonic. Therefore, by Theorem 1.4 and using the maximum principal the required result holds, (see [14, page 264]). \square

Remark 2.12. The above two results hold for log-convex functions on the coordinates, simply, replacing f by $\log f$, to obtain the results.

3. Some Inequalities and Applications

In the following we develop a Hadamard-type inequality for coordinated log-convex functions.

Corollary 3.1. Suppose that $f : \Delta = [a, b] \times [c, d] \rightarrow \mathbf{R}_+$ is log-convex on the coordinates of Δ , then

$$\begin{aligned} \log f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \log f(x, y) dy dx \\ &\leq \log \sqrt[4]{f(a, c)f(a, d)f(b, c)f(b, d)}. \end{aligned} \quad (3.1)$$

For a positive coordinated log-concave function f , the inequalities are reversed.

Proof. In Theorem 1.2, replace f by $\log f$ and we get the required result. \square

Lemma 3.2. For $A, B, C \in \mathbb{R}^+$ with $A, B, C > 1$, the function

$$\psi(\beta) = C^\beta \frac{A^\beta B - 1}{\ln(A^\beta B)}, \quad 0 \leq \beta \leq 1 \quad (3.2)$$

is convex for all $\beta \in [0, 1]$. Moreover,

$$\int_0^1 \psi(\beta) d\beta \leq \frac{\psi(0) + \psi(1)}{2}, \quad (3.3)$$

for all $A, B, C > 1$.

Proof. Since ψ is twice differentiable for all $\beta \in (0, 1)$ with $A, B, C > 1$, we note that for all $0 < \beta_1 \leq \beta_2 < 1$, $\psi(\beta_1) \leq \psi(\beta_2)$, which shows that ψ is increasing and thus ψ' is nonnegative which

is equivalent to saying that ψ' is increasing and hence ψ is convex. Now, using inequality (1.1), we get

$$\int_0^1 C^\beta \frac{A^\beta B - 1}{\ln(A^\beta B)} d\beta = \int_0^1 \psi(\beta) d\beta \leq \frac{\psi(0) + \psi(1)}{2} = \frac{1}{2} \left[\frac{B-1}{\ln(B)} + C \cdot \frac{AB-1}{\ln(AB)} \right], \quad (3.4)$$

which completes the proof. \square

Theorem 3.3. Suppose that $f : \Delta = [a, b] \times [c, d] \rightarrow \mathbf{R}_+$ is log-convex on the coordinates of Δ . Let

$$A = \frac{f(a, c) f(b, d)}{f(b, c) f(a, d)}, \quad B = \frac{f(a, d)}{f(b, d)}, \quad C = \frac{f(b, c)}{f(b, d)}, \quad (3.5)$$

then the inequalities

$$I = \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) dx dy \leq f(b, d) \times \begin{cases} 1, & A = B = C = 1, \\ \left(\frac{B-1}{\ln B} \right) \left(\frac{C-1}{\ln C} \right), & A = 1, \\ H(C), & B = 1, \\ H(B), & C = 1, \\ \frac{C-1}{\ln C}, & A = B = 1, \\ \frac{B-1}{\ln B}, & A = C = 1, \\ -\frac{\gamma + \ln(-\ln A) + Ei(1, -\ln A)}{\ln A}, & B = C = 1, \\ \frac{1}{2} \left[\frac{B-1}{\ln(B)} + C \cdot \frac{AB-1}{\ln(AB)} \right], & A, B, C > 1, \\ \int_0^1 C^\beta \frac{A^\beta B - 1}{\ln(A^\beta B)} d\beta, & \text{otherwise} \end{cases} \quad (3.6)$$

hold, where γ is the Euler constant,

$$H(x) = \frac{Ei(1, -\ln x) + \ln(\ln x) - Ei(1, -\ln(Ax)) - \ln(\ln(Ax))}{\ln A} + \begin{cases} \frac{2 \ln(\ln A) - \ln(-\ln A)}{\ln A}, & \frac{\ln x}{\ln A} < 0, \quad -\frac{\ln x}{\ln A} < 1, \\ 0, & \text{otherwise,} \end{cases} \quad (3.7)$$

$$Ei(x) = V.P. \int_{-x}^{\infty} \frac{e^{-t}}{t} dt$$

is the exponential integral function. For a coordinated log-concave function f , the inequalities are reversed.

Proof. Since $f : \Delta = [a, b] \times [c, d] \rightarrow \mathbf{R}_+$ is log-convex on the coordinates of Δ , then

$$\begin{aligned}
 & f(\alpha a + (1 - \alpha)b, \beta c + (1 - \beta)d) \\
 & \leq f^{\alpha\beta}(a, c) f^{\beta(1-\alpha)}(b, c) f^{\alpha(1-\beta)}(a, d) f^{(1-\alpha)(1-\beta)}(b, d) \\
 & = f^{\alpha\beta}(a, c) f^{\beta}(b, c) f^{-\alpha\beta}(b, c) f^{\alpha}(a, d) f^{-\alpha\beta}(a, d) \\
 & \quad \times f(b, d) f^{-\beta}(b, d) f^{-\alpha}(b, d) f^{\alpha\beta}(b, d) \\
 & = \left[\frac{f(a, c) f(b, d)}{f(b, c) f(a, d)} \right]^{\alpha\beta} \left[\frac{f(a, d)}{f(b, d)} \right]^{\alpha} \left[\frac{f(b, c)}{f(b, d)} \right]^{\beta} f(b, d).
 \end{aligned} \tag{3.8}$$

Integrating the previous inequality with respect to α and β on $[0, 1]^2$, we have,

$$\begin{aligned}
 & \int_0^1 \int_0^1 f(\alpha a + (1 - \alpha)b, \beta c + (1 - \beta)d) d\alpha d\beta \\
 & \leq f(b, d) \int_0^1 \int_0^1 \left[\frac{f(a, c) f(b, d)}{f(b, c) f(a, d)} \right]^{\alpha\beta} \left[\frac{f(a, d)}{f(b, d)} \right]^{\alpha} \left[\frac{f(b, c)}{f(b, d)} \right]^{\beta} d\alpha d\beta.
 \end{aligned} \tag{3.9}$$

Therefore, by (3.9) and for nonzero, positive A, B, C , we have the following cases.

- (1) If $A = B = C = 1$, the result is trivial.
- (2) If $A = 1$, then

$$\begin{aligned}
 & \int_0^1 \int_0^1 f(\alpha a + (1 - \alpha)b, \beta c + (1 - \beta)d) d\alpha d\beta \\
 & \leq f(b, d) \int_0^1 \int_0^1 \left[\frac{f(a, d)}{f(b, d)} \right]^{\alpha} \left[\frac{f(b, c)}{f(b, d)} \right]^{\beta} d\alpha d\beta \\
 & = f(b, d) \left(\int_0^1 B^{\alpha} d\alpha \right) \left(\int_0^1 C^{\beta} d\beta \right) \\
 & = f(b, d) \left(\frac{B-1}{\ln B} \right) \left(\frac{C-1}{\ln C} \right).
 \end{aligned} \tag{3.10}$$

(3) If $B = 1$, then

$$\begin{aligned}
 & \int_0^1 \int_0^1 f(\alpha a + (1 - \alpha)b, \beta c + (1 - \beta)d) d\alpha d\beta \\
 & \leq f(b, d) \int_0^1 \int_0^1 A^{\alpha\beta} C^\beta d\alpha d\beta \\
 & = f(b, d) \int_0^1 \frac{A^\alpha C - 1}{\ln(A^\alpha C)} d\alpha \\
 & = f(b, d) \left[\left(\begin{cases} \frac{2 \ln(\ln A) - \ln(-\ln A)}{\ln A}, & \frac{\ln C}{\ln A} < 0, -\frac{\ln C}{\ln A} < 1 \\ 0, & \text{otherwise} \end{cases} \right) \right. \\
 & \quad \left. + \frac{Ei(1, -\ln C) + \ln(\ln C) - Ei(1, -\ln(AC)) - \ln(\ln(AC))}{\ln A} \right].
 \end{aligned} \tag{3.11}$$

(4) If $C = 1$, then

$$\begin{aligned}
 & \int_0^1 \int_0^1 f(\alpha a + (1 - \alpha)b, \beta c + (1 - \beta)d) d\alpha d\beta \\
 & \leq f(b, d) \int_0^1 \int_0^1 A^{\alpha\beta} B^\alpha d\alpha d\beta \\
 & = f(b, d) \int_0^1 \frac{A^\beta B - 1}{\ln(A^\beta B)} d\beta \\
 & = f(b, d) \left[\left(\begin{cases} \frac{2 \ln(\ln A) - \ln(-\ln A)}{\ln A}, & \frac{\ln B}{\ln A} < 0, -\frac{\ln B}{\ln A} < 1 \\ 0, & \text{otherwise} \end{cases} \right) \right. \\
 & \quad \left. + \frac{Ei(1, -\ln B) + \ln(\ln C) - Ei(1, -\ln(AB)) - \ln(\ln(AB))}{\ln A} \right].
 \end{aligned} \tag{3.12}$$

(5) If $A = B = 1$, then

$$\begin{aligned}
 & \int_0^1 \int_0^1 f(\alpha a + (1 - \alpha)b, \beta c + (1 - \beta)d) d\alpha d\beta \\
 & \leq f(b, d) \int_0^1 \int_0^1 A^{\alpha\beta} B^\alpha C^\beta d\alpha d\beta = f(b, d) \int_0^1 C^\beta d\beta = f(b, d) \frac{C - 1}{\ln C}.
 \end{aligned} \tag{3.13}$$

(6) If $A = C = 1$, then

$$\begin{aligned} & \int_0^1 \int_0^1 f(\alpha a + (1 - \alpha)b, \beta c + (1 - \beta)d) d\alpha d\beta \\ & \leq f(b, d) \int_0^1 \int_0^1 A^{\alpha\beta} B^\alpha C^\beta d\alpha d\beta = f(b, d) \int_0^1 B^\alpha d\alpha = f(b, d) \frac{B - 1}{\ln B}. \end{aligned} \quad (3.14)$$

(7) If $B = C = 1$, then

$$\begin{aligned} & \int_0^1 \int_0^1 f(\alpha a + (1 - \alpha)b, \beta c + (1 - \beta)d) d\alpha d\beta \\ & \leq f(b, d) \int_0^1 \int_0^1 A^{\alpha\beta} B^\alpha C^\beta d\alpha d\beta \\ & = f(b, d) \int_0^1 \int_0^1 (A^\beta)^\alpha d\alpha d\beta \\ & = f(b, d) \int_0^1 \frac{A^\alpha - 1}{\ln A^\alpha} d\alpha \\ & = -f(b, d) \frac{\gamma + \ln(-\ln A) + Ei(1, -\ln A)}{\ln A}. \end{aligned} \quad (3.15)$$

(8) If $A, B, C > 1$, then

$$f(b, d) \int_0^1 \int_0^1 A^{\alpha\beta} B^\alpha C^\beta d\alpha d\beta = f(b, d) \int_0^1 C^\beta \left[\frac{A^\beta B - 1}{\ln(A^\beta B)} \right] d\beta. \quad (3.16)$$

Therefore, by Lemma 3.2, we deduce that

$$f(b, d) \int_0^1 \int_0^1 A^{\alpha\beta} B^\alpha C^\beta d\alpha d\beta \leq \frac{f(b, d)}{2} \left[\frac{B - 1}{\ln(B)} + C \cdot \frac{AB - 1}{\ln(AB)} \right]. \quad (3.17)$$

(9) If $A, B, C \neq 1$, we have

$$f(b, d) \int_0^1 \int_0^1 A^{\alpha\beta} B^\alpha C^\beta d\alpha d\beta = f(b, d) \int_0^1 C^\beta \left[\frac{A^\beta B - 1}{\ln(A^\beta B)} \right] d\beta, \quad (3.18)$$

which is difficult to evaluate because it depends on the values of A, B , and C . \square

Remark 3.4. The integrals in (3), (4), and (7) in the proof of Theorem 2.11 are evaluated using Maple Software.

Corollary 3.5. *In Theorem 3.3, if*

(1) $f(x, y) = f(x)$, then

$$\frac{1}{b-a} \int_a^b f(x) dx \leq L(f(a), f(b)), \quad (3.19)$$

and for instance, if $f_1(x) = e^{x^p}$, $p \geq 1$ we deduce

$$\frac{1}{b-a} \int_a^b e^{x^p} dx \leq L(e^{a^p}, e^{b^p}). \quad (3.20)$$

(2) $f(x, y) = f_1(x)f_2(y)$, then

$$I \leq L(f_1(a), f_1(b))L(f_2(c), f_2(d)), \quad (3.21)$$

and for instance, if $f_1(x, y) = e^{x^p+y^q}$, $p, q \geq 1$, we deduce

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d e^{x^p+y^q} dx dy \leq L(e^{a^p}, e^{b^p})L(e^{c^q}, e^{d^q}). \quad (3.22)$$

Proof. Follows directly by applying inequality (1.4). □

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