ON THE HADAMARD'S INEQULALITY FOR CONVEX FUNCTIONS ON THE CO-ORDINATES IN A RECTANGLE FROM THE PLANE

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Abstract. An inequality of Hadamard's type for convex functions and convex functions on the co-ordinates defined in a rectangle from the plane and some applications are given.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality:

$$(1.1) \qquad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}$$

is known in the literature as Hadamard's inequality for convex mappings. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f.

In the paper [4] (see also [5] and [9]) is considered the following mapping naturally connected with Hadamard's result:

$$H : [0, 1] \to \mathbb{R}, H(t) := \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx.$$

The following properties of H hold:
(h) H is convex and monotonic nondecreasing.

(hh) One has the bounds

$$\sup_{t \in [0,1]} H(t) = H(1) = \frac{1}{b-a} \int_a^b f(x)dx$$

and

$$\inf_{t \in [0,1]} H(t) = H(0) = f\left(\frac{a+b}{2}\right).$$

Another mapping also closely connected with Hadamard's inequality is the following one [5] (see also [9]):

$$F : [0,1] \to \mathbb{R}, \ F(t) := \frac{1}{(b-a)^2} \int_a^b \int_a^b f(tx + (1-t)y)dxdy.$$

The properties of this mapping are the following ones:

(f) F is convex and monotonic nonincreasing on $[0, \frac{1}{2}]$ and nondecreasing on $[\frac{1}{2}, 1]$.

(ff) F is symmetrical relative to the element $\frac{1}{2}$, that is,

$$F(t) = F(1-t) \ \text{for all } t \in [0,1].$$

(fff) One has the bounds

$$\sup_{t \in [0,1]} F(t) = F(0) = F(1) = \frac{1}{b-a} \int_a^b f(x)dx$$

and

$$\inf_{t \in [0,1]} F(t) = F\left(\frac{1}{2}\right) = \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+y}{2}\right) dxdy \geq f\left(\frac{a+b}{2}\right).$$

(ffff) The following inequality holds:

$$F(t) \geq \max\{H(t), H(1-t)\} \ \text{for all } t \in [0,1].$$

In this paper we will point out a similar inequality to Hadamard's one that will work for convex mappings on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$. We will also consider some mappings similar in a sense to the mappings H and F and establish their main properties.

For recent refinements, counterparts, generalizations and new Hadamard - type inequalities, see the papers [1]-[12] and [14]-[15] and the book [13].

2. Hadamard's Inequality

Let us consider the bidimensional interval $\Delta := [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A function $f : \Delta \to \mathbb{R}$ will be called *convex on the co-ordinates* if the partial mappings $f_y : [a, b] \to \mathbb{R}$, $f_y(u) := f(u, y)$, and $f_x : [c, d] \to \mathbb{R}$, $f_x(v) := f(u, v)$, are convex where defined for all $y \in [c, d]$ and $x \in [a, b]$.

Recall that the mapping $f : \Delta \to \mathbb{R}$ is convex in Δ if the following inequality:

$$(2.1) \qquad f(\lambda x + (1 - \lambda)z, \lambda y + (1 - \lambda)w) \leq \lambda f(x, y) + (1 - \lambda)f(z, w)$$

holds, for all (x, y), $(z, w) \in \Delta$ and $\lambda \in [0, 1]$.

The following lemma holds:

Lemma 1. *Every convex mapping $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates, but the converse is not generally true.*

Proof. Suppose that $f : \Delta \to \mathbb{R}$ is convex in Δ. Consider $f_x : [c, d] \to \mathbb{R}$, $f_x(v) := f(x, v)$. Then for all $\lambda \in [0, 1]$ and $v, w \in [c, d]$ one has:

$$\begin{aligned} f_x(\lambda v + (1 - \lambda)w) &= f(x, \lambda v + (1 - \lambda)w) \\ &= f(\lambda x + (1 - \lambda)x, \lambda v + (1 - \lambda)w) \\ &\leq \lambda f(x, v) + (1 - \lambda)f(x, w) \\ &= \lambda f_x(v) + (1 - \lambda)f_x(w) \end{aligned}$$

which shows the convexity of f_x.

The fact that $f_y : [a, b] \to \mathbb{R}$, $f_y(u) := f(u, y)$, is also convex on $[a, b]$ for all $y \in [c, d]$ goes likewise and we shall omit the details.

Now, consider the mapping $f_0 : [0, 1]^2 \to [0, \infty)$ given by $f_0(x, y) = xy$. It's obvious that f is convex on the co-ordinates but is not convex on $[0, 1]^2$.

Indeed, if $(u, 0), (0, w) \in [0, 1]^2$ and $\lambda \in [0, 1]$, we have:

$$f(\lambda(u, 0) + (1 - \lambda)(0, w)) = f(\lambda u, (1 - \lambda)w) = \lambda(1 - \lambda)xw$$

and

$$\lambda f(u, 0) + (1 - \lambda)f(0, w) = 0.$$

Thus, for all $\lambda \in (0, 1), u, w \in (0, 1)$, we have

$$f(\lambda(u, 0) + (1 - \lambda)(0, w)) > \lambda f(u, 0) + (1 - \lambda)f(0, w),$$

which shows that f is not convex on $[0, 1]^2$. ∎

The following inequalities of Hadamard type hold:

Theorem 1. *Suppose that* $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ *is convex on the co-ordinates on* Δ. *Then one has the inequalities*:

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$$

$$\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right]$$

$$\leq \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x,y)dxdy$$

(2.2)

$$\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx\right.$$

$$\left. + \frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(b,y)dy\right]$$

$$\leq \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}.$$

The above inequalities are sharp.

Proof. Since $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates, it follows that the mapping $g_x : [c,d] \to \mathbb{R}$, $g_x(y) = f(x,y)$, is convex on $[c,d]$ for all $x \in [a,b]$. Then by Hadamard's inequality (1.1) one has:

$$g_x\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d g_x(y)dy \leq \frac{g_x(c) + g_x(d)}{2}, \ x \in [a,b].$$

That is,

$$f\left(x, \frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d f(x,y)dy \leq \frac{f(x,c) + f(x,d)}{2}, \ x \in [a,b].$$

Integrating this inequality on $[a,b]$, we have:

$$\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx$$

(2.3)

$$\leq \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x,y)dxdy$$

$$\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx\right].$$

By a similar argument applied for the mapping $g_y : [a, b] \to \mathbb{R}$, $g_y(x) := f(x, y)$, we get

(2.4)
$$\frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy$$
$$\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d (x, y) dx dy$$
$$\leq \frac{1}{2} \left[\frac{1}{d-c} \int_c^d f(a, y) dy + \frac{1}{d-c} \int_c^d f(b, y) dy \right].$$

Summing the inequalities (2.3) and (2.4), we get the second and the third inequalities in (2.2).

By Hadamard's inequality, we also have:

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx$$

and

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy,$$

which give, by addition, the first inequality in (2.2).

Finally, by the same inequality we can also state:

$$\frac{1}{b-a} \int_a^b f(x, c) dx \leq \frac{f(a, c) + f(b, c)}{2},$$
$$\frac{1}{b-a} \int_a^b f(x, d) dx \leq \frac{f(a, d) + f(b, d)}{2},$$
$$\frac{1}{d-c} \int_c^d f(a, y) dy \leq \frac{f(a, c) + f(a, d)}{2},$$

and

$$\frac{1}{d-c} \int_c^d f(b, y) dy \leq \frac{f(b, c) + f(d, b)}{2},$$

which give, by addition, the last inequality in (2.2).

If in (2.2) we choose $f(x) = xy$, then (2.2) becomes an equality, which shows that (2.2) are sharp.   ∎

3. Some Mappings Associated to Hadamard's Inequality

Now, for a mapping $f : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R}$ as above, we can define the mapping $H : [0, 1]^2 \rightarrow \mathbb{R}$,

$$H(t, s) := \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) dxdy.$$

The properties of this mapping are embodied in the following theorem.

Theorem 2. *Suppose that* $f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ *is convex on the co-ordinates on* $\Delta := [a, b] \times [c, d]$. *Then*:

(i) *The mapping* H *is convex on the co-ordinates on* $[0, 1]^2$.

(ii) *We have the bounds*:

$$\sup_{(t,s)\in[0,1]^2} H(t, s) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)dxdy = H(0, 0),$$

$$\inf_{(t,s)\in[0,1]^2} H(t, s) = f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) = H(1, 1).$$

(iii) *The mapping* H *is monotonic nondecreasing on the co-ordinates*.

Proof. (i) Fix $s \in [0, 1]$. Then for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $t_1, t_2 \in [0, 1]$, we have:

$$H(\alpha t_1 + \beta t_2, s) = \frac{1}{(b-a)(d-c)}$$
$$\times \int_a^b \int_c^d f\left((\alpha t_1 + \beta t_2)x + [1 - (\alpha t_1 + \beta t_2)]\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) dxdy$$
$$= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(\alpha\left(t_1 x + (1-t_1)\frac{a+b}{2}\right)\right.$$
$$\left. + \beta\left(t_2 x + (1-t_2)\frac{a+b}{2}\right), sy + (1-s)\frac{c+d}{2}\right) dxdy$$
$$\leq \alpha \cdot \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(t_1 x + (1-t_1)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) dxdy$$
$$+ \beta \cdot \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(t_2 x + (1-t_2)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) dxdy$$
$$= \alpha H(t_1, s) + \beta H(t_2, s).$$

If $t \in [0, 1]$ is fixed, then for all $s_1, s_2 \in [0, 1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, we also have:

$$H(t, \alpha s_1 + \beta s_2) \leq \alpha H(t, s_1) + \beta H(t, s_2)$$

and the statement is proved.

(ii) Since f is convex on the co-ordinates, we have, by Jensen's inequality for integrals, that:

$$H(t, s) = \frac{1}{b-a} \int_a^b \left[\frac{1}{d-c} \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) dy \right] dx$$

$$\geq \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}, \frac{1}{d-c} \int_c^d \left[sy + (1-s)\frac{c+d}{2}\right] dy\right) dx$$

$$= \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}, \frac{c+d}{2}\right) dx$$

$$\geq f\left(\frac{1}{b-a} \int_a^b \left[tx + (1-t)\frac{a+b}{2}\right], \frac{c+d}{2}\right) dx$$

$$= f\left(\frac{a+b}{2}, \frac{c+d}{2}\right).$$

By the convexity of H on the co-ordinates, we have:

$$H(t, s) \leq \frac{1}{b-a} \int_a^b \left[\frac{1}{d-c} \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, y\right) dy \right.$$

$$\left. + (1-s) \cdot \frac{1}{d-c} \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, \frac{c+d}{2}\right) dy \right] dx$$

$$\leq s \cdot \frac{1}{d-c} \int_c^d \left[t \cdot \frac{1}{b-a} \int_a^b f(x, y) dx dy \right.$$

$$+ (1-t) \cdot \frac{1}{b-a} \int_a^b f\left(\frac{a+b}{2}, y\right) dx \right] dy$$

$$+ (1-s) \cdot \frac{1}{d-c} \int_c^d \left[t \cdot \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx \right.$$

$$+ (1-t) \cdot f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right] dy$$

$$= st \cdot \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dx dy + s(1-t) \cdot \frac{1}{d-c} \int_a^b f\left(\frac{a+b}{2}, y\right) dy$$

$$+ (1-s)t \cdot \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx + (1-s)(1-t) \cdot f\left(\frac{a+b}{2}, \frac{c+d}{2}\right).$$

By Hadamard's inequality, we also have:

$$f\left(\frac{a+b}{2}, y\right) \le \frac{1}{b-a}\int_a^b f(x,y)dx, \ y \in [c,d]$$

and

$$f\left(x, \frac{c+d}{2}\right) \le \frac{1}{d-c}\int_c^d f(x,y)dy, \ x \in [a,b].$$

Thus, by integration, we get that:

$$\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy$$

and

$$\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy.$$

Using the above inequality, we deduce that

$$H(t,s)$$

$$\le [st + s(1-t) + (1-s)t + (1-s)(1-t)]\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy$$

$$= \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy, \ (s,t) \in [0,1]^2,$$

and the second bound in (ii) is proved.

(iii) Firstly, we will show that

(3.1) $$H(t,s) \ge H(0,s) \text{ for all } (t,s) \in [0,1]^2.$$

By Hadamard's inequality, we have:

$$H(t,s) \ge \frac{1}{d-c}\int_c^d f\left(\frac{1}{b-a}\int_a^b\left[tx + (1-t)\frac{a+b}{2}\right]dx, sy + (1-s)\frac{c+d}{2}\right)dy$$

$$= \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right)dy = H(0,s)$$

for all $(t,s) \in [0,1]^2$.

Now let $0 \le t_1 < t_2 \le 1$. By the convexity of the mapping $H(\cdot, s)$ for all $s \in [0,1]$, we have

$$\frac{H(t_2, s) - H(t_1, s)}{t_2 - t_1} \ge \frac{H(t_1, s) - H(0, s)}{t_1} \ge 0.$$

Note that, for the last inequality we have used (3.1).   ∎

The following theorem also holds.

Theorem 3. *Suppose that $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ is convex on Δ. Then*

(i) *The mapping H is convex on Δ.*

(ii) *Define the mapping $h : [0, 1] \to \mathbb{R}$, $h(t) = H(t, t)$. Then h is convex, monotonic nondecreasing on $[0, 1]$ and one has the bounds:*

$$\sup_{t \in [0,1]} h(t) = h(1) = \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) dx dy$$

and

$$\inf_{t \in [0,1]} h(t) = h(0) = f\left(\frac{a + b}{2}, \frac{c + d}{2}\right).$$

Proof.

(i) Let $(t_1, s_1), (t_2, s_2) \in [0, 1]^2$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Since $f : \Delta \to \mathbb{R}$ is convex on Δ we have:

$$H(\alpha(t_1, s_1) + \beta(t_2, s_2))$$

$$= H(\alpha t_1 + \beta t_2, \alpha s_1 + \beta s_2)$$

$$= \frac{1}{(b - a)(d - c)}$$

$$\times \int_a^b \int_c^d f\left(\alpha\left(t_1 x + (1 - t_1)\frac{a + b}{2}, s_1 y + (1 - s_1)\frac{c + d}{2}\right)\right.$$

$$\left. + \beta\left(t_2 x + (1 - t_2)\frac{a + b}{2}, s_2 y + (1 - s_2)\frac{c + d}{2}\right)\right) dx dy$$

$$\leq \alpha \cdot \frac{1}{(b - a)(d - c)}$$

$$\times \int_a^b \int_c^d f\left(t_1 x + (1 - t_1)\frac{a + b}{2}, s_1 y + (1 - s_1)\frac{c + d}{2}\right) dx dy$$

$$+ \beta \cdot \frac{1}{(b - a)(d - c)}$$

$$\times \int_a^b \int_c^d f\left(t_2 x + (1 - t_2)\frac{a + b}{2}, s_2 y + (1 - s_2)\frac{c + d}{2}\right) dx dy$$

$$= \alpha H(t_1, s_1) + \beta H(t_2, s_2),$$

which shows that H is convex on $[0, 1]^2$.

(ii) Let $t_1, t_2 \in [0, 1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Then

$$
\begin{aligned}
h(\alpha t_1, \beta t_2) &= H(\alpha t_1 + \beta t_2, \alpha t_1 + \beta t_2) \\
&= H(\alpha(t_1, t_1) + \beta(t_2, t_2)) \\
&\leq \alpha H(t_1, t_1) + \beta H(t_2, t_2) \\
&= \alpha h(t_1) + \beta h(t_2),
\end{aligned}
$$

which shows the convexity of h on $[0, 1]$.

We have, by the above theorem, that

$$
h(t) = H(t, t) \geq H(0, 0) = f\left(\frac{a+b}{2}, \frac{c+d}{2}\right), \ t \in [0, 1],
$$

and

$$
h(t) = H(t, t) \leq H(1, 1) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dx dy, \ t \in [0, 1],
$$

which prove the required bounds.

Now, let $0 \leq t_1 < t_2 \leq 1$. Then, by the convexity of h, we have that

$$
\frac{h(t_2) - h(t_1)}{t_2 - t_1} \geq \frac{h(t_1) - h(0)}{t_1} \geq 0,
$$

and the theorem is proved.   ∎

Next, we shall consider the following mapping, which is closely connected with Hadamard's inequality: $\tilde{H} : [0, 1]^2 \to [0, \infty)$ given by

$$
\begin{aligned}
&\tilde{H}(t, s) \\
&:= \frac{1}{(b-a)^2(d-c)^2} \int_a^b \int_a^b \int_c^d \int_c^d f(tx + (1-t)y, sz + (1-s)u) dx dy dz du.
\end{aligned}
$$

The next theorem contains the main properties of this mapping.

Theorem 4. *Suppose that $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ is convex on the co-ordinates on Δ. Then*:

(i) *We have the equalities*:

$$
\tilde{H}\left(t + \frac{1}{2}, s\right) = \tilde{H}\left(\frac{1}{2} - t, s\right) \ \text{for all } t \in \left[0, \frac{1}{2}\right], s \in [0, 1];
$$

$$
\tilde{H}\left(t, s + \frac{1}{2}\right) = \tilde{H}\left(t, \frac{1}{2} - s\right) \ \text{for all } t \in [0, 1], s \in \left[0, \frac{1}{2}\right];
$$

$$
\tilde{H}(1 - t, s) = \tilde{H}(t, s) \text{ and } \tilde{H}(t, 1 - s) = \tilde{H}(t, s) \ \text{for all } (t, s) \in \Delta.
$$

(ii) $\tilde{H}$ *is convex on the co-ordinates.*

(iii) *We have the bounds*

$$\inf_{(t,s)\in[0,1]^2} \tilde{H}(t,s) = \tilde{H}\left(\frac{1}{2},\frac{1}{2}\right)$$

$$= \frac{1}{(b-a)^2(d-c)^2} \int_a^b \int_a^b \int_c^d \int_c^d f\left(\frac{x+y}{2},\frac{z+u}{2}\right) dxdydzdu$$

and

$$\sup_{(t,s)\in[0,1]^2} \tilde{H}(t,s) = \tilde{H}(0,0) = \tilde{H}(1,1)$$

$$= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,z)dxdz.$$

(iv) *The mapping $\tilde{H}(\cdot,s)$ is monotonic nonincreasing on $[0,\frac{1}{2})$ and nondecreasing on $[\frac{1}{2},1]$ for all $s \in [0,1]$. A similar property has the mapping $\tilde{H}(t,\cdot)$ for all $t \in [0,1]$.*

(v) *We have the inequality*

(3.2) $\qquad \tilde{H}(t,s) \geq \max\{H(t,s), H(1-t,s), H(t,1-s), H(1-t,1-s)\}$

for all $(t,s) \in [0,1]^2$.

Proof. (i), (ii) are obvious.

(iii) By the convexity of f in the first variable, we get that

$$\frac{1}{2}[f(tx+(1-t)y, sz+(1-s)u) + f((1-t)x+ty, sz+(1-s)u)]$$
$$\geq f\left(\frac{x+y}{2}, sz+(1-s)u\right)$$

for all $(x,y) \in [a,b]^2$, $(z,u) \in [c,d]^2$ and $(t,s) \in [0,1]^2$.
Integrating on $[a,b]^2$, we get

$$\frac{1}{(b-a)^2} \int_a^b \int_a^b f(tx+(1-t)y, sz+(1-s)u)dxdy$$

$$\geq \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+y}{2}, sz+(1-s)u\right) dxdy.$$

Similarly,

$$\frac{1}{(d-c)^2} \int_c^d \int_c^d f\left(\frac{x+y}{2}, sz + (1-s)u\right) dz du$$

$$\geq \frac{1}{(d-c)^2} \int_c^d \int_c^d f\left(\frac{x+y}{2}, \frac{z+u}{2}\right) dz du.$$

Now, integrating this inequality on $[a, b]^2$ and taking into account the above inequality, we deduce:

$$\tilde{H}(t, s) \geq \frac{1}{(b-a)^2(d-c)^2} \int_a^b \int_a^b \int_c^d \int_c^d f\left(\frac{x+y}{2}, \frac{z+u}{2}\right) dx dy dz du$$

for $(t, s) \in [0, 1]$. The first bound in (iii) is therefore proved.

The second bound goes likewise and we shall omit the details.

(iv) The monotonicity of $\tilde{H}(\cdot, s)$ follows by a similar argument as in the proof of Theorem 2, (iii) and we shall omit the details.

(v) By Jensen's inequality, we have successively for all $(t, s) \in [0, 1]^2$ that

$$\tilde{H}(t, s)$$

$$\geq \frac{1}{(b-a)(d-c)^2}$$

$$\times \int_a^b \int_c^d \int_c^d f\left(\frac{1}{b-a} \int_a^b [tx + (1-t)y] dy, sz + (1-s)u\right) dx dz du$$

$$= \frac{1}{(b-a)(d-c)^2} \int_a^b \int_c^d \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, sz + (1-s)u\right) dx dz du$$

$$\geq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, sz + (1-s)\frac{c+d}{2}\right) dx dz$$

$$= H(t, s).$$

In addition, as

$$\tilde{H}(t, s) = \tilde{H}(1-t, s) = \tilde{H}(t, 1-s) = \tilde{H}(1-t, 1-s) \text{ for all } (t, s) \in [0, 1]^2,$$

by the above inequality we deduce (3.2).

The theorem is thus proved.   ■

Finally, we can also state the following theorem which can be proved in a similar fashion to Theorem 3 and we will omit the details.

Theorem 5. *Suppose that* $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ *is convex on* Δ. *Then we have*:

(i) *The mapping $\tilde{H}$ is convex on Δ.*

(ii) *Define the mapping $\tilde{h} : [0,1] \rightarrow \mathbb{R}$, $\tilde{h}(t) := \tilde{H}(t,t)$. Then $\tilde{h}$ is convex, monotonic nonincreasing on $\left[0, \frac{1}{2}\right]$ and nondecreasing on $\left[\frac{1}{2}, 1\right]$ and one has the bounds*:

$$\sup_{t, \in [0,1]} \tilde{h}(t) = \tilde{h}(1) = \tilde{h}(0) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dx dy$$

and

$\inf_{t \in [0,1]} \tilde{h}(t) = \tilde{h}\left(\frac{1}{2}\right)$

$$= \frac{1}{(b-a)^2(d-c)^2} \int_a^b \int_a^b \int_c^d \int_c^d f\left(\frac{x+y}{2}, \frac{z+u}{2}\right) dx dy dz du.$$

(iii) *One has the inequality*:

$$\tilde{h}(t) \geq \max\{h(t), h(1-t)\} \; for \; all \; t \in [0,1].$$

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