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On the Hermite–Hadamard type inequality for ψ-Riemann–Liouville fractional integrals via convex functions

Kui Liu[1], JinRong Wang[2,3*] and Donal O'Regan[4]

*Correspondence:
jrwang@gzu.edu.cn
[2]Department of Mathematics, Guizhou University, Guiyang, China
[3]School of Mathematical Sciences, Qufu Normal University, Qufu, China
Full list of author information is available at the end of the article

Abstract

In this paper, we establish a new Hermite–Hadamard inequality involving left-sided and right-sided ψ-Riemann–Liouville fractional integrals via convex functions. We also show two basic ψ-Riemann–Liouville fractional integral identities including the first order derivative of a given convex function, and these will be used to derive estimates for some fractional Hermite–Hadamard inequalities. Finally, we give some applications to special means of real numbers.

Keywords: Hermite–Hadamard inequality; ψ-Riemann–Liouville fractional integrals

1 Introduction and preliminaries

It is well known that Hermite established the following Hermite–Hadamard integral inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(s)\,ds \leq \frac{f(a)+f(b)}{2}, \tag{1}$$

where $f : [a,b] \subset R \to R$ is a convex function (see [6]). This inequality provides a lower and an upper estimate for the integral average of any convex function defined on a compact interval. For generalizations of the classical Hermite–Hadamard inequality, see [1–4, 6–8, 10–12] and the references therein.

In the last decade, fractional calculus [5] has played an important role in various scientific fields since it is a good tool to describe long-memory processes. In [7], the authors established Hermite–Hadamard's inequalities for Riemann–Liouville fractional integrals and some Hermite–Hadamard type integral inequalities for fractional integrals; in [8], the authors obtained some new inequalities of Ostrowski type involving fractional integrals; and in [9], the authors presented some properties and results on fractional calculus using the ψ-Hilfer fractional derivative. Fractional Hermite–Hadamard inequalities for Riemann–Liouville and Hadamard fractional integrals have been studied extensively in the literature, but there are only a few results concerning Hermite–Hadamard inequalities for ψ-Riemann–Liouville fractional integrals via convex functions. In [10–12], the authors extended the classical Hermite–Hadamard type inequalities to Riemann–Liouville and Hadamard fractional integral cases, which can be used to find lower and upper bounds for fractional integral for some given convex functions.

Definition 1.1 (see [5] or [9, Definition 4]) Let (a,b) $(-\infty \leq a < b \leq \infty)$ be a finite or infinite interval of the real line R and $\alpha > 0$. Also let $\psi(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(x)$ on (a,b). The left- and right-sided ψ-Riemann–Liouville fractional integrals of a function f with respect to another function ψ on $[a,b]$ are defined by

$$I_{a^+}^{\alpha:\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)\bigl(\psi(x) - \psi(t)\bigr)^{\alpha-1} f(t)\,dt,$$
$$I_{b^-}^{\alpha:\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t)\bigl(\psi(t) - \psi(x)\bigr)^{\alpha-1} f(t)\,dt,$$

respectively; here $\Gamma(\cdot)$ is the gamma function.

The aim of this paper is to establish Hermite–Hadamard's inequality for fractional integrals $I_{a^+}^{\alpha:\psi} f(x)$ and $I_{b^-}^{\alpha:\psi} f(x)$ and derive some related integral inequalities by using new identities for ψ-fractional integrals.

2 Hermite–Hadamard inequality for ψ-Riemann–Liouville fractional integrals

Theorem 2.1 *Let $0 \leq c < d$, $g : [c,d] \to R$ be a positive function and $g \in L_1[c,d]$. Also suppose that g is a convex function on $[c,d]$, $\psi(x)$ is an increasing and positive monotone function on $(c,d]$, having a continuous derivative $\psi'(x)$ on (a,b) and $\alpha \in (0,1)$. Then the following fractional integral inequalities hold*:

$$\begin{aligned} g\left(\frac{c+d}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{2(d-c)^\alpha}\bigl[I_{\psi^{-1}(c)^+}^{\alpha:\psi}(g\circ\psi)\bigl(\psi^{-1}(d)\bigr) + I_{\psi^{-1}(d)^-}^{\alpha:\psi}(g\circ\psi)\bigl(\psi^{-1}(c)\bigr)\bigr] \\ &\leq \frac{g(c)+g(d)}{2}. \end{aligned} \tag{2}$$

Proof Let $x, y \in [c,d]$. Since $g : [c,d] \to R$ is a convex function, from (1) we have

$$g\left(\frac{x+y}{2}\right) \leq \frac{g(x)+g(y)}{2}. \tag{3}$$

Let $x = tc + (1-t)d$, $y = (1-t)c + td$, and put x, y into (3), so we have

$$2g\left(\frac{c+d}{2}\right) \leq g\bigl(tc + (1-t)d\bigr) + g\bigl((1-t)c + td\bigr). \tag{4}$$

Multiply both sides of (4) by $t^{\alpha-1}$ and then integrate, so we have

$$\int_0^1 t^{\alpha-1} g\bigl(tc + (1-t)d\bigr)\,dt + \int_0^1 t^{\alpha-1} g\bigl((1-t)c + td\bigr)\,dt \geq \frac{2}{\alpha} g\left(\frac{c+d}{2}\right). \tag{5}$$

Next,

$$\frac{\Gamma(\alpha+1)}{2(d-c)^\alpha}\bigl[I_{\psi^{-1}(c)^+}^{\alpha:\psi}(g\circ\psi)\bigl(\psi^{-1}(d)\bigr) + I_{\psi^{-1}(d)^+}^{\alpha:\psi}(g\circ\psi)\bigl(\psi^{-1}(c)\bigr)\bigr]$$

$$
\begin{aligned}
&= \frac{\Gamma(\alpha+1)}{2(d-c)^{\alpha}} \frac{1}{\Gamma(\alpha)} \biggl[\int_{\psi^{-1}(c)}^{\psi^{-1}(d)} \psi'(\nu)\bigl(d-\psi(\nu)\bigr)^{\alpha-1}(g\circ\psi)(\nu)\,d\nu \\
&\quad + \int_{\psi^{-1}(c)}^{\psi^{-1}(d)} \psi'(\nu)\bigl(\psi(\nu)-c\bigr)^{\alpha-1}(g\circ\psi)(\nu)\,d\nu \biggr] \\
&= \frac{\alpha}{2} \biggl[\int_{\psi^{-1}(c)}^{\psi^{-1}(d)} \biggl(\frac{d-\psi(\nu)}{d-c}\biggr)^{\alpha-1} g\bigl(\psi(\nu)\bigr) \frac{\psi'(\nu)}{d-c}\,d\nu \\
&\quad + \int_{\psi^{-1}(c)}^{\psi^{-1}(d)} \biggl(\frac{\psi(\nu)-c}{d-c}\biggr)^{\alpha-1} g\bigl(\psi(\nu)\bigr) \frac{\psi'(\nu)}{d-c}\,d\nu \biggr] \\
&= \frac{\alpha}{2} \biggl[\int_0^1 t^{\alpha-1} g\bigl(tc+(1-t)d\bigr)\,dt + \int_0^1 s^{\alpha-1} g\bigl((1-s)c+sd\bigr)\,ds \biggr] \\
&\quad \biggl(\text{let } t = \frac{\psi(\nu)d}{c-d}, s = \frac{\psi(\nu)-c}{d-c}\biggr) \\
&= \frac{\alpha}{2} \biggl[\int_0^1 t^{\alpha-1} g\bigl(tc+(1-t)d\bigr)\,dt + \int_0^1 t^{\alpha-1} g\bigl((1-t)c+td\bigr)\,dt \biggr] \\
&\geq g\biggl(\frac{c+d}{2}\biggr),
\end{aligned}
$$

where (5) is used, so the left-hand side inequality in (2) is proved.

To prove the right-hand side inequality in (2), since g is a convex function, then for $t \in [0,1]$, we have

$$
g\bigl(tc+(1-t)d\bigr) \leq tg(c) + (1-t)g(d)
$$

and

$$
g\bigl((1-t)c+td\bigr) \leq (1-t)g(c) + tg(d).
$$

Now

$$
g\bigl(tc+(1-t)d\bigr) + g\bigl((1-t)c+td\bigr) \leq tg(c) + (1-t)g(d) + (1-t)g(c) + tg(d),
$$

i.e.,

$$
g\bigl(tc+(1-t)d\bigr) + g\bigl((1-t)c+td\bigr) \leq g(c) + g(d). \tag{6}
$$

Multiply both sides of (6) by $t^{\alpha-1}$ and then integrate, so we obtain

$$
\int_0^1 t^{\alpha-1} g\bigl(tc+(1-t)d\bigr)\,dt + \int_0^1 t^{\alpha-1} g\bigl((1-t)c+td\bigr)\,dt \leq \frac{g(c)+g(d)}{\alpha},
$$

i.e.,

$$
\frac{\Gamma(\alpha)}{(d-c)^{\alpha}} \bigl[I^{\alpha:\psi}_{\psi^{-1}(c)^+}(f\circ\psi)\bigl(\psi^{-1}(d)\bigr) + I^{\alpha:\psi}_{\psi^{-1}(d)^+}(f\circ\psi)\bigl(\psi^{-1}(c)\bigr) \bigr] \leq \frac{g(c)+g(d)}{\alpha}.
$$

The proof is complete. □

3 Hermite–Hadamard type inequalities for ψ-Riemann–Liouville fractional integrals

Lemma 3.1 *Let $c < d$ and $g : [c,d] \to R$ be a differentiable mapping on (c,d). Also suppose that $g' \in L[c,d]$, $\psi(x)$ is an increasing and positive monotone function on $(c,d]$, having a continuous derivative $\psi'(x)$ on (c,d) and $\alpha \in (0,1)$. Then the following equality for fractional integrals holds*:

$$\begin{aligned}&\frac{g(c)+g(d)}{2} - \frac{\Gamma(\alpha+1)}{2(d-c)^\alpha}\bigl[I^{\alpha:\psi}_{\psi^{-1}(c)^+}(g\circ\psi)\bigl(\psi^{-1}(d)\bigr) + I^{\alpha:\psi}_{\psi^{-1}(d)^-}(g\circ\psi)\bigl(\psi^{-1}(c)\bigr)\bigr]\\ &\quad= \frac{1}{2(d-c)^\alpha}\int_{\psi^{-1}(c)}^{\psi^{-1}(d)} \bigl[\bigl(\psi(v)-c\bigr)^\alpha - \bigl(d-\psi(v)\bigr)^\alpha\bigr]\bigl(g'\circ\psi\bigr)(v)\psi'(v)\,dv.\end{aligned}$$

Proof Let $I_1 = \frac{\Gamma(\alpha+1)}{2(d-c)^\alpha} I^{\alpha:\psi}_{\psi^{-1}(c)^+}(g\circ\psi)(\psi^{-1}(d))$ and $I_2 = \frac{\Gamma(\alpha+1)}{2(d-c)^\alpha} I^{\alpha:\psi}_{\psi^{-1}(d)^-}(g\circ\psi)(\psi^{-1}(c))$. Then

$$\begin{aligned}I_1 &= \frac{\alpha}{2(d-c)^\alpha}\int_{\psi^{-1}(c)}^{\psi^{-1}(d)} \psi'(v)\bigl(d-\psi(v)\bigr)^{\alpha-1}(g\circ\psi)(v)\,dv\\ &= -\frac{1}{2(d-c)^\alpha}\int_{\psi^{-1}(d)}^{\psi^{-1}(c)} (g\circ\psi)(v)\,d\bigl(d-\psi(v)\bigr)^\alpha\\ &= \frac{1}{2(d-c)^\alpha}[(d-c)^\alpha g(c) + \int_{\psi^{-1}(c)}^{\psi^{-1}(d)} \psi'(v)\bigl(d-\psi(v)\bigr)^\alpha\bigl(g'\circ\psi\bigr)(v)\,dv,\end{aligned}$$

and

$$\begin{aligned}I_2 &= \frac{\alpha}{2(d-c)^\alpha}\int_{\psi^{-1}(c)}^{\psi^{-1}(d)} \psi'(v)\bigl(\psi(v)-c\bigr)^{\alpha-1}(g\circ\psi)(v)\,dv\\ &= \frac{1}{2(d-c)^\alpha}\int_{\psi^{-1}(c)}^{\psi^{-1}(d)} (g\circ\psi)(v)\,d\bigl(\psi(v)-c\bigr)^\alpha\\ &= \frac{1}{2(d-c)^\alpha}\left[(d-c)^\alpha g(d) - \int_{\psi^{-1}(c)}^{\psi^{-1}(d)} \psi'(v)\bigl(\psi(v)-c\bigr)^\alpha\bigl(g'\circ\psi\bigr)(v)\,dv\right].\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{g(c)+g(d)}{2} - I_1 - I_2\\ &\quad= \frac{1}{2(d-c)^\alpha}\int_{\psi^{-1}(c)}^{\psi^{-1}(d)} \bigl[\bigl(\psi(v)-c\bigr)^\alpha - \bigl(d-\psi(v)\bigr)^\alpha\bigr]\bigl(g'\circ\psi\bigr)(v)\psi'(v)\,dv.\end{aligned}$$

The proof is complete. □

Lemma 3.2 *Let $c < d$ and $g : [c,d] \to R$ be a differentiable mapping on (c,d). Also suppose that $g' \in L[c,d]$, $\psi(x)$ is a positive monotone function increasing on $(c,d]$, having a continuous derivative $\psi'(x)$ on (c,d) and $\alpha \in (0,1)$. Then the following equality for fractional integrals holds*:

$$\frac{\Gamma(\alpha+1)}{2(d-c)^\alpha}\bigl[I^{\alpha:\psi}_{\psi^{-1}(c)^+}(g\circ\psi)\bigl(\psi^{-1}(d)\bigr) + I^{\alpha:\psi}_{\psi^{-1}(d)^-}(g\circ\psi)\bigl(\psi^{-1}(c)\bigr)\bigr] - g\left(\frac{c+d}{2}\right)$$

$$= \int_{\psi^{-1}(c)}^{\psi^{-1}(d)} k\bigl(g' \circ \psi\bigr)(v)\psi'(v)\,dv$$
$$+ \frac{1}{2(d-c)^\alpha} \int_{\psi^{-1}(c)}^{\psi^{-1}(d)} \bigl[\bigl((d-\psi(v)\bigr)^\alpha - \bigl(\psi(v)-c\bigr)^\alpha\bigr]\bigl(g' \circ \psi\bigr)(v)\psi'(v)\,dv, \tag{7}$$

where

$$k = \begin{cases} \frac{1}{2}, & \psi^{-1}(\frac{c+d}{2}) \le v \le \psi^{-1}(d), \\ -\frac{1}{2}, & \psi^{-1}(c) < v < \psi^{-1}(\frac{c+d}{2}). \end{cases} \tag{8}$$

Proof Let

$$J_1 = \int_{\psi^{-1}(c)}^{\psi^{-1}(\frac{c+d}{2})} -\frac{1}{2}\bigl(g' \circ \psi\bigr)(v)\psi'(v)\,dv = -\frac{1}{2}g\left(\frac{c+d}{2}\right) + \frac{1}{2}g(c),$$
$$J_2 = \int_{\psi^{-1}(\frac{c+d}{2})}^{\psi^{-1}(d)} \frac{1}{2}\bigl(g' \circ \psi\bigr)(v)\psi'(v)\,dv = \frac{1}{2}g(d) - \frac{1}{2}g\left(\frac{c+d}{2}\right),$$
$$J_3 = \frac{1}{2(d-c)^\alpha} \int_{\psi^{-1}(c)}^{\psi^{-1}(d)} \psi'(v)\bigl(d-\psi(v)\bigr)^\alpha \bigl(g' \circ \psi\bigr)(v)\,dv$$
$$= -\frac{1}{2}g(c) + \frac{1}{2(d-c)^\alpha} \int_{\psi^{-1}(c)}^{\psi^{-1}(d)} \alpha\psi'(v)\bigl(d-\psi(v)\bigr)^{\alpha-1}(g \circ \psi)(v)\,dv$$
$$= -\frac{1}{2}g(c) + \frac{\alpha\Gamma(\alpha)}{2(d-c)^\alpha} I^{\alpha:\psi}_{\psi^{-1}(c)^+}(g \circ \psi)\bigl(\psi^{-1}(d)\bigr),$$

and

$$J_4 = -\frac{1}{2(d-c)^\alpha} \int_{\psi^{-1}(c)}^{\psi^{-1}(d)} \psi'(v)\bigl(\psi(v)-c\bigr)^\alpha \bigl(g' \circ \psi\bigr)(v)\,dv$$
$$= -\frac{1}{2}g(d) + \frac{1}{2(d-c)^\alpha} \int_{\psi^{-1}(c)}^{\psi^{-1}(d)} \alpha\psi'(v)\bigl(\psi(v)-c\bigr)^{\alpha-1}(g \circ \psi)(v)\,dv$$
$$= -\frac{1}{2}g(d) + \frac{\alpha\Gamma(\alpha)}{2(d-c)^\alpha} I^{\alpha:\psi}_{\psi^{-1}(d)^-}(g \circ \psi)\bigl(\psi^{-1}(c)\bigr).$$

Note that

$$J_1 + J_2 + J_3 + J_4$$
$$= \frac{\Gamma(\alpha+1)}{2(d-c)^\alpha}\bigl[I^{\alpha:\psi}_{\psi^{-1}(c)^+}(g \circ \psi)\bigl(\psi^{-1}(d)\bigr) + I^{\alpha:\psi}_{\psi^{-1}(d)^-}(g \circ \psi)\bigl(\psi^{-1}(c)\bigr)\bigr] - g\left(\frac{c+d}{2}\right).$$

The proof is complete. □

Example 3.3 Let $a = 1$, $b = 2$, $\alpha = \frac{1}{2}$, $f(x) = x^2$, $\psi(x) = x$. Then all the assumptions in Theorem 2.1 are satisfied. Clearly, $f(\frac{a+b}{2}) = \frac{9}{4}$ and

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\bigl[I^{\alpha:\psi}_{\psi^{-1}(a)^+}(f \circ \psi)\bigl(\psi^{-1}(b)\bigr) + I^{\alpha:\psi}_{\psi^{-1}(b)^-}(f \circ \psi)\bigl(\psi^{-1}(a)\bigr)\bigr]$$

$$\begin{aligned}
&= \frac{\Gamma(\frac{3}{2})}{2}\left[\frac{1}{\Gamma(\frac{1}{2})}\int_1^2 (2-t)^{-\frac{1}{2}}t^2\,dt + \frac{1}{\Gamma(\frac{1}{2})}\int_1^2 (t-1)^{-\frac{1}{2}}t^2\,dt\right] \\
&= \frac{71}{30},
\end{aligned}$$

and then the left-hand side term of (7) $\Longleftrightarrow \frac{71}{30} - \frac{9}{4} = \frac{7}{60}$.

On the other hand,

$$\int_{\psi^{-1}(a)}^{\psi^{-1}(b)} k\bigl(f' \circ \psi\bigr)(\nu)\psi'(\nu)\,d\nu = \int_1^2 k2\nu\,d\nu = \frac{1}{4}$$

and k is defined in (8). Next,

$$\begin{aligned}
&\frac{1}{2(b-a)^\alpha}\int_{\psi^{-1}(a)}^{\psi^{-1}(b)} \bigl[\bigl(b-\psi(\nu)\bigr)^\alpha - \bigl(\psi(\nu)-a\bigr)^\alpha\bigr]\bigl(f' \circ \psi\bigr)(\nu)\psi'(\nu)\,d\nu \\
&= -\frac{1}{2}\int_1^2 (\sqrt{\nu-1} - \sqrt{2-\nu})2\nu\,d\nu \\
&= -\frac{4}{5} - \frac{2}{3} = -\frac{2}{15},
\end{aligned}$$

and then the right-hand side term of (7) $\Longleftrightarrow \frac{1}{4} - \frac{2}{15} = \frac{7}{60}$.

Theorem 3.4 *Let $c < d$ and $g : [c,d] \to R$ be a differentiable mapping on (c,d). Also suppose that $|g'|$ is convex on $[c,d]$, $\psi(x)$ is a positive monotone function increasing on $(c,d]$, having a continuous derivative $\psi'(x)$ on (c,d) and $\alpha \in (0,1)$. Then the following inequality for fractional integrals holds*:

$$\begin{aligned}
&\left|\frac{g(c)+g(d)}{2} - \frac{\Gamma(\alpha+1)}{2(d-c)^\alpha}\bigl[I^{\alpha:\psi}_{\psi^{-1}(c)^+}(g \circ \psi)\bigl(\psi^{-1}(d)\bigr) + I^{\alpha:\psi}_{\psi^{-1}(d)^-}(g \circ \psi)\bigl(\psi^{-1}(c)\bigr)\bigr]\right| \\
&\le \frac{d-c}{2(\alpha+1)}\left(1 - \frac{1}{2^\alpha}\right)\bigl[\bigl|g'(c)\bigr| + \bigl|g'(d)\bigr|\bigr].
\end{aligned}$$

Proof For every $\nu \in (\psi^{-1}(c), \psi^{-1}(d))$, we have $c < \psi(\nu) < d$. Let $t = \frac{d-\psi(\nu)}{d-c}$, and then $\psi(\nu) = ct + (1-t)d$. Using Lemma 3.1 and the convexity of $|g'|$, we obtain

$$\begin{aligned}
&\left|\frac{g(c)+g(d)}{2} - \frac{\Gamma(\alpha+1)}{2(d-c)^\alpha}\bigl[I^{\alpha:\psi}_{\psi^{-1}(c)^+}(g \circ \psi)\bigl(\psi^{-1}(d)\bigr) + I^{\alpha:\psi}_{\psi^{-1}(d)^-}(g \circ \psi)\bigl(\psi^{-1}(c)\bigr)\bigr]\right| \\
&\le \frac{1}{2(d-c)^\alpha}\int_{\psi^{-1}(c)}^{\psi^{-1}(d)} \bigl|\bigl(\psi(\nu)-c\bigr)^\alpha - \bigl(d-\psi(\nu)\bigr)^\alpha\bigr|\bigl|\bigl(g' \circ \psi\bigr)(\nu)\bigr|\,d\psi(\nu) \\
&= \frac{d-c}{2}\int_0^1 \bigl|(1-t)^\alpha - t^\alpha\bigr|\bigl|g'\bigl(tc+(1-t)d\bigr)\bigr|\,dt \\
&\le \frac{d-c}{2}\int_0^1 \bigl|(1-t)^\alpha - t^\alpha\bigr|\bigl[t\bigl|g'(c)\bigr| + (1-t)\bigl|g'(d)\bigr|\bigr]\,dt \\
&:= \frac{d-c}{2}(T_1 + T_2),
\end{aligned}$$

where

$$T_1 := \int_0^{\frac{1}{2}} \big[(1-t)^\alpha - t^\alpha\big]\big[t\big|g'(c)\big| + (1-t)\big|g'(d)\big|\big]\,dt,$$
$$T_2 := \int_{\frac{1}{2}}^1 \big[t^\alpha - (1-t)^\alpha\big]\big[t\big|g'(c)\big| + (1-t)\big|g'(d)\big|\big]\,dt.$$

Note

$$\begin{aligned} T_1 &= \big|g'(c)\big|\bigg[\int_0^{\frac{1}{2}} t(1-t)^\alpha\,dt - \int_0^{\frac{1}{2}} t^{\alpha+1}\,dt\bigg] \\ &\quad + \big|g'(d)\big|\bigg[\int_0^{\frac{1}{2}} (1-t)^{\alpha+1}\,dt - \int_0^{\frac{1}{2}} (1-t)t^\alpha\,dt\bigg] \\ &= \big|g'(c)\big|\bigg[\frac{1}{(\alpha+1)(\alpha+2)} - \frac{(\frac{1}{2})^{\alpha+1}}{\alpha+1}\bigg] + \big|g'(d)\big|\bigg[\frac{1}{(\alpha+2)} - \frac{(\frac{1}{2})^{\alpha+1}}{\alpha+1}\bigg], \end{aligned}$$

and

$$\begin{aligned} T_2 &= \big|g'(c)\big|\bigg[\int_{\frac{1}{2}}^1 t^{\alpha+1}\,dt - \int_{\frac{1}{2}}^1 t(1-t)^\alpha\,dt\bigg] + \big|g'(d)\big|\bigg[\int_{\frac{1}{2}}^1 (1-t)t^\alpha\,dt - \int_{\frac{1}{2}}^1 (1-t)^{\alpha+1}\,dt\bigg] \\ &= \big|g'(c)\big|\bigg[\frac{1}{(\alpha+2)} - \frac{(\frac{1}{2})^{\alpha+1}}{\alpha+1}\bigg] + \big|g'(d)\big|\bigg[\frac{1}{(\alpha+1)(\alpha+2)} - \frac{(\frac{1}{2})^{\alpha+1}}{\alpha+1}\bigg]. \end{aligned}$$

The proof is complete. □

Theorem 3.5 *Let $g : [c,d] \to R$ be a differentiable mapping on (c,d) with $c < d$. Also suppose that $|g'|$ is convex on $[c,d]$, $\psi(x)$ is an increasing and positive monotone function on $(c,d]$, having a continuous derivative $\psi'(x)$ on (c,d) and $\alpha \in (0,1)$. Then the following inequality for fractional integrals holds*:

$$\begin{aligned} &\bigg|\frac{\Gamma(\alpha+1)}{2(d-c)^\alpha}\big[I^{\alpha:\psi}_{\psi^{-1}(c)^+}(g\circ\psi)\big(\psi^{-1}(d)\big) + I^{\alpha:\psi}_{\psi^{-1}(d)^-}(g\circ\psi)\big(\psi^{-1}(c)\big)\big] - g\bigg(\frac{c+d}{2}\bigg)\bigg| \\ &\quad \le \frac{|g(d)-g(c)|}{2} + \frac{d-c}{2(\alpha+1)}\bigg(1-\frac{1}{2^\alpha}\bigg)\big[\big|g'(c)\big| + \big|g'(d)\big|\big]. \end{aligned} \tag{9}$$

Proof Using Lemma 3.2 and the convexity of $|g'|$, we obtain

$$\begin{aligned} &\bigg|\frac{\Gamma(\alpha+1)}{2(d-c)^\alpha}\big[I^{\alpha:\psi}_{\psi^{-1}(d)^+}(g\circ\psi)\big(\psi^{-1}(d)\big) + I^{\alpha:\psi}_{\psi^{-1}(d)^-}(g\circ\psi)\big(\psi^{-1}(c)\big)\big] - g\bigg(\frac{c+d}{2}\bigg)\bigg| \\ &\quad = \bigg|\int_{\psi^{-1}(c)}^{\psi^{-1}(d)} k\big(g'\circ\psi\big)(v)\psi'(v)\,dv \\ &\qquad + \frac{1}{2(d-c)^\alpha}\int_{\psi^{-1}(c)}^{\psi^{-1}(d)} \big[\big(\big(d-\psi(v)\big)^\alpha - \big(\psi(v)-c\big)^\alpha\big]\big(g'\circ\psi\big)(v)\psi'(v)\,dv\bigg| \\ &\quad \le \bigg|\int_{\psi^{-1}(c)}^{\psi^{-1}(d)} k\big(g'\circ\psi\big)(v)\psi'(v)\,dv\bigg| \\ &\qquad + \bigg|\frac{1}{2(d-c)^\alpha}\int_{\psi^{-1}(c)}^{\psi^{-1}(d)} \big[\big(\big(d-\psi(v)\big)^\alpha - \big(\psi(v)-c\big)^\alpha\big]\big(g'\circ\psi\big)(v)\psi'(v)\,dv\bigg| \\ &\quad := K_1 + K_2, \end{aligned} \tag{10}$$

where

$$K_1 := \left| \int_{\psi^{-1}(c)}^{\psi^{-1}(d)} k\big(g' \circ \psi\big)(v)\psi'(v)\,dv \right|,$$

$$K_2 := \left| \frac{1}{2(d-c)^\alpha} \int_{\psi^{-1}(c)}^{\psi^{-1}(d)} \big[\big(\big(d-\psi(v)\big)^\alpha - \big(\psi(v)-c\big)^\alpha\big]\big(g' \circ \psi\big)(v)\psi'(v)\,dv \right|,$$

and k is defined in (8).

From Theorem 3.4,

$$K_2 \le \frac{d-c}{2(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\big[\big|g'(c)\big| + \big|g'(d)\big|\big]. \tag{11}$$

Also we easily obtain

$$K_1 = \frac{|g(d)-g(c)|}{2}. \tag{12}$$

Then put (11) and (12) in (10), and we obtain inequality (9). This completes the proof. □

4 Examples

We consider the following special means for arbitrary real numbers α, β, $\alpha \neq \beta$:

$$H(\alpha,\beta) = \frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}, \quad \alpha,\beta \in R \setminus \{0\},$$

$$A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \quad \alpha,\beta \in R,$$

$$L(\alpha,\beta) = \frac{\beta-\alpha}{\ln|\beta| - \ln|\alpha|}, \quad |\alpha| \neq |\beta|, \alpha\beta \neq 0,$$

$$L_n(\alpha,\beta) = \left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}}, \quad n \in Z \setminus \{-1,0\}, \alpha,\beta \in R, \alpha \neq \beta.$$

Now, using the results in Sect. 3, we have some applications to the special means of real numbers.

Proposition 4.1 *Let $a, b \in R^+$, $a < b$. Then*

$$\big|A\big(a^2,b^2\big) - L_2^2(a,b)\big| \le \frac{b^2-a^2}{4}.$$

Proof Apply Theorem 3.4 with $f(x) = x^2$, $\psi(x) = x$, $\alpha = 1$, and we obtain the result immediately. □

Let $f(x) = x^n$, $\psi(x) = x$, $\alpha = 1$, $a, b \in R^+$, $a < b$. Then we have the general result

$$\big|A\big(a^n,b^n\big) - L_n^n(a,b)\big| \le \frac{b-a}{8}\big(na^{n-1} + nb^{n-1}\big).$$

Proposition 4.2

$$\left|A\left(e^a, e^b\right) - L\left(e^a, e^b\right)\right| \leq \frac{b-a}{8}\left(e^a + e^b\right).$$

Proof Apply Theorem 3.4 with $f(x) = e^x$, $\psi(x) = x$, $\alpha = 1$, $a, b \in R^+$, $a < b$. Then we obtain the result immediately. □

Proposition 4.3

$$\left|H^{-1}(a,b) - L^{-1}(a,b)\right| \leq \frac{b-a}{8}\left(\frac{1}{a^2} + \frac{1}{b^2}\right).$$

Proof Apply Theorem 3.4 with $f(x) = \frac{1}{x}$, $\psi(x) = x$, $\alpha = 1$, $a, b \in R^+$, $a < b$. Then we obtain the result immediately. □

Proposition 4.4

$$\left|L^{-1}(a,b) - A^{-1}(a,b)\right| \leq \frac{b-a}{8}\left(4 + \frac{1}{a^2} + \frac{1}{b^2}\right).$$

Proof Apply Theorem 3.5 with $f(x) = \frac{1}{x}$, $\psi(x) = x$, $\alpha = 1$, $a, b \in R^+$, $a < b$. Then we obtain the result immediately. □

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

Author details

[1] College of Science, Guizhou Institute of Technology, Guiyang, China. [2] Department of Mathematics, Guizhou University, Guiyang, China. [3] School of Mathematical Sciences, Qufu Normal University, Qufu, China. [4] School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland.

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