ON THE HESSIAN OF A FUNCTION AND THE CURVATURES OF ITS GRAPH

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INTRODUCTION

In a well-known paper [3], S. S. Chern constructs certain complicated integrands, denoted by B_{m-h} (see [3, p. 84]), on a nonparametric hypersurface $x_{m+1} = f(x_1, \cdots, x_m)$ in $\mathbb{R}^{m+1}$. The purpose of this note is to interpret these forms in two ways. First, we show that they are closely related to the elementary invariants of the Hessian matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$. This is proved in two cases by Chern in [3] and by H. Flanders in [4]. For our second interpretation, valid in the cases when h is even, we require a concept of a Ricci tensor of order q in the theory of q-sectional curvature of J. A. Thorpe [10].

Several other results are scattered through the note. For example, we give a formula for the kth mean curvature function of a nonparametric hypersurface; it generalizes the well-known formula

$$m\sigma_1 = \sum_{j=1}^{m} \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_j}\Big/\left(1 + \sum_{k=1}^{m}\left(\frac{\partial f}{\partial x_k}\right)^2\right)^{1/2}\right).$$

1. THE INVARIANTS OF A SYMMETRIC TRANSFORMATION

In this section, we review some elementary facts. Let us consider a symmetric linear transformation A: V → V, where V is an m-dimensional inner-product space. We denote the eigenvalues of A by $\lambda_1, \lambda_2, \cdots, \lambda_m$.

1.1 *Definition.* If $0 \leq q \leq m$, then *the* q-*th invariant* $S_q(A)$ *of* A is the qth elementary symmetric function of the numbers $\lambda_1, \cdots, \lambda_m$. That is,

$$S_q(A) = \sum_{1 \leq i_1 < \cdots < i_q \leq m} \lambda_{i_1} \cdots \lambda_{i_q}.$$

Furthermore, the q*th Newton transformation* $T_q(A)$ associated with A is

$$T_q(A) = S_q(A)I - S_{q-1}(A) \cdot A + \cdots + (-1)^q A^q.$$

For the sake of convenience, we gather the principal facts concerning $S_q(A)$ and $T_q(A)$ into a single proposition.

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Notation. For $1 \le q \le m$ and $1 \le i_1, \cdots, i_q, j_1, \cdots, j_q \le m$, the Kronecker symbol $\delta\begin{pmatrix} i_1 \cdots i_q \\ j_1 \cdots j_q \end{pmatrix}$ has the value $+1$ (respectively, -1) if $i_1, \cdots, i_q$ are distinct and $(j_1, \cdot, j_q)$ is an even permutation (respectively, an odd permutation) of $(i_1, \cdot, i_q)$. Otherwise, it has the value 0.

1.2 PROPOSITION. *Suppose that, relative to some basis of* V, *the transformation* A *has matrix* (A_i^j). *Then*

(a) $S_q(A) = \frac{1}{q!} \sum \delta\begin{pmatrix} i_1 \cdots i_q \\ j_1 \cdots j_q \end{pmatrix} A_{i_1}^{j_1} \cdots A_{i_q}^{j_q}$,

(b) $T_q(A)_j^i = \frac{1}{q!} \sum \delta\begin{pmatrix} i_1 \cdots i_q\, i \\ j_1 \cdots j_q\, j \end{pmatrix} A_{i_1}^{j_1} \cdots A_{i_q}^{j_q}$,

(c) $\operatorname{Trace}(T_q(A) \cdot A) = (q+1)\, S_{q+1}(A)$,

(d) $T_q(A) = S_q(A) I - T_{q-1}(A) \cdot A$,

(e) $\operatorname{Trace} T_q(A) = (m-q)\, S_q(A)$.

Proof. First observe that both sides of the equation transform similarly under a change of basis in (a) and (b). Thus (a) and (b) follow easily if we choose an orthonormal basis of eigenvectors of A. Now (c) follows from (a) and (b), the relation (d) is trivial, and (e) follows from (c) and (d). ∎

1.3 *Remark.* We shall encounter "mixed invariants" and "mixed Newton tensors" once or twice in this note. We get these objects by partially polarizing the homogeneous polynomials $S_q(A)$ and $T_q(A)$. Precisely, if $0 \le r \le q \le m$ and A and B are symmetric transformations of V, we set

$$S_{qr}(A, B) = \frac{1}{q!} \sum \delta\begin{pmatrix} i_1 \cdots i_q \\ j_1 \cdots j_q \end{pmatrix} A_{i_1}^{j_1} \cdots A_{i_r}^{j_r} B_{i_{r+1}}^{j_{r+1}} \cdots B_{i_q}^{j_q}$$

and

$$T_{qr}(A, B)_j^i = \frac{1}{q!} \sum \delta\begin{pmatrix} i_1 \cdots i_q\, i \\ j_1 \cdots j_q\, j \end{pmatrix} A_{i_1}^{j_1} \cdots A_{i_r}^{j_r} B_{i_{r+1}}^{j_{r+1}} \cdots B_{i_q}^{j_q}.$$

Clearly, we can extend Proposition 1.2 (c) to the following result.

1.2 PROPOSITION (c'). $\operatorname{Trace}(T_{qr}(A, B) \cdot A) = (q+1)\, S_{q+1,r+1}(A, B)$ *and* $\operatorname{Trace}(T_{qr}(A, B) \cdot B) = (q+1)\, S_{q+1,r}(A, B)$.

2. THE HESSIAN MATRIX

Notation. Throughout the rest of this paper, f will denote a real-valued function of class C^3 defined on the closure $\overline{\mathscr{D}}$ of a domain $\mathscr{D} \subset \mathbb{R}^m$. We assume that $\overline{\mathscr{D}}$ is compact and that the boundary $\partial\mathscr{D} = \overline{\mathscr{D}} \sim \mathscr{D}$ is a smooth closed hypersurface embedded in $\mathbb{R}^m$. We denote the partial derivatives $\frac{\partial f}{\partial x_j}, \frac{\partial^2 f}{\partial x_j \partial x_k}, \cdots$, by $f_j, f_{jk}, \cdots$. We treat the Hessian matrix $H(f) = (f_{jk})$ as a field of symmetric linear transformations in $\mathbb{R}^m$, but we write $S_q(f)$ and $T_q(f)$ instead of the more proper $S_q(H(f))$ and $T_q(H(f))$. Finally, we set $W = \sqrt{1 + \sum_k f_k^2}$.

Remarks. (a) Observe that, by (a) and (b) of Proposition 1.2, $S_q(f)$ and $T_q(f)$ are C^1 on $\overline{\mathscr{D}}$.

(b) From now on our index notation will not always agree with classical tensor notation. However, repeated indices will normally still be summed.

Let us consider a pair of smooth functions f and g on $\overline{\mathscr{D}}$. The mixed Newton tensors $T_{qr}(f, g)$ $(= T_{qr}(H(f), H(g)))$ enjoy a pleasant differentiation property.

2.1 PROPOSITION. *For all* r *and* q $(0 \le r \le q \le m)$, $\operatorname{div}(T_{qr}(f, g)) = 0$. *That is*, $\sum_i \frac{\partial}{\partial x_i}(T_{qr}(f, g)^i_j) = 0$ $(j = 1, \cdots, m)$.

Proof. We have the relations

$$\sum_i \frac{\partial}{\partial x_i}(T_{qr}(f, g)^i_j) = \sum_i \frac{\partial}{\partial x_i}\frac{1}{q!}\sum \delta\begin{pmatrix} i_1 \cdots i_q\, i \\ j_1 \cdots j_q\, j \end{pmatrix} f_{i_1 j_1} \cdots f_{i_r j_r} g_{i_{r+1} j_{r+1}} \cdots g_{i_q j_q}$$

$$= \sum_i \frac{1}{q!}\sum \delta\begin{pmatrix} i_1 \cdots i_q\, i \\ j_1 \cdots j_q\, j \end{pmatrix}$$

$$\cdot (r f_{i_1 j_1} \cdots f_{i_r j_r i} g_{i_{r+1} j_{r+1}} \cdots g_{i_q j_q} + (q - r) f_{i_1 j_1} \cdots f_{i_r j_r} g_{i_{r+1} j_{r+1}} \cdots g_{i_q j_q i}).$$

Now $f_{i_r j_r i}$ and $g_{i_q j_q i}$ are symmetric in $i_r i$ and $i_q i$, respectively, while the Kronecker symbols are skew-symmetric in those indices. Thus the sums over $i_r i$ and $i_q i$ vanish. ∎

2.2 COROLLARY. *Suppose that* f *and* g *are smooth functions on* $\overline{\mathscr{D}}$ *such that* grad f = grad g *on the boundary* $\partial\mathscr{D}$. *Then*

$$\int_{\mathscr{D}} S_{q+1}(f)\, dx_1 \cdots dx_m = \int_{\mathscr{D}} S_{q+1}(g)\, dx_1 \cdots dx_m .$$

Proof. Consider the vector field $Y = (Y_1, \cdots, Y_m)$ on $\overline{\mathscr{D}}$, where

$$Y_i = \sum_{j=1}^{m} \sum_{r=0}^{q} T_{qr}(f, g)^i_j (f_j - g_j).$$

Then Proposition 2.1 and Proposition 1.1(c') imply that

$$\operatorname{div} Y = \sum_i \frac{\partial}{\partial x_i} Y_i = \sum_{ij} \sum_r T_{qr}(f, g)^i_j (f_{ij} - g_{ij})$$

$$= \sum_r ((q + 1) S_{q+1, r+1}(f, g) - (q + 1) S_{q+1, r}(f, g))$$

$$= (q+1)S_{q+1,q+1}(f,g) - (q+1)S_{q+1,0}(f,g) = (q+1)S_{q+1}(f) - (q+1)S_{q+1}(g)\ .$$

Now apply Stokes' theorem to $\int_{\mathscr{D}} ((q+1)S_{q+1}(f) - (q+1)S_{q+1}(g))\, dx_1 \cdots dx_m$ and use the hypothesis that $Y = 0$ on $\partial\mathscr{D}$. ∎

2.3 *Remarks.* (a) By setting $g = 0$ in the proof of Corollary 2.2, we get the general formula

$$\int_{\mathscr{D}} (q+1)S_{q+1}(f)\, dx_1 \cdots dx_m = \int_{\partial\mathscr{D}} \sum_{ij} T_q(f)^i_j (f_j t_i)\, dA\ ,$$

where $t = (t_1 \cdots t_m)$ is the outward unit normal to $\partial\mathscr{D}$ and dA is the volume element for the hypersurface $\partial\mathscr{D}$.

(b) By breaking up $\operatorname{grad}(f-g)|_{\partial\mathscr{D}}$ into its components normal to $\partial\mathscr{D}$ and tangent to $\partial\mathscr{D}$, we see that the hypothesis $\operatorname{grad}(f-g)|_{\partial\mathscr{D}} = 0$ in Corollary 2.2 means $(f-g)|_{\partial\mathscr{D}} = \text{constant}$ and $\frac{\partial}{\partial n}(f-g) = 0$ on $\partial\mathscr{D}$ (where $\frac{\partial}{\partial n}$ denotes the outward normal derivative). Thus $\int_{\mathscr{D}} S_q(f)\, dx_1 \cdots dx_m$ depends only on the values of $f|_{\partial\mathscr{D}}$ and $\frac{\partial f}{\partial n}$ along $\partial\mathscr{D}$. We can easily make this dependence explicit. Indeed, let us set $z = f|_{\partial\mathscr{D}}$ and $z_m = \frac{\partial f}{\partial n}$ along $\partial\mathscr{D}$. It is a simple task to express the quantities f_{ij} and f_k appearing in the boundary integral in Remark (a) in terms of the components $z_{,\alpha}$, $z_{m,\alpha}$, and $z_{,\alpha\beta}$ of the covariant derivatives of z and z_m (relative to $\partial\mathscr{D}$) and the components $A_{\alpha\beta}$ of the second fundamental form of $\partial\mathscr{D}$ in $\mathbb{R}^m$. All these components are computed relative to an orthonormal frame field tangent to $\partial\mathscr{D}$. The indices $\alpha, \beta, \cdots$ run from 1 to $m-1$. For the sake of completeness, we state the formula.

2.4 PROPOSITION. *Let* f, z, z_m, $z_{,\alpha\beta}$, $z_{m,\alpha}$ *have the same meaning as in Remark* 2.3 (b) *above. Set*

$$\tilde{S}_q(z) = \frac{1}{q!} \sum \delta\begin{pmatrix}\alpha_1 & \cdots & \alpha_q\\ \beta_1 & \cdots & \beta_q\end{pmatrix} (z_{,\alpha_1\beta_1} - z_m A_{\alpha_1\beta_1}) \cdots (z_{,\alpha_q\beta_q} - z_m A_{\alpha_q\beta_q})$$

and

$$\tilde{T}_q(z)_{\alpha\beta} = \frac{1}{q!} \sum \delta\begin{pmatrix}\alpha_1 & \cdots & \alpha_q & \alpha\\ \beta_1 & \cdots & \beta_q & \beta\end{pmatrix} (z_{,\alpha_1\beta_1} - z_m A_{\alpha_1\beta_1}) \cdots (z_{,\alpha_q\beta_q} - z_m A_{\alpha_q\beta_q})\ .$$

Then

$$\int_{\mathscr{D}} (q+1)S_{q+1}(f)\, dx_1 \cdots dx_m = \int_{\partial\mathscr{D}} \Big(\tilde{S}_q(z) z_m - \sum_{\alpha\beta} \tilde{T}_{q-1}(z)_{\alpha\beta}\, z_{,\alpha} z_{m,\beta}\Big)\, dA\ .$$

2.5 COROLLARY. *Suppose that* $z = f|_{\partial\mathscr{D}}$ *is constant on* $\partial\mathscr{D}$. *Then*

$$\int_{\mathscr{D}} (q+1)\, S_{q+1}(f)\, dx_1 \cdots dx_m = (-1)^q \binom{m-1}{q} \int_{\partial\mathscr{D}} z_m^{q+1} \sigma_q \, dA ,$$

where σ_q is the qth mean curvature function on $\partial\mathscr{D}$. In particular, if $q+1$ is even and $\partial\mathscr{D}$ is strictly convex, then $\int_{\mathscr{D}} S_{q+1}(f)\, dx_1 \cdots dx_m \geq 0$, with equality if and only if $z_m \equiv 0$.

Proof of Corollary. If z = constant, then all terms involving the derivatives $z_{,\alpha}$ and $z_{,\alpha\beta}$ in $\tilde{S}_q$ and $\tilde{T}_q$ will vanish. What remains, namely the expression

$$\frac{1}{q!} \sum \delta \begin{pmatrix} \alpha_1 & \cdots & \alpha_q \\ \beta_1 & \cdots & \beta_q \end{pmatrix} (-z_m A_{\alpha_1\beta_1})(-z_m A_{\alpha_2\beta_2}) \cdots (-z_m A_{\alpha_q\beta_q})$$

in $\tilde{S}_q(z)$, multiplied by z_m, yields the formula. As for the inequality, if $q+1$ is even, then $z_m^{q+1} \geq 0$, $(-1)^q = -1$, and (since $\partial\mathscr{D}$ is strictly convex) $\sigma_q < 0$. The inequality now follows. ■

Remark. S. Bernstein [2] proved the corollary in the special case where $m = 2$, $q+1 = 2$, and $\mathscr{D}$ is the ball of radius R.

3. THE CURVATURES OF A GRAPH

Suppose that M is the graph $x_{m+1} = f(x_1 \cdots x_m)$. We denote the natural coordinatization of M by

$$X(x_1 \cdots x_m) = (x_1, \cdots, x_m, f(x_1 \cdots x_m)) = \sum_{j=1}^{m} x_j A_j + f(x_1 \cdots x_m) A ,$$

where $A_j = (0, \cdots, 0, 1, \cdots, 0)$ (1 in the jth place) and $A = (0, \cdots, 0, 1)$. We recall the standard calculations, without proof, in a single proposition.

3.1 PROPOSITION. (a) *The natural frame field on* M *is* $X_1, \cdots, X_m$, *where* $X_i = \frac{\partial X}{\partial x_i} = A_i + f_i A$, *and the unit normal is*

$$N = \sum_{j=1}^{m} -\frac{f_j}{W} A_j + \frac{1}{W} A .$$

(b) *The matrix of the first fundamental form* I *of the induced metric on* M (*relative to the natural frame*) *is* $g_{ij} = \langle X_i, X_j \rangle = \delta_{ij} + f_i f_j$.

(c) *The volume element on* M *is*

$$dV = \sqrt{\det g_{ij}}\, dx_1 \wedge \cdots \wedge dx_m = W\, dx_1 \wedge \cdots \wedge dx_m .$$

(d) *The inverse matrix* (g^{ij}) *of* (g_{ij}) *is* $g^{ij} = \delta_{ij} - \frac{f_i f_j}{W^2}$.

(e) *The matrix of the second fundamental form* II *on* M *is* $b_{ij} = \frac{\partial^2 X}{\partial x_i \partial x_j} \cdot N = \frac{f_{ij}}{W}$.

(f) *The matrix of the shape operator* B *is* $b_i^j = \sum_k b_{ik} g^{kj} = \frac{f_{ij}}{W} - \sum_k \frac{f_{ik} f_k f_j}{W^3}$.

3.2 *Remark.* The concepts from Section 1 are quite familiar in the context of the field B of symmetric linear transformations of the tangent spaces of M. Thus $S_q(B) = \binom{m}{q} \sigma_q$, where σ_q is the qth mean curvature function for M. Similarly, the Newton tensors $T_q(B)$ have also been studied, for example in [6], [8], and [9]. We shall generally write $S_q(M)$ and $T_q(M)$, or, if no confusion can arise, simply S_q and T_q, in place of $S_q(B)$ and $T_q(B)$.

In [6] we proved that $\operatorname{div}_M T_q(B) = 0$. Because our proof in [6] was quite awkward, we reprove the result here.

3.3 PROPOSITION. *If* div_M *denotes the divergence operator in the induced metric on* M, *then* $\operatorname{div}_M T_q(M) = 0$; *that is,* $\sum_i T^i_{qj,i} = 0$ (*where the comma denotes covariant differentiation*).

Proof. Use a proof of the same style as that in Proposition 2.1, with the Codazzi equations $b^j_{i,k} = b^j_{k,i}$ replacing the symmetry $f_{ijk} = f_{ikj}$.

3.4 *Remark.* If q is even, then $S_q(M)$ and $T_q(M)$ actually depend only on the induced metric on M. In fact, if we use the Gauss curvature equations $R^{jk}_{hi} = b^j_i b^k_h - b^j_h b^k_i$ together with the skew-symmetries of the Kronecker symbols, we can easily verify that

$$S_q(M) = \left(-\frac{1}{2}\right)^{q/2} \frac{1}{q!} \sum \delta \begin{pmatrix} i_1 & \cdots & i_q \\ j_1 & \cdots & j_q \end{pmatrix} R^{j_1 j_2}_{i_1 i_2} \cdots R^{j_{q-1} j_q}_{i_{q-1} i_q}$$

and

$$T_q(M)^i_j = \left(-\frac{1}{2}\right)^{q/2} \frac{1}{q!} \sum \delta \begin{pmatrix} i_1 & \cdots & i_q & i \\ j_1 & \cdots & j_q & j \end{pmatrix} R^{j_1 j_2}_{i_1 i_2} \cdots R^{j_{q-1} j_q}_{i_{q-1} i_q}.$$

In particular, $-T_2$ is the classical Einstein tensor, and if q is even, then up to a constant factor $T_q(M)$ is the qth generalized Einstein tensor of D. Lovelock [5].

Now we relate $S_q(f)$ to $S_q(M)$ and $T_{q-1}(M)$. Let A^T denote the vector field on M obtained by projecting the unit vector $A = (0, 0, \cdots, 0, 1)$ orthogonally onto M. For later use, we state the following result.

3.5 PROPOSITION. (a) $A^T = \sum_j \lambda^j X_j$, *where* $\lambda^j = f_j / W^2$,

(b) $\operatorname{div}_M (T_q(M)(A^T)) = (q+1) S_{q+1}(M) \cdot \langle A, N \rangle = (q+1) S_{q+1}(M) \cdot (1/W)$.

Proof. (a) Clearly, $f_k = \langle A, X_k \rangle = \langle A^T, X_k \rangle = \sum_j \lambda^j g_{jk}$. Thus

$$\lambda^i = \sum_k f_k g^{kj} = \sum_k f_k (\delta_{kj} - f_k f_j / W^2)$$

$$= f_j - \left(\sum_k f_k^2\right) f_j / W^2 = f_j (1 - (W^2 - 1)/W^2) = f_j / W^2.$$

(b) Use Proposition 3.3 (see [6]).

3.6 THEOREM. $S_q(f) = W^q(S_q(M) + W^2 \langle (T_{q-1}(M) \cdot B)(A^T), A^T \rangle)$.

Proof. We know that

$$S_q(f) = \frac{1}{q!}\sum \delta\begin{pmatrix} i_1 \cdots i_q \\ j_1 \cdots j_q \end{pmatrix} f_{i_1j_1} \cdots f_{i_qj_q} = W^q \cdot \frac{1}{q!}\sum \delta\begin{pmatrix} i_1 \cdots i_q \\ j_1 \cdots j_q \end{pmatrix} b_{i_1j_1} \cdots b_{i_qj_q}$$

$$= W^q \cdot \frac{1}{q!}\sum \delta\begin{pmatrix} i_1 \cdots i_q \\ j_1 \cdots j_q \end{pmatrix} b_{i_1}^{k_1} \cdots b_{i_q}^{k_q} g_{k_1j_1} \cdots g_{k_qj_q}$$

$$= W^q \frac{1}{q!}\sum \delta\begin{pmatrix} i_1 \cdots i_q \\ j_1 \cdots j_q \end{pmatrix} b_{i_1}^{k_1} \cdots b_{i_q}^{k_q} (\delta_{k_1j_1} + f_{k_1}f_{j_1}) \cdots (\delta_{k_qj_q} + f_{k_q}f_{j_q})$$

$$= W^q S_q(M) + q \cdot W^q \cdot \frac{1}{q!}\sum \delta\begin{pmatrix} i_1 \cdots i_q \\ j_1 \cdots j_q \end{pmatrix} b_{i_1}^{k} f_k f_{j_1} b_{i_2}^{j_2} \cdots b_{i_q}^{j_q}$$

$$= W^q \left(S_q(M) + \sum T_{q-1}(M)_j^i\, b_i^k f_k f_j \right).$$

By the proof of Proposition 3.5 (a), we can write $f_k = \sum_\ell \lambda^\ell g_{\ell k}$, while $f_j = W^2\lambda^j$ $\left(A^T = \sum_j \lambda^j X_j\right)$. Thus, $S_q(f) = W^q(S_q(M) + \langle (T_{q-1}(M) \cdot B)(A^T), A^T \rangle)$. ∎

3.7 *Remarks.* (a) It is easy to verify that the form B_q, introduced by Chern on p. 84 of [3], can be identified with

$$(-1)^q q!\,(m-q)!\,W^{q+1}(S_q(M) \cdot W^{-2} + \langle (T_{q-1}(M) \cdot B)(A^T), A^T \rangle)\, dV$$

$$= (-1)^q q!\,(m-q)!\,W^q(S_q(M) + W^2 \langle (T_{q-1}(M) \cdot B)(A^T), A^T \rangle)\, dx_1 \wedge \cdots \wedge dx_m$$

(since $dV = W\,dx_1 \wedge \cdots \wedge dx_m$). Thus, by Theorem 3.6,

$$B_q = (-1)^q q!\,(m-q)!\,S_q(f)\, dx_1 \wedge \cdots \wedge dx_m .$$

(b) We can now see that bounds on the curvatures of the nonparametric hypersurface M imply bounds on the invariants of the Hessian. Suppose, for example, that there exists a positive constant C such that no eigenvalue of $T_{q-1}(M) \cdot B$ is less than C. Then

$$S_q(M) = \frac{1}{q}\,\mathrm{Trace}\,(T_{q-1}(M) \cdot B) \geq \frac{m}{q} C$$

and

$$\langle (T_{q-1}(M)B)(A^T), A^T \rangle \geq C\,|A^T|^2 .$$

Thus, using Theorem 3.6, we see that

$$\frac{S_q(f)}{W^{q+2}} \geq \frac{m}{q} C \langle A, N \rangle^2 + C\,|A^T|^2 \geq C(\langle A, N \rangle^2 + |A^T|^2) = C$$

(since $\langle A, N \rangle = 1/W$ and $|A| = 1$). Similarly, if no eigenvalue of $T_{q-1}(M) \cdot B$ is greater than $-C$, we see that $S_q(f)/W^{q+2} \leq -C$.

(c) Suppose now that $\mathscr{D}$ is a disc of radius R. In Theorem 4 of [3], Chern proves that a bound on the eigenvalues of $T_1(M) \cdot B$ (as in Remark (b)) determines an upper bound for R. Similarly, in [4] Flanders proves that a condition of the form $\left|\frac{S_2(f)}{W^{2+\varepsilon}}\right| \geq C > 0$ places an upper bound on R. Thus, by our remark (b), we see that the result of Flanders implies Chern's result. More importantly, an extension of Flanders' analytical result to inequalities such as $\left|\frac{S_q(f)}{W^{q+2}}\right| \geq C > 0$ would automatically extend Chern's geometric theorem.

(d) By Proposition 1.2 (d) and Remark 3.4, we see that if q is even, then $T_{q-1}(M) \cdot B$ depends only on the induced metric on M. It is easy to see that $T_1 B$ is simply the Ricci tensor on M. (Later, we shall see that in general $T_{q-1} \cdot B$ can be interpreted as a kind of Ricci tensor, if q is even.) Thus we can rephrase Chern's Theorem 4 [3] in a more striking form: if no eigenvalue of the Ricci tensor of M is greater than $-C$, and in addition the domain is a disc of radius R, then $R \leq K/\sqrt{C}$, where K is independent of C. In this formulation, Chern's Theorem is highly relevant to the following question: is it true that there is no complete hypersurface in $\mathbb{R}^{m+1}$ such that all the eigenvalues of the Ricci tensor are negative and remain uniformly bounded away from 0? The celebrated Hilbert-Efimov theorem provides an affirmative answer to this question in the case $m = 2$.

We now interpret the tensor $T_{q-1}(M) \cdot B$ (q even) in terms of the higher-order sectional curvatures (see [10]).

3.8 *Definition.* Let M be a Riemannian manifold of dimension $m \geq 2$. Let q be an even integer $(2 \leq q \leq m)$. If L is a q-dimensional subspace of the tangent space M_x at $x \in M$, and if $(e_1, \cdot, e_m)$ is an orthonormal frame at x with $e_1, \cdot, e_q \in L$, then we define the *q-sectional curvature* $\gamma(L)$ of M at L by

$$\gamma(L) = C_q \cdot \sum_1^q \delta\begin{pmatrix} i_1 & \cdots & i_q \\ j_1 & \cdots & j_q \end{pmatrix} R_{i_1 i_2}^{j_1 j_2} \cdots R_{i_{q-1} i_q}^{j_{q-1} j_q},$$

where $C_q = \left(-\frac{1}{2}\right)^{q/2} \cdot \frac{1}{q!}$. Notice that this expression is actually independent of the choice of frame, as long as $e_1, \cdot, e_q$ lie in L. We further define the *Ricci tensor of degree* q, denoted by R_q, to be the symmetric tensor of type (1, 1) such that if v is a unit tangent vector at $x \in M$ and $(e_1, \cdot, e_m)$ is an orthonormal basis at x such that $v = e_m$, then

$$\langle R_q(v), v \rangle = \sum_{1 \leq i_1 < \cdots < i_{q-1} \leq (m-1)} \gamma(e_{i_1} \wedge \cdots \wedge e_{i_{q-1}} \wedge v).$$

The independence of R_q from the choice of frame will become evident. Our construction generalizes the usual one for the classical Ricci tensor R_2.

It is easy to relate these generalized Ricci tensors to the generalized Einstein tensors of Lovelock. Indeed, let $(e_1, \cdot, e_m)$ be as above, with $e_m = v$. Then

$$\left\langle T_q(v),\, v\right\rangle = T^m_{qm} = C_q \sum_1^m \delta\begin{pmatrix} i_1 & \cdots & i_q & m \\ j_1 & \cdots & j_q & m \end{pmatrix} R^{j_1 j_2}_{i_1 i_2} \cdots R^{j_{q-1} j_q}_{i_{q-1} i_q}$$

$$= C_q \sum_1^m \delta\begin{pmatrix} i_1 & \cdots & i_q \\ j_1 & \cdots & j_q \end{pmatrix} R^{j_1 j_2}_{i_1 i_2} \cdots R^{j_{q-1} j_q}_{i_{q-1} i_q}$$

$$- \sum_1^{m}{}' \; C_q\, \delta\begin{pmatrix} i_1 & \cdots & i_q \\ j_1 & \cdots & j_q \end{pmatrix} R^{j_1 j_2}_{i_1 i_2} \cdots R^{j_{q-1} j_q}_{i_{q-1} i_q} .$$

Here $\sum'$ means "sum over values of the indices such that $m \in \{i_1 \cdots i_q\}$". Now the first sum of the right-hand side clearly gives us $S_q(M)$. The $\sum'$-sum can be decomposed conveniently into summands as follows. For each q-tuple $k_1 < k_2 < \cdots < k_{q-1} < k_q = m$, gather all the terms in the $\sum'$-sum such that $\{i_1 \cdots i_q\} = \{k_1 \cdots k_q\}$. These terms add up to $\gamma(e_{k_1} \wedge \cdots \wedge e_{k_{q-1}} \wedge v)$. Now sum over all such q-tuples $k_1 < k_2 < \cdots < k_q = m$. The total sum is $\left\langle R_q(v),\, v\right\rangle$. Thus we have proved the following result.

3.9 PROPOSITION. *If* M *is an* m*-dimensional Riemannian manifold and* q *is even* $(2 \le q \le m)$, *then* $T_q(M) = S_q(M)\,I - R_q(M)$.

3.10 COROLLARY. *Suppose that* M *is an Einstein manifold of degree* q *in the sense that* $R_q = \lambda I$. *Then* S_q *is constant if* $q < m$.

Proof. If $R_q = \lambda I$, then

$$\text{Trace } R_q = \text{Trace}\,(S_q I - T_q) = m S_q - (m - q) S_q = q S_q = m\lambda ,$$

so that $\lambda = \frac{q}{m} S_q$. Thus $T_q = (S_q - \lambda)I = S_q(1 - q/m)I$. Now the fact that $\text{div}_M T_q = 0$ implies that $\text{grad}_M(S_q(1 - q/m)) = 0$, in other words, that $S_q = \text{constant}$ if $q/m \ne 1$. ∎

3.11 COROLLARY. *If* M *is a nonparametric hypersurface and* q *is even, then* $T_{q-1}(M) \cdot B = R_q$.

4. MISCELLANEOUS RESULTS

Suppose that M is a nonparametric hypersurface, as before. We shall derive a formula in the coordinates $x_1, \cdots, x_m$ for the qth mean curvature. It will be in divergence form.

4.1 PROPOSITION. *If* M *is a nonparametric hypersurface, then*

$$(q + 1)\, S_{q+1}(M) = \sum_{ij} \frac{\partial}{\partial x_i}\left(\frac{1}{W}\, T_q(M)^i_j\, f_j\right) .$$

Remark. The point of this proposition is that we can express the right-hand side in terms of f and its derivatives, using Proposition 3.1 (f). Notice that when

$q = 1$, we recover the well-known formula $m\sigma_1 = \sum \frac{\partial}{\partial x_j}\left(\frac{f_j}{W}\right)$. The proof of the proposition requires the following lemma (see [1, p. 77]).

4.2 LEMMA. *Let* N_1 *and* N_2 *be smooth manifolds of the same dimension, and let* $\alpha: N_1 \to N_2$ *be a diffeomorphism. Suppose that* N_1 *and* N_2 *are equipped with volume elements* Ω_1 *and* Ω_2, *respectively, and that*

$$\alpha^*(\Omega_2) = \rho \cdot \Omega_1, \qquad (\alpha^{-1})^*(\Omega_1) = \lambda\Omega_2$$

(*so that* $\lambda = (\rho \circ \alpha)^{-1}$). *Let* Z_1 *and* Z_2 *be vector fields on* N_1 *and* N_2, *respectively, such that* $Z_2 = \alpha_*(Z_1)$. *Also let* div_1 *and* div_2 *be the divergence operators associated with the volume elements* Ω_1 *and* Ω_2. *Then* $\mathrm{div}_1(\rho Z_1) = \rho((\mathrm{div}_2 Z_2) \circ \alpha)$.

In our situation, we have $N_1 = \mathscr{D}$ and $N_2 = M$, while $\Omega_1 = dx_1 \wedge \cdots \wedge dx_m$ and $\Omega_2 = dV$. Also, $\alpha = X$ and $\rho = \sqrt{g} = W$. We set $Z_2 = T_q(A^T)$ and $Z_1 = (\alpha^{-1})_*(Z_2)$. The components of the vector Z_2 relative to the natural frame $X_1, \cdots, X_m$ on M are $\sum_j T_q(M)^i_j f_j / W^2$ $(i = 1, \cdots, m)$ (see Proposition 3.5 (a)). Thus, since $\alpha_*(A_j) = X_j$, the vector field Z_1 has the same components. Now, applying the lemma and Proposition 3.5 (b), we get the equation

$$\sum_i \frac{\partial}{\partial x_i}\left(W \cdot \left(\sum_j T^i_{qj} f_j / W^2\right)\right) = W \cdot ((q+1) S_{q+1}(M) \cdot \langle A, N \rangle) = (q+1) S_{q+1}(M)$$

(since $\langle A, N \rangle = 1/W$). ■

4.2 COROLLARY. *If* $\mathrm{grad}\, f = 0$ *on* $\partial\mathscr{D}$, *then* $\int_{\mathscr{D}} S_q(M)\, dx_1 \cdots dx_m = 0$.

Proof. Use Proposition 4.1, Stokes' theorem and the fact that $\mathrm{grad}\, f = 0$ implies $A^T = 0$. ■

We end this note by proving a uniqueness theorem.

4.3 THEOREM. *Suppose that* f_0 *and* f_1 *are functions on* $\overline{\mathscr{D}}$ *with graphs* M_0 *and* M_1. *Suppose further that*

(a) $f_0 = f_1$ *and* $\mathrm{grad}\, f_0 = \mathrm{grad}\, f_1$ *on* $\partial\mathscr{D}$,

(b) $f_1 \geq f_0$ *on* $\mathscr{D}$,

(c) f_0 *and* f_1 *are convex functions*,

(d) $\int_{\mathscr{D}} S_q(M_0)\, dV_0 = \int_{\mathscr{D}} S_q(M_1)\, dV_1$ *for some* $q < m$.

Then $f_0 \equiv f_1$ *on* $\mathscr{D}$.

Proof. Consider the 1-parameter family of functions $f_t = tf_1 + (1 - t)f_0$ $(0 \leq t \leq 1)$. Corresponding to these functions we have the graphs M_t, curvatures $S_q(t)$, volume elements dV_t, normals N_t, and so forth. Let

$$X_t = (x_1 \cdots x_m, f_t(x_1 \cdots x_m))$$

be the corresponding family of immersions. Then $\xi = \frac{\partial X_t}{\partial t}$ is the deformation vector.

Hypothesis (a) implies that ξ and grad ξ vanish on $\partial\mathcal{D}$. Thus we can use the variational formulas in [7] and assert that

$$\frac{d}{dt}\int_{\mathcal{D}} S_q(t)\,dV_t = -(q+1)\int_{\mathcal{D}} S_{q+1}(t)\,\langle \xi, N_t\rangle\,dV_t .$$

Now $\langle \xi, N_t\rangle = \dfrac{f_1 - f_0}{W_t} \geq 0$, by (b), and $S_{q+1}(t)$ never vanishes, by (c). However by (d),

$$0 = \int_{\mathcal{D}} S_q(M_1)\,dV_1 - \int_{\mathcal{D}} S_q(M_0)\,dV_0 = \int_0^1 \frac{d}{dt}\int_{\mathcal{D}} S_q(t)\,dV_t\,dt .$$

Since we have just seen that the integrand for the t-integral never changes sign, and that it vanishes only if $f_1 \equiv f_0$, the result follows. ■

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