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# ON THE HILBERT SCHEME COMPACTIFICATION OF THE SPACE OF TWISTED CUBICS

By RAGNI PIENE and MICHAEL SCHLESSINGER\*

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**1. Introduction.** One of the enumerative problems treated by Schubert in his book “Kalkul der abzählenden Geometrie” [S] is that of determining the number of twisted cubic curves which satisfy various given conditions. The complete solution to this problem should contain a description of the intersection ring of some compactification of the space of twisted cubics. In this paper we make a step in this direction by undertaking a study of the compactification given by the Hilbert scheme (see also [P]).

A *twisted cubic* curve is a rational, smooth curve of degree 3 in  $\mathbf{P}^3$ . The space  $H_0$  of such curves has the structure of a smooth, 12-dimensional, noncompact variety—in fact,  $H_0$  can be identified with the homogeneous space  $SL(4)/SL(2)$ . Let  $\text{Hilb}^{P(m)}(\mathbf{P}^3)$  denote the Hilbert scheme parametrizing closed subschemes of  $\mathbf{P}^3$  with Hilbert polynomial  $P(m)$ . Then  $H_0 \subset \text{Hilb}^{3m+1}(\mathbf{P}^3)$ , and we denote by  $H$  the closure of  $H_0$ . Our main result is the following theorem.

**THEOREM.**  $\text{Hilb}^{3m+1}(\mathbf{P}^3)$  consists of two irreducible components,  $H$  and  $H'$ , of dimension 12 and 15 respectively. Both  $H$  and  $H'$  are smooth and rational, they intersect transversally, and their intersection is non-singular, rational, of dimension 11.

The component  $H'$  which does not contain the twisted cubics contains the points corresponding to plane cubic curves union a point in  $\mathbf{P}^3$ . The intersection  $H \cap H'$  consists of plane, singular cubic curves, with a “spatial” embedded point at a singular point, “emerging from” the plane. The most degenerated such curve (in the sense that all curves corresponding to points in  $H \cap H'$  specialize to one of that kind) consists of a line tripled in the plane, with a spatial embedded point. A main ingredient in the proof of the theorem is the explicit construction of the deformation space of such

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a curve. We use a comparison theorem which enables us to identify the deformation theory of a projective variety with that of its associated homogeneous ideal, provided that suitable linear systems on the variety are complete (Section 3). The degenerate curve has a  $G_m$  action and its universal deformation is easy to compute (Section 5).

**2. Preliminary description of  $\text{Hilb}^{3m+1}(\mathbf{P}^3)$ .** Let  $C \subset \mathbf{P}_k^3 = \mathbf{P}^3$  ( $k$  is an algebraically closed field of characteristic  $\neq 2, 3$ ) be a twisted cubic curve, i.e.,  $C$  is smooth, rational, of degree 3. All such curves are projectively equivalent, hence we may fix one, say  $C_0 = \phi(\mathbf{P}^1)$ , where  $\phi: \mathbf{P}^1 \rightarrow \mathbf{P}^3$  is given by  $\phi(u, v) = (u^3, u^2v, uv^2, v^3)$ , and identify the space  $H_0$  of twisted cubics with automorphisms of  $\mathbf{P}^3$  modulo automorphisms of  $\mathbf{P}^1$ . So  $H_0 = \text{SL}(4)/\text{SL}(2)$  is a homogeneous space, hence smooth and irreducible, of dimension 12.

Since a twisted cubic curve has Hilbert polynomial  $P(m) = 3m + 1$ , we have  $H_0 \subset \text{Hilb}^{3m+1}(\mathbf{P}^3)$ ; let  $H = \bar{H}_0$  denote its closure. Set  $H'_0 = \{C' : C' = \text{a plane, smooth cubic curve in } \mathbf{P}^3 \text{ union a point in } \mathbf{P}^3 \text{ not on the curve}\}$ ; then  $H'_0 \subset \text{Hilb}^{3m+1}(\mathbf{P}^3)$ , and we denote by  $H' = \bar{H}'_0$  its closure. Since  $H'_0$  is irreducible, so is  $H'$ , and  $H'$  has dimension 15.

LEMMA 1.  $\text{Hilb}^{3m+1}(\mathbf{P}^3) = H \cup H'$ .

*Proof.* Suppose  $C \subset \mathbf{P}^3$  is a closed subscheme with Hilbert polynomial  $P(m) = \chi(\mathcal{O}_C(m)) = 3m + 1$ . We must show that  $C$  is a specialization of a curve in  $H_0$  or  $H'_0$ . Let  $\bar{C} \subset C$  be the maximal closed subscheme of  $C$  which is Cohen-Macaulay and of pure dimension 1. There are three cases to consider: i)  $\bar{C} = C$ . Then  $C$  is projectively Cohen-Macaulay and there is a projective resolution of the maximal homogeneous ideal  $I \subset P = k[x, y, z, w]$  defining  $C$ ,

$$0 \rightarrow P(-3)^2 \rightarrow P(-2)^3 \rightarrow I \rightarrow 0 \quad [\text{E, Ex.1, p. 430}].$$

By [loc. cit, Thm. 2]  $C$  can be deformed to a twisted cubic. ii)  $C = \bar{C} \cup Y$ , where  $Y \cap \bar{C} = \emptyset$  and  $\text{lg } \mathcal{O}_Y = r \geq 1$ . Since  $\chi(\mathcal{O}_C(m)) = \chi(\mathcal{O}_{\bar{C}}(m)) + r = 3m + 1$  and  $\chi(\mathcal{O}_{\bar{C}}(m)) \geq 3m$ , we have  $r = 1$  and  $\chi(\mathcal{O}_{\bar{C}}(m)) = 3m$ . Hence  $\bar{C}$  is a plane cubic curve, and  $Y$  is a reduced point, so  $C \in H'$ . iii)  $C$  has embedded points. Set  $K = \text{Ker}(\mathcal{O}_C \rightarrow \mathcal{O}_{\bar{C}})$ . Reasoning as in the previous case, we conclude  $\text{lg } K = 1$ , and  $\bar{C}$  is plane, so that  $C \in H'$ .

LEMMA 2. *If  $C \in H \cap H'$ , then  $C$  is a plane, singular cubic curve with a spatial embedded point, "emerging from" the plane, at a singular*

point. More precisely,  $C$  is projectively equivalent to the curve defined by an ideal  $I \subset k[x, v, z, w]$  of the form  $I = (xz, yz, z^2, q(x, y, w))$  where  $q(x, y, w)$  is a cubic form which is singular at  $(0, 0, 1)$ .

*Proof.* With the notation of the proof of Lemma 1,  $\bar{C}$  is plane and  $C$  is connected, so we're in case iii) of that proof. Moreover, it follows from a lemma of Hironaka [H, p. 360] that the embedded point must occur at a singular point of  $\bar{C}$ . It remains to describe the structure of  $C$  at the embedded point. First we observe that if  $C$  is contained in some surface  $S \subset \mathbf{P}^3$ , then  $S$  has to be singular at the embedded point  $p$  of  $C$ . In fact, we may assume that  $C \subset S$  are the closed fibres of families  $C_R \subset S_R \subset \mathbf{P}_R^3$ , over a discrete valuation ring  $R$  with traction field  $K$ , s.t.  $C_K \subset \mathbf{P}_K^3$  is a twisted cubic (if  $\deg S = 1$ , replace  $S$  by  $S$  union a plane not containing  $p$ ). If  $p$  were a smooth point on  $S$ , then it would be smooth on  $S_R$ , since  $S$  is a Cartier divisor on  $S_R$ . Then  $C_R$  would be a local complete intersection at  $p$ , hence so would  $C$ , and so  $p$  could not be an embedded point on  $C$ . Assume the embedded point is  $(0, 0, 0, 1)$ , and that the ideal of  $C$  in the affine coordinate ring  $k[x, y, z]$  is equal to  $I_a = (z, q) \cap \mathcal{Q}$ , where  $q(x, y)$  is singular at  $(0, 0)$ , and  $\mathcal{Q}$  is an  $(x, y, z)$ -primary ideal. Consider the exact sequence

$$0 \rightarrow K = (z, q)/I_a \rightarrow k[x, y, z]/I_a \rightarrow k[x, y]/(q) \rightarrow 0.$$

We know  $lgK = 1$ , so that either (a)  $z \in \mathcal{Q}$ , or (b) there is a  $q' \in \mathcal{Q}$  with  $q' = q \pmod{z}$ . In case (a),  $z \in I_a$ , hence  $C$  is plane and cannot by the observation above, be the specialization of a twisted cubic. (In this case,  $I = (z, xq, yq)$ , and the ideal of  $C$  is obtained by homogenizing  $q$  with respect to  $w$ .) In case (b),  $z \in I_a$ , but necessarily  $(xz, yz, z^2, q') \subset I_a$ , and these ideals are equal. Now  $q' = q + \alpha z$ ,  $\alpha \in k$ , and if  $\alpha \neq 0$ , then the surface defined by  $q'$  would be smooth at  $(0, 0, 0, 1)$ . By the observation above, we must therefore have  $\alpha = 0$  and hence  $q' = q$ .

Note that it follows from Lemma 2, by counting parameters, that the dimension of  $H \cap H'$  is equal to 11.

**3. Local Description of the Hilbert Scheme.** To a subscheme  $X$  of  $\mathbf{P}^n$  corresponds the homogeneous ideal  $I$  in the polynomial ring  $P = k[x_0, \dots, x_n]$  such that  $X = \text{Proj}(P/I)$  and  $I$  is maximal with respect to this. We thus have a map

$$u: M \rightarrow M'$$

from the universal deformation space  $M$ , which parametrizes all homogeneous ideals with Hilbert *function* equal to that of  $I$ , to the Hilbert scheme  $M'$  which parametrizes subschemes of  $\mathbf{P}^n$  with Hilbert *polynomial* equal to that of  $X$ . We shall show here that  $M$  and  $M'$  are isomorphic near the base points  $I$  and  $X$ , provided that the linear systems cut out on  $X$  by hypersurfaces of suitable degrees are complete.

**COMPARISON THEOREM.** *If the ideal of polynomials defining  $X \subset \mathbf{P}^n$  is generated by homogeneous polynomials  $f_1, \dots, f_r$ , of degrees  $d_1, d_2, \dots, d_r$ , for which*

$$(k[x_0, \dots, x_n]/I)_d \simeq H^0(X, \mathcal{O}_X(d))$$

$$d = d_1, d_2, \dots, d_r,$$

then the map  $u: M \rightarrow M'$  is an analytic isomorphism at the basepoints  $I, X$ .

We remark that in general, when the completeness condition is not satisfied, one must replace  $I$  by a high truncation, as Curtin [C] does for Mumford's obstructed curve.

*Proof of the Comparison Theorem.* We compare the Zariski tangent and normal spaces of  $M$  and  $M'$ . Let  $R = k[[t_1, \dots, t_m]]/J, J \subseteq (t)^2$ , be the completion of the local ring of  $M$  at its base point. We have

$$t^1(M) = ((t)/(t)^2)^*$$

$$t^2(M) = (J/tJ)^*,$$

the Zariski tangent and normal (i.e. "obstruction") spaces of  $M$  (In general  $t^i(M) = T^i(k/R, k) i \geq 1$  are the "homotopy" of  $R[A]$ .) Now  $u$  induces  $u^i: t^i(M) \rightarrow t^i(M')$ , all  $i$ ; as in [S, p. 153] we find easily that  $u: M \rightarrow M'$  is an analytic isomorphism provided that  $u$  is a "two equivalence" in the sense that  $u^1$  is an *isomorphism* and  $u^2$  is a *monomorphism*.

If we now take  $T^i(I) = T^i(A/P, A)$ ,  $T^i(X) = T^i(X/\mathbf{P}^n, \mathcal{O}_X)$ , the appropriate cotangent cohomology, we get a commutative diagram

$$\begin{array}{ccc} t^*(M) & \longrightarrow & t^*(M') \\ \downarrow & & \downarrow \\ T^*(I) & \longrightarrow & T^*(X) \end{array}$$

where the vertical “Kodaira-Spencer” maps form two equivalences, by versality of  $M$  and  $M'$ . We must show that  $T^i(I) \rightarrow T^i(X)$  is a two equivalence.

To compute  $T^i(I)$  for  $i = 1, 2$  we take a free resolution

$$\dots \longrightarrow H \xrightarrow{\nu} G \xrightarrow{\mu} F \xrightarrow{\lambda} P$$

of the module  $P/I$  over the polynomial ring  $P = k[x_0, \dots, x_n]$ . Here  $F = \Sigma P(-d_i)$  and  $\lambda = (f_1, \dots, f_r)$ . We map  $\Lambda^2 F \rightarrow G$  by sending  $u \wedge v$  to  $w$  in  $G$  with  $\mu(w) = \lambda(u)v - \lambda(v)u \in \ker \lambda$ . The *cotangent complex*, in low degrees, is then  $L : L_3 \rightarrow L_2 \rightarrow L_1 = \Lambda^2 F \otimes A \otimes H \otimes A \rightarrow G \otimes A \rightarrow F \otimes A$  with  $A = P/I$  (see [L.S.]), and  $T^i(I)$  is the cohomology of  $L^i = \text{Hom}(L^i, A)$ .

Now the complex  $\mathcal{L} = \tilde{L}$  restricts, over each affine open subset  $U$  of  $X$  to the relative cotangent complex of  $U$  in  $\mathbb{P}^n$ . Following Illusie [I] we then have  $T^i(X) = \text{Ext}_{\mathcal{O}_X}(\mathcal{L}^i, \mathcal{O}_X)$ . If we consider instead the cohomology  $S^i(X)$  of the complex of vector spaces  $\text{Hom}(\mathcal{L}^i, \mathcal{O}_X)$ , the edge homomorphism  $S^i(X) \rightarrow T^i(X)$  is a two equivalence and we need only show that  $T^i(I) \rightarrow S^i(X)$  is a two equivalence. The map in question comes from taking cohomology of the horizontal rows of the diagram

$$\begin{array}{ccccc} L_0^3 & \longleftarrow & L_0^2 & \longleftarrow & L_0^1 \\ \downarrow \alpha_3 & & \downarrow \alpha_2 & & \downarrow \alpha_1 \\ H^0(\tilde{L}^3) & \longleftarrow & H^0(\tilde{L}^2) & \longleftarrow & H^0(\tilde{L}^1) \end{array}$$

By hypothesis  $\alpha_1$  is an isomorphism, so that  $\alpha$  induces a two-equivalence and the proof is thus complete.

We remark that the cohomology sheaves  $\mathfrak{F}^i$  of  $\text{Hom}(\mathcal{L}, \mathcal{O}_X)$  consist of the normal sheaf  $\mathfrak{N} = \mathfrak{F}^1$  to  $X$  in  $\mathbb{P}^n$ , which determines local deformations of  $X$ , and the sheaf  $\mathfrak{F}^2$  which contains obstructions to local deformations of  $X$ .  $\mathfrak{F}^2$  is supported on the noncomplete intersection locus of  $X$ . We have  $H^0(X, \mathfrak{F}^1) \cong T^1(X)$  and  $0 \rightarrow H^1(X, \mathfrak{F}^1) \rightarrow T^2(X) \rightarrow H^0(X, \mathfrak{F}^2)$  which decomposes  $T^2(X)$  into local and global obstructions.

The hypotheses of the comparison theorem are certainly met for the smooth space curves in  $H_0$  or  $H'_0$ . For a curve  $C$  in  $H \cap H'$  let  $I = (xz, yz, z^2, q), J = (z, q)$  be the ideal of the Cohen-Macaulay curve  $\bar{C}$  and  $K = J/I$ , which is isomorphic to  $P/(x, y, z)$  twisted once as a  $P$  module. The local

cohomology sequence associated to the exact sequence  $0 \rightarrow K \rightarrow P/I \rightarrow P/J \rightarrow 0$  now shows that  $(P/I)_d \rightarrow H^0(\mathcal{O}_C(d))$  is an isomorphism for all  $d > 0$ . By the comparison theorem above we find that the completion of the Hilbert scheme at the point  $C$  is given by the universal deformation of the ideal  $I$  associated to  $C$ .

Alternatively, we may show directly that deformations of  $I$  and  $C$  agree by computing that the tangent space  $T^1(I) = H^0(C, \mathfrak{N}_C) = H^0(\mathfrak{J}^1)$  has dimension 16 and consists entirely of nonpositively weighted (thus globalizable) deformations of the singular point. Moreover,  $H^1(\mathfrak{J}^1) = 0$  and  $T^2(I) = H^0(\mathfrak{J}^2) \cong T^2(X)$  (has dimension two). The deformations of  $C$  thus coincide with the nonpositive deformations of its singular point, which coincide with homogeneous deformations of the affine cone over  $C$ .

**4. The tangent spaces to  $\text{Hilb}^{3m+1}(\mathbf{P}^3)$ .** Let  $C \in \text{Hilb}^{3m+1}(\mathbf{P}^3) = H \cup H'$ , let  $I \subset P = k[x, y, z, w]$  denote the maximal homogeneous ideal defining  $C$ , and set  $A = P/I$ , so that  $\mathcal{O}_C = \tilde{A}$ . Set  $\mathfrak{G} = \tilde{I}$ , so that  $\mathfrak{N} = \underline{\text{Hom}}(\mathfrak{G}, \mathcal{O}_C) = (\mathfrak{G}/\mathfrak{G}^2)$  is the normal sheaf of  $C$  in  $\mathbf{P}^3$ . With this notation the tangent space to  $\text{Hilb}^{3m+1}(\mathbf{P}^3)$  at  $C$  is given by  $T_{H \cup H', C} = H^0(C, \mathfrak{N})$ , and we now want to compute this space, which, as we have seen in Section 3, is isomorphic to  $T^1(I) = \text{Hom}_P(I, A)_0$  (the degree 0 piece of the graded module  $\text{Hom}_P(I, A)$ .) By [E, loc. cit] we know that  $H - H \cap H'$  is smooth, so that  $\dim T_{H \cup H', C} = 12$  if  $C \in H - H \cap H'$ . (This can also be computed directly from a presentation of  $I$ , as will be done below in the other cases.)

**LEMMA 3.** *If  $C \in H \cap H'$ , then  $\dim T_{H \cup H', C} = 16$ .*

*Proof.* We may assume  $C$  is defined by a homogeneous ideal  $I$  as in Lemma 2. It suffices to show  $\dim \text{Hom}_P(I, A)_0 = 16$ . Set  $J = (z, q)$ ; then  $J$  defines a plane curve  $\bar{C} \subset C$ . Set  $\bar{A} = P/J$  and  $K = J/I$ ; then we have an exact sequence  $0 \rightarrow K \rightarrow A \rightarrow \bar{A} \rightarrow 0$ . Consider the following presentation of  $J$ :

$$0 \longrightarrow P(-4) \longrightarrow P(-1) \oplus P(-3) \longrightarrow J \longrightarrow 0.$$

By applying  $\text{Hom}_P(-, A)$  we obtain a long exact sequence which yields  $\text{Hom}_P(J, A) = A(1) \oplus M(3)$  (where  $M = (x, y, z)A$ ) and  $\text{Ext}_P^1(J, A) = \bar{A}(4)$ . The presentation of  $K$ ,

$$0 \rightarrow P(-4) \xrightarrow{\begin{pmatrix} z \\ -y \\ x \end{pmatrix}} P(-3)^3 \xrightarrow{\begin{pmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{pmatrix}} P(-2)^3 \xrightarrow{(x,y,z)} P(-1) \xrightarrow{z} K \rightarrow 0$$

shows  $\text{Hom}_P(K, A) = K(1)$  and that  $\text{Ext}_P^1(K, A)_0 \subset A(2)_0^3 = A_2^3$  is generated by

$$\left\{ \begin{pmatrix} zw \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ zw \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ zw \end{pmatrix}, \begin{pmatrix} -q_1 \\ -q_2 \\ 0 \end{pmatrix} \right\},$$

where  $q = xq_1 + yq_2$  (and  $q_1, q_2 \in (x, y)A$ , since  $q$  is singular at  $(0, 0, 1)$ ).

From  $0 \rightarrow I \rightarrow J \rightarrow K \rightarrow 0$  we therefore obtain the following exact sequence:  $0 \rightarrow K(1) \rightarrow A(1) \oplus M(3) \rightarrow \text{Hom}_P(I, A) \rightarrow \text{Ext}_P^1(K, A) \xrightarrow{\beta} \bar{A}(4)$ . A diagram chase shows that the map  $\beta$  is the restriction of the map  $(q_1, q_2, 0): A(2)^3 \rightarrow \bar{A}(4)$ , and hence is 0. Thus we obtain a short exact sequence

$$0 \rightarrow \bar{A}(1) \oplus M(3) \rightarrow \text{Hom}_P(I, A) \rightarrow \text{Ext}_P^1(K, A) \rightarrow 0,$$

which yields  $\dim \text{Hom}_P(I, A)_0 = \dim \bar{A}_1 + \dim M_3 + \dim \text{Ext}_P^1(K, A)_0 = 3 + 9 + 4 = 16$ .

LEMMA 4. *If  $C \in H' - H \cap H'$ , then  $\dim T_{H \cup H', C} = 15$ .*

*Proof.* Case (i):  $C = \bar{C} \cup Y$ , where  $Y$  is a reduced point not on  $\bar{C}$ . Then  $H^0(C, \mathfrak{U}) = h^0(\bar{C}, \mathcal{O}_{\bar{C}}(3) \oplus \mathcal{O}_{\bar{C}}(1)) + 3 = 15$ . Case (ii):  $C$  has an embedded point emerging from the plane of  $\bar{C}$ , at a nonsingular point of  $\bar{C}$ . We may assume  $I = (xz, yz, z^2, q)$ , where  $q \in k[x, y, w]$  is a cubic form which goes through, but is nonsingular at, the point  $(0, 0, 1)$ . The computation of its dimension is similar to the one above, except that in this case  $\text{Ext}_P^1(K, A)_0$  is generated by

$$\begin{pmatrix} zw \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ zw \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ zw \end{pmatrix},$$

which completes the count. Case (iii):  $C$  has an embedded point and is contained in a plane, i.e., we are in case (a) of the proof of Lemma 2. Then we may assume  $I = (z, xq, yq)$ , where  $q \in k[x, y, w]$  is a cubic form vanishing at  $(0, 0, 1)$ . Set  $P' = k[x, y, w]$  and  $I' = (xq, yq) \subset P'$ , and let  $\mathfrak{U}'$  denote the normal sheaf of  $C$  in the plane  $z = 0$ . Reasoning as in the proof of the comparison thm. one shows  $H^0(C, \mathfrak{U}') = \text{Hom}_P(I, A)_0$ . Since



$h^0(C, \mathfrak{N}) = h^0(C, \mathfrak{N}') + h^0(C, \mathcal{O}_C(1))$  and  $h^0(C, \mathcal{O}_C(1)) = 4$ , it suffices to show  $\dim \text{Hom}_{P'}(I', A)_0 = 11$ .

Set  $J' = (q) \subset P'$ ,  $\bar{A} = P'/J'$ , and  $K = J'/I'$ . The  $P'$ -module  $K$  has a presentation

$$0 \longrightarrow P'(-5) \xrightarrow{\begin{pmatrix} -y \\ -x \end{pmatrix}} P'(-4)^2 \xrightarrow{(x,y)} P'(-3) \xrightarrow{q} K \longrightarrow 0.$$

From this, we obtain  $\text{Hom}_{P'}(K, A) = K(3)$ , and that  $\text{Ext}_{P'}^1(K, A)_0$  is generated by  $\begin{pmatrix} q^w \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ q^w \end{pmatrix}$ . Thus there is an exact sequence

$$0 \longrightarrow \bar{A}(3) \longrightarrow \text{Hom}_{P'}(I', A) \longrightarrow \text{Ext}_{P'}^1(K, A) \longrightarrow 0,$$

and we conclude:  $\dim \text{Hom}_{P'}(I', A)_0 = \dim \bar{A}_3 + 2 = 11$ .

Thus we have shown that  $H \cup H'$  is smooth outside  $H \cap H'$ , and that  $\dim T_{H \cup H', C} = 16$  if  $C \in H \cap H'$ .

**LEMMA 5.** For all  $C \in \text{Hilb}^{3m+1}(\mathbf{P}^3)$ ,  $H^1(C, \mathfrak{N}) = 0$ .

*Proof.* We shall consider separately four cases and show that in each case we have  $\chi(\mathfrak{N}) = h^0(C, \mathfrak{N})$ , by the above computations. i)  $C \in H - H \cap H'$ . The exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^3}(-3)^2 \longrightarrow \mathcal{O}_{\mathbf{P}^3}(-2)^3 \longrightarrow \mathfrak{J} \longrightarrow 0$$

gives

$$0 \longrightarrow \mathfrak{N} \longrightarrow \mathcal{O}_C(2)^3 \longrightarrow \mathcal{O}_C(3)^2 \longrightarrow \omega_C(4) \longrightarrow 0,$$

where  $\omega_C$  denotes the dualizing sheaf on  $C$ , from which we get  $\chi(\mathfrak{N}) = 12 = h^0(C, \mathfrak{N})$ . Hence  $h^1(C, \mathfrak{N}) = 0$ . ii)  $C \in H \cap H'$ . The exact sequence

$$0 \longrightarrow \mathcal{O}_{\bar{C}}(1) \oplus \tilde{M}(3) \longrightarrow \mathfrak{N} \longrightarrow \text{Ext}_{\mathbf{P}^3}^1(K, \mathcal{O}_C) \longrightarrow 0$$

gives  $\chi(\mathfrak{N}) = 7 + \chi(\tilde{M}(3)) = 7 + \chi(\mathcal{O}_C(3)) - 1 = 16 = h^0(C, \mathfrak{N})$ . iii)  $C \in H' - H \cap H'$ ,  $C = \bar{C} \cup Y$ ,  $Y \cap \bar{C} = \emptyset$ . Then  $\chi(\mathfrak{N}) = \chi(\mathfrak{N}_{\bar{C}/\mathbf{P}^3}) + 3 = 15 = h^0(C, \mathfrak{N})$ . iv)  $C \in H' - H \cap H'$ ,  $C \subset P^2 \subset \mathbf{P}^3$ ,  $C$  has an embedded point. Then  $\chi(\mathfrak{N}) = \chi(\mathfrak{N}_{C/\mathbf{P}^2}) = \chi(\mathcal{O}_C(1))$ . The exact sequence (Lemma 4, (iii))

$$0 \longrightarrow \mathcal{O}_{\bar{C}}(3) \longrightarrow \mathfrak{N}_{C/\mathbf{P}^2} \longrightarrow \text{Ext}_{\mathbf{P}^2}^1(\tilde{K}, \mathcal{O}_C) \longrightarrow 0$$

gives  $\chi(\mathfrak{N}_{C/\mathbf{P}^2}) = 11$ , hence  $\chi(\mathfrak{N}) = 15 = h^0(C, \mathfrak{N})$ .

**5. The universal deformation of  $k[x, y, z, w]/(xz, yz, z^2, x^3)$ .** From the description given in Lemma 2, we know that every  $C \in H \cap H'$  specializes to a curve of the form: a line tripled in a plane, with a spatial embedded point. Such a curve is completely determined by its associated (point-line-plane) flag, and all such curves are projectively equivalent. (They form a closed orbit—isomorphic to the flag variety, hence of dimension 6—under the action of  $SL(4)$  on  $\text{Hilb}^{3m+1}(\mathbf{P}^3)$ .) Thus, in order to study deformations of some  $C \in H \cap H'$ , it suffices to study deformations of a curve of the above degenerated form, e.g. whose maximal homogeneous ideal is  $I = (xz, yz, z^2, x^3)$ .

**LEMMA 6.** *Suppose  $I = (xz, yz, z^2, x^3)$ . Then  $I$  has a universal deformation space of the form  $M = \mathbf{A}^{12} \cup \mathbf{A}^{15}$ , where  $\mathbf{A}^{12} \cap \mathbf{A}^{15} = \mathbf{A}^{11}$  and the intersection is transversal.*

*Proof.* Consider the following presentation of  $A = P/I$  over  $P = k[x, y, z, w]$ :

$$0 \longrightarrow P(-4) \xrightarrow{\nu} P(-4) \oplus P(-3)^3 \xrightarrow{\mu} P(-2)^3 \oplus P(-3) \xrightarrow{\lambda} P \longrightarrow A \longrightarrow 0,$$

where the maps are given by

$$\lambda = (xz, yz, z^2, x^3), \quad \mu = \begin{pmatrix} x^2 & y & z & 0 \\ 0 & -x & 0 & z \\ 0 & 0 & -x & -y \\ -z & 0 & 0 & 0 \end{pmatrix}, \quad \nu = \begin{pmatrix} 0 \\ z \\ -y \\ x \end{pmatrix}$$

We have already seen (Lemma 3) that  $\dim \text{Hom}(I, A)_0 = 16$ , and one checks that the following 10 elements (of  $A_2^3 \oplus A_3$ ),

$$\frac{\partial}{\partial u_1} = \begin{pmatrix} 0_2 \\ x \\ 0 \\ 0 \end{pmatrix}, \quad \frac{\partial}{\partial u_2} = \begin{pmatrix} 0 \\ 0 \\ zw \\ 0 \end{pmatrix}, \quad \frac{\partial}{\partial u_3} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ xw^2 \end{pmatrix}, \quad \frac{\partial}{\partial u_4} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ yw^2 \end{pmatrix},$$

$$\frac{\partial}{\partial u_5} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ zw^2 \end{pmatrix}, \quad \frac{\partial}{\partial u_6} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ x^2w \end{pmatrix}, \quad \frac{\partial}{\partial u_7} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ xyw \end{pmatrix}, \quad \frac{\partial}{\partial u_8} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ y^2w \end{pmatrix},$$

$$\frac{\partial}{\partial u_9} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ xy^2 \end{pmatrix}, \quad \frac{\partial}{\partial u_{10}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ y^3 \end{pmatrix},$$

together with the 6 “trivial deformations” (corresponding to moving the flag determined by  $C$ ),

$$\frac{\partial}{\partial t_1} = w \frac{\partial}{\partial x} = \begin{pmatrix} zw \\ 0 \\ 0 \\ 3x^2w \end{pmatrix}, \quad \frac{\partial}{\partial t_2} = y \frac{\partial}{\partial x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 3x^2y \end{pmatrix},$$

$$\frac{\partial}{\partial t_3} = w \frac{\partial}{\partial y} = \begin{pmatrix} 0 \\ zw \\ 0 \\ 0 \end{pmatrix}, \quad \frac{\partial}{\partial t_4} = w \frac{\partial}{\partial z} = \begin{pmatrix} xw \\ yw \\ 2zw \\ 0 \end{pmatrix},$$

$$\frac{\partial}{\partial t_5} = x \frac{\partial}{\partial z} = \begin{pmatrix} x^2 \\ xy \\ 0 \\ 0 \end{pmatrix}, \quad \frac{\partial}{\partial t_6} = y \frac{\partial}{\partial z} = \begin{pmatrix} xy \\ y^2 \\ 0 \\ 0 \end{pmatrix},$$

form a basis of  $\text{Hom}(I, A)_0$ .

To obtain homogeneous deformations of  $I$ , we consider homogeneous perturbations  $\lambda', \mu', \nu'$  of the maps  $\lambda, \mu, \nu$ :

$$\lambda' = (xz + u_1(bx + cy), yz - u_1(ax + by), (z + u_2w)z - u_1^2(b^2 - ac),$$

$$ax^2 + 2bxy + cy^2 + (u_3x + u_4y + u_5(z + u_2w))w^2),$$

$$\mu' = \begin{pmatrix} ax + by + u_3w^2 & y & z - u_1b + u_2w & u_1a \\ bx + cy + u_4w^2 & -x & -u_1c & z + u_1b + u_2w \\ u_5w^2 & 0 & -x & -y \\ -z & -u_1 & 0 & 0 \end{pmatrix},$$

$$\nu' = \begin{pmatrix} -u_1 \\ z + u_2w \\ -y \\ x \end{pmatrix}$$

where we have set  $a = x + u_6w, b = u_9y + u_7w, c = u_{10}y + u_8w$ , and where the variables  $u_i, i = 1, \dots, 10$  give infinitesimal deformations tangent to the basis elements of  $\text{Hom}_P(I, A)_0$  denoted by  $\partial/\partial u_i$ .

One checks that  $\lambda' \cdot \mu' \equiv 0 \pmod{(u_1u_2, u_1u_3, u_1u_4, u_1u_5)}$ . Moreover,  $\mu' \cdot \nu' = (-u_1u_3w^2, -u_1u_4w^2, -u_1u_5w^2, -u_1u_2w) \equiv 0 \pmod{(u_1u_2, u_1u_3, u_1u_4, u_1u_5)}$ , and no additional higher order terms can cancel these entries. Therefore, the flat deformation that we can obtain over the union of the 6-space  $u_2 = u_3 = u_4 = u_5 = 0$  with the 9-space  $u_1 = 0$  cannot be extended to any larger parameter space. (Alternatively, the entries arising from  $\lambda' \cdot \mu'$  may be shown to span  $T^2$ .) We have thus exhibited a versal deformation of  $I$  (and hence of  $C$ ).

A universal deformation is now obtained from the above by adding the trivial deformations; this is done by performing everywhere the following substitutions:  $x = x + t_1w + t_2y, y = y + t_3w, z = z + t_4w + t_5x + t_6y$ . Hence we have shown:

$$M = \text{Spec}(k[u_1, \dots, u_{10}, t_1, \dots, t_6]/(u_1u_2, u_1u_3, u_1u_4, u_1u_5)).$$

*Remark.* Recalling the exact sequence (proof of Lemma 3)

$$0 \longrightarrow \bar{A}(1) \oplus M(3) \longrightarrow \text{Hom}(I, A) \longrightarrow \text{Ext}^1(K, A) \longrightarrow 0$$

and remarking that

$$a \in \bar{A}_1 \text{ goes to } \begin{pmatrix} xa \\ ya \\ za \\ 0 \end{pmatrix} \in \text{Hom}(I, A)_0 \subset A_2^3 \oplus A_3$$

and

$$b \in M_3 \text{ goes to } \begin{pmatrix} 0 \\ 0 \\ 0 \\ b \end{pmatrix},$$

we observe that  $\text{Hom}(I, A)_0$  is generated, modulo the trivial deformations, by the elements coming from  $M_3$ , together with the elements  $\partial/\partial u_1$  and  $\partial/\partial u_2$ . The former corresponds to “twisting the curve into space” (hence making the embedded point disappear), whereas the latter corresponds to moving the embedded point out of the plane.

**6. The Hilbert Scheme  $\text{Hilb}^{3m+1}(\mathbf{P}^3)$ .** We shall now prove the theorem stated in the introduction.

**THEOREM.** *The scheme  $\text{Hilb}^{3m+1}(\mathbf{P}^3)$  is the union of two nonsingular rational varieties  $H$  and  $H'$ , of dimension 12 and 15; their transversal intersection is nonsingular rational of dimension 11.*

*Proof.* By the previous Lemmas, we need only demonstrate the rationality of  $H, H', H \cap H'$ . Consider the point  $C_0 \in H \cap H'$  whose ideal is  $I = (xz, yz, z^2, x^3)$ , and the universal family of deformations of  $I$  constructed in section 5. We get a flat family  $X \rightarrow \mathbf{A}^{12} \cup \mathbf{A}^{15}$  of subschemes of  $\mathbf{P}^3$  and hence a classifying map  $\phi: \mathbf{A}^{12} \cup \mathbf{A}^{15} \rightarrow \text{Hilb}(\mathbf{P}^3)$ . We have seen that  $\phi$  is an analytic isomorphism at each point of its domain. The ideal  $I$  occurs only at the base point of  $\mathbf{A}^{12} \cup \mathbf{A}^{15}$ , and does not reappear as the

parameters  $(u, t)$  approach infinity.  $\phi$  has degree one over the Hilbert point  $C_0$ , and therefore has degree one over each point of its image, as any such point specializes to  $C_0$ .  $\phi$  is therefore an open immersion, and the theorem follows.

Alternatively, one may compute directly that the ideals corresponding to parameter values  $(u, t)$  and  $(u', t')$  are not equal unless  $u = u', t = t'$ , and proceed as above.

Also, as Robert Varley has kindly pointed out to us, the rationality of  $H$  is classical. Fix two planes  $p_1, p_2$  in  $\mathbf{P}^3$ . They intersect a general twisted cubic  $C$  in two pairs of three points, and these six points in turn determine  $C$ .  $H$  is thus birationally equivalent to the product of  $\text{Symm}_3(\mathbf{P}^2)$  with itself. A modern proof of the rationality of a symmetric product was given by Mattuck [M]; this may also be seen from the versal deformation of a suitable thick point.

**COROLLARY.** *The scheme  $H$  decomposes as a finite disjoint union of affine spaces,  $H = \mathbf{A}^{12} \cup \mathbf{U}\mathbf{A}^{n_i}$ , where  $0 \leq n_i \leq 11$  and all integers between 0 and 11 occur.*

*Proof.*  $H$  is smooth, complete (in fact projective), and has a finite number of orbits under the action of  $SL(4)$ . By a result of D. Luna (see [D-P], 7.2) the set of fixed points of a maximal torus of  $SL(4)$  is finite, and therefore we can apply the results of Bialynicki-Birula [B, section 4]. (The  $\mathbf{A}^{12}$  found in the proof of the theorem can be taken as the beginning of such a decomposition.)

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