ON THE HISTORY OF THE KRONECKER PRODUCT

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Abstract

History reveals that what is today called the Kronecker product should be called the Zehfuss product.

1. Introduction

A matrix operation of wide application, defined for matrices $\underset{\sim}{A} = \{a_{ij}\}$ and $\underset{\sim}{B}$ of any order to be

$$\underset{\sim}{A} \otimes \underset{\sim}{B} = \{a_{ij} \underset{\sim}{B}\} , \tag{1}$$

is generally referred to as the Kronecker or direct product. The intriguing history of Kronecker's name being associated with this product is outlined, and development of its properties discussed, thus extending the earlier work in Searle (1966) and Henderson and Searle (1980).

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2. The Zehfuss Determinant Result

Our story begins with one Johann Georg Zehfuss (-) at the University of Heidelberg who, according to biographical notes by Poggendorf (1863, 1898), published papers on determinants until at least 1868 before moving to studies in astronomy. In particular, Zehfuss (1858) contains the determinant result

$$|\underset{\sim}{A} \otimes \underset{\sim}{B}| = |\underset{\sim}{A}|^b |\underset{\sim}{B}|^a , \tag{2}$$

for square matrices $\underset{\sim}{A}$ and $\underset{\sim}{B}$ of order a and b, respectively. Zehfuss wrote in terms of determinants rather than matrices, following the then customary practice of employing the term 'determinant' both for what we now call a square matrix as well as for its determinant. [The idea that a matrix could have its own identity was barely beginning, in the seminal work of Sylvester (1850) and Cayley (1858).]

3. Matrix Results

It took some time for the distinctiveness of matrices to be adopted in a manner that we would now recognize. Concerning (1), Hurwitz (1894) uses the symbol $\times$, as a <u>producttransformation</u> of matrices, while Stéphanos (1899a) uses the term conjunction. Hurwitz (1894) also develops the determinant result (2) and the now very familiar matrix equalities:

$$\begin{aligned}
\underset{\sim}{I}_m \otimes \underset{\sim}{I}_n &= \underset{\sim}{I}_{mn} , \\
(\underset{\sim}{A} \otimes \underset{\sim}{B})(\underset{\sim}{C} \otimes \underset{\sim}{D}) &= (\underset{\sim}{AC}) \otimes (\underset{\sim}{BD}) , \\
(\underset{\sim}{A} \otimes \underset{\sim}{B})^{-1} &= \underset{\sim}{A}^{-1} \otimes \underset{\sim}{B}^{-1} , \\
\text{and} \qquad (\underset{\sim}{A} \otimes \underset{\sim}{B})' &= \underset{\sim}{A}' \otimes \underset{\sim}{B}' ,
\end{aligned} \tag{3}$$

where, in these expressions, the necessary rank and conformability conditions for their existence are assumed satisfied, and where $\underset{\sim}{I}_n$ is the identity matrix of order n. In addition, Stéphanos (1899a,b, 1900) formulates eigenvalues of $\underset{\sim}{A} \otimes \underset{\sim}{B}$ as all products of those of $\underset{\sim}{A}$ and $\underset{\sim}{B}$, thus providing indirect derivation of the determinant result (2). In terms of (3), it follows directly, of course, from $|\underset{\sim}{A} \otimes \underset{\sim}{B}| = |\underset{\sim}{A} \otimes \underset{\sim}{I}_b||\underset{\sim}{I}_a \otimes \underset{\sim}{B}|$ because $\underset{\sim}{I}_a \otimes \underset{\sim}{B}$ is the block diagonal matrix of a $\underset{\sim}{B}$'s with determinant $|\underset{\sim}{I}_a \otimes \underset{\sim}{B}| = |\underset{\sim}{B}|^a$, and $\underset{\sim}{A} \otimes \underset{\sim}{I}_b = \underset{\sim}{I}_{b,a}(\underset{\sim}{I}_b \otimes \underset{\sim}{A})\underset{\sim}{I}_{a,b}$, using vec-permutation matrices of Henderson and Searle (1980), and so $|\underset{\sim}{A} \otimes \underset{\sim}{I}_b| = |\underset{\sim}{I}_b \otimes \underset{\sim}{A}| = |\underset{\sim}{A}|^b$.

4. Zehfuss Rediscovered

Unfortunately, the Zehfuss (1858) determinant result (2) seems to have been overlooked for more than fifty years, until its rediscovery by Muir (1911, pp.102-3) who claims it for Zehfuss and accordingly calls $|\underset{\sim}{A} \otimes \underset{\sim}{B}|$ the Zehfuss determinant of $\underset{\sim}{A}$ and $\underset{\sim}{B}$. Others, notably Rutherford (1933), and Aitken (1935) and his student Ledermann (1936), following Muir's lead, have gone further and called $\underset{\sim}{A} \otimes \underset{\sim}{B}$ the Zehfuss matrix of $\underset{\sim}{A}$ and $\underset{\sim}{B}$. Aitken's adoption of this name is of interest in light of Ledermann's later (1968) comment that Aitken "was particularly fond of stressing the claims of lesser known mathematicians of former times for discoveries erroneously attributed to their more famous contemporaries. Many of his historical references were gleaned from Sir Thomas Muir's monumental work on determinants, for which Aitken had a profound admiration." More recently, Henderson and Searle (1980) also refer to Zehfuss (1858) for (2), this being the sole citation of that reference in Science Citation Index since its inception in 1961. Generally speaking, though, the name of Zehfuss has been forgotten in this connection.

5. Hensel's Claim for Kronecker

In contrast, the name of Kronecker (1823-91) has long been associated with the $\otimes$ operation of (1), and with the determinant result (2). This association originated with Hensel (1889, 1891) who, in presenting (2), notes that Kronecker had for some time given the result and a proof in algebra lectures, presumably during Hensel's student days (1880-1884) in Berlin. (Hensel completed his dissertation in 1884 and joined the department in 1886.) According to Hass (1950), Kronecker's lectures were very difficult, so much so that Seliwarroff, one of the circle of young mathematicians in Berlin, recounts "And when the lecture was over, we all exclaimed 'Wonderful' but had not understood a thing." Hensel, whom Hass (1950) considers probably got most from Kronecker's lectures, subsequently edited Kronecker's (1903) Vorlesungen über die Theorie der Determinanten, Band I, which consisted of 21 lectures, part of a university course on Allgemeine Arithmetik belonging to the period 1883-1891. Although Hensel announced this work to be only a first volume, subsequent volumes did not appear, nor did Vorlesungen über die Theorie der algebraischen Gleischungen announced by the publisher on the last page of the published work. This superb volume does not contain the material to which Hensel (1889, 1891) refers, nor do any of Kronecker's research papers, which is hardly surprising as there is no reference to any paper in this connection in Hensel's editing of Kronecker's collected works (in five volumes, 1895, 1897, 1899, 1929, 1930). [As an aside, Hensel can also be remembered for his statistical spoonerism (sic):

> "In diesem kleinen Gartenhaus
> Bezwing' ich selbst den harten Gauss."]

6. Claim and Counter-claim

Numerous other writers of the late 1800's also developed the determinant result (2), all without reference to Zehfuss (1858). First comes Rados (1886), believing his work (using Grassmann's theory) to be original. Although Hensel (1889) attributed (2) solely to Kronecker, for Rados this was apparently not enough because, almost a decade later, Rados (1900) claims the result for himself, questioning Hensel's claim for Kronecker since he had found no trace of the result in a notebook of Kronecker's course, which Kronecker had himself reviewed. For Muir (1927, p. 259) this apparent lack of evidence is conclusive; he reports that Rados (1900) refers "to Hensel's claim for Kronecker and effectively disposes of it". But maybe Muir is overzealous in disposing of Kronecker, to the extent that he seems to deny the possibility that

Kronecker developed the result at all. It is reasonable to accept, without doubt, Hensel's testimony that Kronecker "the doubter" presented (2) in lectures. But also, without doubt, Zehfuss (1858) had priority over Kronecker: he published the result before Kronecker began lecturing. Kronecker, it will be recalled, completed his Ph.D. in 1845 and spent the next eight years as a successful business man, during which time he maintained a lively scientific correspondence with his former master, Kummer. Further, Bell (1937, p. 478) notes that "from 1861 to 1883 Kronecker [as a member of the Berlin Academy] conducted regular courses at the university, principally on his personal researches, after the necessary introductions. In 1883 Kummer ··· retired, and Kronecker succeeded his old master as ordinary professor" until his death in 1891.

Was Kronecker indebted to Zehfuss? Kronecker's interest in determinants was long-standing, according to Frobenius (1893), and even in his early scientific days he had dealt with determinants. He may have read the short note by Zehfuss (1858) and noticed the result, but it would seem that he derived the result independently, since at least his method of proof, noted by Hensel as an elegant reformulation of the determinants using multiplicative rules, seems to differ from that of Zehfuss.

7. The Neglect of Zehfuss and Association of Kronecker

The claim made by Rados (1900) did not appear to affect the growing association of Kronecker with the determinant result (2). This association which, as we have seen, originated with Hensel (1889, 1891), was referred to by Netto (1893, 1898) and was included in popular text-books of the day. Evidence of this is Muir's lament (1927, p. 259) in reporting Rados (1900) that "the text-book of Scott and Mathews [1904, p. 72], which appeared four years after the publication of Rados' paper, gave new life to the old error. This was probably due to the teaching of Pascal, whose second edition (1923) still propagates the error" of the first edition (1897). In addition, Loewy's (1903) extensions of Hurwitz (1894) follow Netto's (1898) reference to Kronecker and later, in Pascal's Repertorium, Loewy (1910) refers without discussion to (2) as Kronecker's theorem, which is surprising since he cites relevant works of both Hensel and Rados.

Other writers of the 1890's, such as Igel (1892), Mertens (1893), Escherich (1892), Hurwitz (1894), Sterneck (1895) and Stéphanos (1899a,b, 1900), also developed (2), and Metzler (1899), Moore (1900) and Petr (1906) effected certain generalizations thereto – all of them, without reference to Zehfuss which, as Muir (1923, p. 49) points out in reporting Igel (1892) and Escherich (1892), was a neglect that "not a few subsequent writers were equally guilty of".

This neglect of Zehfuss lasted for some twenty years after Hensel's papers, until Muir's (1911) rediscovery of Zehfuss (1858), in which he strongly condemned propagating the error of attributing (2) to Kronecker and of overlooking Zehfuss. But this condemnation has apparently fallen on deaf ears; from 1911 until now there has been only one reference (see the end of section 4) to Zehfuss, so that after this passage of time the attachment to (2) of the Zehfuss name in place of Kronecker may never succeed.

8. The Kronecker Name for Other Matrix Results

The strong association of the Kronecker name with the determinant result (2), inappropriate as we believe it to be, has brought with it the attachment of the name to several matrix operations allied to (2).

8.1. The direct (or Zehfuss, not Kronecker) product

Murnaghan (1938, p. 68) is an early example of making the easy step from associating the Kronecker name with (2) to attaching it to the matrix operation (1), which is involved in (2). In this way, the term Kronecker product has, following Murnaghan, been used for $\mathbf{A} \otimes \mathbf{B} = \{a_{ij}\mathbf{B}\}$ by many writers, e.g., Cornish

(1957), Shah (1959), Searle (1966), Neudecker (1968, 1969) and Graybill (1969) in the context of statistics; Barnett (1979) in engineering, and Jacobson (1953), Halmos (1958) and Bellman (1970) in algebra texts. Thus today, the name Kronecker product is generally associated with the ⊗ operation of (1), with little thought as to its historical validity. It seems to us that Muir's (1911) claim of priority for Zehfuss demands that if any name be attached to the ⊗ operation it should be Zehfuss and not Kronecker.

The alternative name of direct product for the ⊗ operation is also in current use (e.g., Lancaster, 1969), and can be traced back to the early thirties, e.g., to MacDuffee (1933), Roth (1934) and Wedderburn (1934). Calling this operation a direct product (of matrices) is descriptive and appropriate because it arises naturally from the concept of direct product in group theory as discussed, for example, by Jordan (1870), Hölder (1893) and Burnside (1911). The ⊗ operation is also a particular case of the tensor product for transformations as in Halmos (1958, p. 97), for example.

8.2. Direct (and/or Zehfuss, not Kronecker) sums

The operation

$$\mathbf{A} \oplus \mathbf{B} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix} \tag{4}$$

has long been known as the direct sum of two matrices and may occasionally be called, quite wrongly, the Kronecker sum. In contrast, at least two other expressions are found in the literature for Kronecker sum, each of them involving the (now-to-be-called) Zehfuss product (1).

The first, for which we introduce the operator ⓚ, is

$$\mathbf{A} \,ⓚ\, \mathbf{B} = \mathbf{A}_{m\times m} \otimes \mathbf{I}_p + \mathbf{I}_m \otimes \mathbf{B}_{p\times p} \tag{5}$$

where $\underset{\sim}{I}_p$ is the identity matrix of order p. Texts such as Lancaster (1969, p. 260) and Bellman (1970, p. 238) both suggest that Kronecker sum is the historical name for ⓚ of (5). Although the origin of this use of Kronecker's name seems unknown, it is an appropriate extension of calling $\underset{\sim}{A} \otimes \underset{\sim}{B}$ the Kronecker product, in the following sense: the eigenvalues of $\underset{\sim}{A} \otimes \underset{\sim}{B}$ and $\underset{\sim}{A} ⓚ \underset{\sim}{B}$ are, respectively, all possible products and sums of those of $\underset{\sim}{A}$ and $\underset{\sim}{B}$. Despite this, Pease (1965, p. 325) comments that

> " ... the name is perhaps unfortunate since the relation does not have the properties that we expect from an additive relation. For example, it is not commutative. Nevertheless, it is an important relation that does play a role that, in many situations, is analogous to a sum."

A second expression named Kronecker sum is, according to Sinha et al. (1979), that defined by Nigam and Mathur (1974) as

$$\underset{\sim}{A} ⓢ \underset{\sim}{B} = \underset{\sim}{A}_{m\times n} \otimes \underset{\sim}{J}_{p\times q} + \underset{\sim}{J}_{m\times n} \otimes \underset{\sim}{B}_{p\times q} . \tag{6}$$

Motivation for using this name for (6) is, presumably, that just as $\underset{\sim}{A} \otimes \underset{\sim}{B}$ is a matrix of all possible products of elements of $\underset{\sim}{A}$ and $\underset{\sim}{B}$, so is $\underset{\sim}{A} ⓢ \underset{\sim}{B}$ of (6) a matrix of all possible sums of elements of $\underset{\sim}{A}$ and $\underset{\sim}{B}$. That is, $\underset{\sim}{A} \otimes \underset{\sim}{B} = \{a_{ij}b_{k\ell}\}$ and $\underset{\sim}{A} ⓢ \underset{\sim}{B} = \{a_{ij} + b_{k\ell}\}$. An application of (6) is given in Karlin and Liberman (1978).

In their different ways, both $\underset{\sim}{A} ⓚ \underset{\sim}{B}$ and $\underset{\sim}{A} ⓢ \underset{\sim}{B}$ are additive analogues of the Zehfuss product $\underset{\sim}{A} \otimes \underset{\sim}{B}$, namely $\underset{\sim}{A} ⓚ \underset{\sim}{B}$ in terms of eigenvalues and $\underset{\sim}{A} ⓢ \underset{\sim}{B}$ in terms of elements. Since $\underset{\sim}{A} ⓚ \underset{\sim}{B}$ takes historical precedence, we feel that the name Zehfuss sum should be reserved for $\underset{\sim}{A} ⓚ \underset{\sim}{B}$ of (5), and we suggest the name complete sum for $\underset{\sim}{A} ⓢ \underset{\sim}{B}$ of (6).

8.3. Powers of a matrix

Murnaghan (1938) and Bellman (1970) both name

$$\underset{\sim}{A}^{[t]} = \bigotimes_{i=1}^{t} \underset{\sim}{A} = \underset{\sim}{A} \otimes \underset{\sim}{A} \otimes \cdots \otimes \underset{\sim}{A} \text{ for } t\ \underset{\sim}{A}\text{'s}$$

as the Kronecker t'th power of $\underset{\sim}{A}$. We suggest the name Zehfuss t'th power. Bellman (1970) also has a second expression for this name, which is actually the t'th Schläflian matrix, associated with which Bellman also defines a Kronecker logarithm. Both are misnomers and should, we feel, be known by the name Zehfuss, unless history can reveal Kronecker's direct association with them. We doubt if this will occur.

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