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SPECIAL SECTION: GEOMETRY IN ALGEBRA AND ALGEBRA IN GEOMETRY

On the Hodge conjecture for quasi-smooth intersections in toric varieties

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Abstract

We establish the Hodge conjecture for some subvarieties of a class of toric varieties. First we study quasi-smooth intersections in a projective simplicial toric variety, which is a suitable notion to generalize smooth complete intersection subvarieties in the toric environment, and in particular quasi-smooth hypersurfaces. We show that under appropriate conditions, the Hodge conjecture holds for a very general quasi-smooth intersection subvariety, generalizing the work on quasi-smooth hypersurfaces of the first author and Grassi in Bruzzo and Grassi (Commun Anal Geom 28: 1773–1786, 2020). We also show that the Hodge Conjecture holds asymptotically for suitable quasi-smooth hypersurface in the Noether–Lefschetz locus, where "asymptotically" means that the degree of the hypersurface is big enough, under the assumption that the ambient variety $\mathbb{P}^{2k+1}_{\Sigma}$ has Picard group $\mathbb{Z}$. This extends to a class of toric varieties Otwinowska's result in Otwinowska (J Alg Geom 12: 307–320, 2003).

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1 Introduction

A projective simplicial toric variety $\mathbb{P}^d_\Sigma$ satisfies the Hodge Conjecture, i.e., every cohomology class in $H^{p,p}(\mathbb{P}^d_\Sigma, \mathbb{Q})$ is a linear combination of algebraic cycles. On the one hand, by the Lefschetz hyperplane theorem, the Hodge conjecture holds true for every hypersurface and $p < \frac{d-1}{2}$ and by the hard Lefschetz theorem also for $p > \frac{d-1}{2}$. Moreover, by Theorem 1.1 in [3], when $p = \frac{d-1}{2}$, $d = 2k+1$ and $\mathbb{P}^{2k+1}_\Sigma$ is an Oda variety with an ample class β such that $k\beta - \beta_0$ is nef, where β_0 is the anticanonical class, the Hodge conjecture with rational coefficients holds for a very general hypersurface in the linear system $|\beta|$.

The notion of Oda varieties was introduced in [2]. Let us recall that the Cox ring of a toric variety $\mathbb{P}_\Sigma$ is graded over the class group $\mathrm{Cl}(\mathbb{P}_\Sigma)$, and that one has an injection $\mathrm{Pic}(\mathbb{P}_\Sigma) \to \mathrm{Cl}(\mathbb{P}_\Sigma)$.

Definition 1.1 Let $\mathbb{P}_\Sigma$ be a toric variety with Cox ring S. $\mathbb{P}_\Sigma$ is said to be an Oda variety if the multiplication morphism $S^{\alpha_1} \otimes S^{\alpha_2} \to S^{\alpha_1+\alpha_2}$ is surjective whenever the classes α_1 and α_2 in $\mathrm{Pic}(\mathbb{P}_\Sigma)$ are ample and nef, respectively.

In [15] Mavlyutov proved a Lefschetz type theorem for quasi-smooth intersection subvarieties, and moreover using the "Cayley trick" he related the cohomology of a quasi-smooth subavariety $X = X_{f_1} \cap \cdots \cap X_{f_s} \subset \mathbb{P}^d_\Sigma$ to the cohomology of a quasi-smooth hypersurface $Y \subset \mathbb{P}^{d+s-1}_\Sigma$. This allows us to prove a Noether–Lefschetz type theorem, namely:

Theorem 2.5. *Let $\mathbb{P}^d_\Sigma$ be an Oda projective simplicial toric variety. For a very general quasi-smooth intersection subvariety X* cut off by *$f_1, \dots f_s$ such that $d + s = 2(\ell + 1)$ and*

$$\sum_{i=1}^{s} \deg(f_i) - \beta_0$$

is nef, one has

$$H^{\ell+1-s,\ell+1-s}(X, \mathbb{Q}) = i^*\left(H^{\ell+1-s,\ell+1-s}(\mathbb{P}^d_\Sigma, \mathbb{Q})\right).$$

From this one obtains the following result about the Hodge conjecture for quasi-smooth intersections.

Corollary 2.7. *If $\mathbb{P}^d_\Sigma$ is an Oda projective simplicial toric variety, the Hodge Conjecture holds for a very general quasi-smooth intersection subvariety X cut off by $f_1, \dots f_s$ such that $d + s$ is even and $\sum_{i=1}^{s} \deg(f_i) - \beta_0$ is nef.*

Let T be the open subset of $|\beta|$ corresponding to quasi-smooth hypersurfaces, and let $\mathcal{H}^{2k} = R^{2k}\pi_*\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_T$ be the Hodge bundle on T; here $\pi : \mathcal{X} \to T$ is the tautological family on T, and $d = 2k + 1$. We restrict $\mathcal{H}^{2k}$ to a contractible open subset $U \subset T$. The bundle $\mathcal{H}^{2k}$ has a Hodge decomposition

$$\mathcal{H}^{2k} = \bigoplus_{p+q=2k} \mathcal{H}^{p,q}$$

but this is not holomorphic. On the other hand, the bundles that make up the Hodge filtration

$$F^p\mathcal{H}^{2k} = \bigoplus_{p=0}^{2k} \mathcal{H}^{2k-p,p}$$

are holomorphic; to see this one can use the *period map* (which in particular we write for $p = k$)

$$\mathcal{P}^{k,2k} : U \to \mathrm{Grass}(b_k, H^{2k}(X_{u_0}, \mathbb{C}))$$

where $b_k = \dim F^kH^{2k}(X_{u_0}, \mathbb{C})$ for a fixed point $u_0 \in U$; this map sends $f \in U$ to the subspace $F^kH^{2k}(X_f, \mathbb{C}) \subset H^{2k}(X_f, \mathbb{C}) = H^{2k}(X_{u_0}, \mathbb{C})$. This map is holomorphic (see [14] and [5, Prop. 3.4]). But, by the very definition of the period map (see also [17], Section 10.2.1 for the smooth case)

$$F^k\mathcal{H}^{2k} \simeq (\mathcal{P}^{k,2k})^*\mathcal{U}_k,$$

where $\mathcal{U}_k$ is the tautological bundle on the Grassmannian $\mathrm{Grass}(b_k, H^{2k}(X_{u_0}, \mathbb{C}))$, so that the bundles $F^k\mathcal{H}^{2k}$ are indeed holomorphic.

Pushing ahead the ideas developed in [5] and [4], let λ_f be a nonzero class in the primitive cohomology $H^{k,k}(X_f, \mathbb{Q})/H^{k,k}(\mathbb{P}^{2k+1}_\Sigma, \mathbb{Q})$, and let U be a contractible open subset of T around f, so that $\mathcal{H}^{2k}_{|U}$ is constant. Moreover, let $\lambda \in \mathcal{H}^{2k}(U)$ be the section defined by λ_f and let $\bar{\lambda}$ be its image in $(\mathcal{H}^{2k}/F^k\mathcal{H}^{2k})(U)$. One has

Proposition 1.2 *The local Noether–Lefschetz loci can be defined as*

$$N^{k,\beta}_{\lambda,U} := \{G \in U \mid \bar{\lambda}_G = 0\}$$

where $\beta = \deg(f)$.

The following result is Theorem 1.2 in [4].

Theorem. *Let* $\mathbb{P}^{2k+1}_\Sigma$ *be an Oda variety with an ample class* β *such that* $k\beta - \beta_0 = n\eta$, *where* β_0 *is the anticanonical class,* η *is a primitive ample class, and* $n \in \mathbb{N}$. *Let*

$$m_\beta = \max\{i \in \mathbb{N} \mid i\eta \le \beta\}. \tag{1}$$

For every positive ϵ *there is a positive* δ *such that for every* $m \ge \max(\frac{1}{\delta}, m_\beta)$ *and* $\hat{d} \in [1, m\delta]$, *and every nontrivial Hodge class* $\lambda \in F^k\mathcal{H}^{2k}(U)$ *such that*

$$\operatorname{codim} N^{k,\beta}_{\lambda,U} \leq \hat{d}\frac{m^k_\beta}{k!},$$

for every $f \in N^{k,\beta}_{\lambda,U}$, there exists a k-dimensional variety $V \subset X_f$ with $\deg V \leq (1+\epsilon)\hat{d}$*. Here* $\deg V$ *is taken with respect to the ample divisor η, i.e.,*

$$\deg V = [V] \cdot \eta^k.$$

Based on this, in this paper we obtain the following result.

Theorem 4.2. *Under the same hypotheses of the previous theorem, assume also that* $\mathrm{Pic}(\mathbb{P}^{2k+1}_\Sigma) = \mathbb{Z}$*. Then, if $V \subset X_f$ is a nonempty quasi-smooth intersection subvariety of $\mathbb{P}^{2k+1}_\Sigma$ for some $f \in N^{k,\beta}_{\lambda,U}$, there exists $c \in \mathbb{Q}^*$ such that $\lambda_f = c\lambda_V$, where λ_V is the class of V in $H^{k,k}_{\mathrm{prim}}(X_f, \mathbb{Q})$.*

In other words, λ_f is algebraic.

In his paper [11] A. Dan proves a form of our Theorem 4.2 for smooth hypersurfaces in odd-dimensional projective spaces $\mathbb{P}^{2k+1}$ which is not asymptotic. Although our result is more general in two ways, as we consider *quasi-smooth intersections* in *toric varieties* with $h^{k,k} = 1$ (for instance, weighted or fake projective spaces); however, our result is asymptotic.

In Sect. 3 we give an extension of the notion of Gorenstein ideal to Cox rings; this may have some interest on its own.

2 Very general quasi-smooth intersections

Let $f_1, \dots, f_s$ be homogeneous polynomials in the Cox ring $S = \mathbb{C}[x_1, \dots, x_n]$ of $\mathbb{P}^d_\Sigma$. Their zero locus $V(f_1, \dots, f_s)$ defines a closed subvariety $X \subset \mathbb{P}^d_\Sigma$. Let $U(\Sigma) = \mathbb{A}^n - Z(\Sigma)$, where $Z(\Sigma)$ is the irrelevant locus, i.e., $Z(\Sigma) = \mathrm{Spec}B$, where B is the irrelevant ideal.

Definition 2.1 [15] X is a codimension s quasi-smooth intersection if $V(f_1, \dots, f_s) \cap U(\Sigma)$ is either empty or a smooth intersection subvariety of codimension s in $U(\Sigma)$.

This notion generalizes that of smooth complete intersection in a projective space. For $s = 1$ it reduces to the notion of *quasi-smoooth hypersurface,* see Def. 3.1 in [1]. If we regard $\mathbb{P}^d_\Sigma$ as an orbifold, then an intersection of hypersurfaces $X_{f_1} \cap \dots \cap X_{f_s}$ is quasi-smooth when it is a sub-orbifold of $\mathbb{P}^d_\Sigma$, see Prop 1.3 [15]; heuristically, "X has only singularities coming from the ambient variety."

We also have a Lefschetz type theorem in this context.

Proposition 2.2 ([15] Proposition 1.4) *Let $X \subset \mathbb{P}^d_\Sigma$ be a closed subset, defined by homogeneous polynomials $f_1, \dots f_s \in B$. Then the natural map $i^* : H^i(\mathbb{P}^d_\Sigma) \to H^i(X)$ is an isomorphism for $i < d - s$ and an injection for $i = d - s$. In particular, this is true if the hypersurfaces cut by the polynomials f_i are ample.*

Hence if $p \neq \frac{d-s}{2}$ every cohomology class in $H^{p,p}(X)$ is a linear combination of algebraic cycles. So let us see what happens when $p = \frac{d-s}{2}$. The idea is to relate the Hodge structure of a quasi-smooth intersection variety $X = X_{f_1} \cap \cdots \cap X_{f_s}$ in $\mathbb{P}^d_\Sigma$ with the Hodge structure of a quasi-smooth hypersurface Y in a toric variety $\mathbb{P}^{d+s-1}_{X,\Sigma}$ whose fan depends on X and Σ.

Proposition 2.3 *Let $X = X_{f_1} \cap \cdots \cap X_{f_s}$ be quasi-smooth intersection subvariety in $\mathbb{P}^d_\Sigma$ cut off by homogeneous polynomials $f_1 \ldots f_s$. There exists a projective simplicial toric variety $\mathbb{P}^{d+s-1}_{X,\Sigma}$ and a quasi-smooth hypersurface $Y \subset \mathbb{P}^{d+s-1}_{X,\Sigma}$ such that for $p \neq \frac{d+s-1}{2}, \frac{d+s-3}{2}$*

$$H^{p-1,d+s-1-p}_{\text{prim}}(Y) \simeq H^{p-s,d-p}_{\text{prim}}(X).$$

Proof One constructs $\mathbb{P}^{d+s-1}_{X,\Sigma}$ via the so-called "Cayley trick". Let $L_1, \ldots, L_s$ be the line bundles associated to the quasi-smooth hypersurfaces $X_1, \ldots X_s$, and so let $\mathbb{P}(E)$ be the projective bundle of $E = L_1 \oplus \cdots \oplus L_s$. It turns out that $\mathbb{P}(E)$ is a $d+s-1$-dimensional projective simplicial toric variety whose Cox ring is

$$\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots y_s]$$

where $S = \mathbb{C}[x_1, \ldots, x_n]$ is the Cox ring of $\mathbb{P}^d_\Sigma$. The hypersurface Y is cut off by the polynomial $F = y_1 f_1 + \cdots + y_s f_s$ and is quasi-smooth by Lemma 2.2 in [15]. Moreover, combining Theorem 10.13 in [1] and Theorem 3.6 in [15], we have that

$$H^{p-1,d+s-1-p}_{\text{prim}}(Y) \simeq R(F)_{(d+s-p)\beta-\beta_0} \simeq H^{p-s,d-p}_{\text{prim}}(X)$$

for $p \neq \frac{d+s-1}{2}, \frac{d+s-3}{2}$ as desired. □

Here $R(F)$ is the Jacobian ring of Y, i.e., the quotient of the Cox ring

$$R(F) = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots y_s]/J(F),$$

where $J(F)$ is the ideal generated by the derivatives of F, see [1].

Remark 2.4 With the same notation of Proposition 2.3, note that we have a well defined map

$$\begin{aligned} \phi : |\beta_1| \times \cdots \times |\beta_s| &\to |\beta| \\ (f_1, \ldots, f_s) &\mapsto f_1 y_1 + \cdots + f_s y_s. \end{aligned}$$

Moreover, by the Noether-Lefschetz theorem NL_β is a countable union of closed sets $\bigcup_i C_i$ and hence $\bigcup \phi^{-1}(C_i)$ is too.

We have a Noether-Lefschetz type theorem, namely,

Theorem 2.5 *Let $\mathbb{P}^d_\Sigma$ be an Oda projective simplicial toric variety. Then for a very general quasi-smooth intersection subvariety X cut off by $f_1, \dots f_s$ such that $d + s = 2(l + 1)$ and $\sum_{i=1}^s \deg(f_i) - \beta_0$ is nef, one has that*

$$H^{l+1-s,l+1-s}(X, \mathbb{Q}) = i^* \left(H^{l+1-s,l+1-s}(\mathbb{P}^d_\Sigma, \mathbb{Q})\right)$$

So we get a natural generalization of the Noether-Lefschetz loci.

Definition 2.6 The Noether-Lefschetz locus $NL_{\beta_1,\dots,\beta_s}$ of quasi-smooth intersection varieties is the locus of s–tuples $(f_1, \dots, f_s)$ such that $X = X_{f_1} \cap \dots X_{f_s}$ is quasi-smooth intersection with $f_i \in |\beta_i|$ and $H^{l+1-s,l+1-s}(X, \mathbb{Q}) \neq i^* \left(H^{l+1-s,l+1-s}(\mathbb{P}^d_\Sigma, \mathbb{Q})\right)$.

Now we consider the Hodge conjecture for very general quasi-smooth intersection subvarieties in $\mathbb{P}^d_\Sigma$.

Corollary 2.7 *If $\mathbb{P}^d_\Sigma$ is a Oda projective simplicial toric variety, the Hodge Conjecture holds for a very general quasi-smooth intersection subvariety X cut off by $f_1, \dots f_s$ such that $d + s = 2(l + 1)$ and $\sum_{i=1}^s \deg(f_i) - \beta_0$ is nef.*

Proof First note that by Thereom 4.1 in [12] the projective simplicial toric variety $\mathbb{P}^{2l+1}_{X,\Sigma}$ is Oda and since X is very general the quasi-smooth hypersurface Y is very general as well. So applying the Noether-Lefschetz theorem one has that $h^{l,l}_{\text{prim}}(Y) = 0 = h^{l+1-s,l+1-s}_{\text{prim}}(X)$ or equivalently every $(l + 1 - s, +1 - s)$ cohomology class is a linear combination of algebraic cycles. □

3 Cox-Gorenstein ideals

We shall need a partial generalization of Macaulay's theorem (see e.g. Thm. 6.19 in [18] for the classical theorem). This generalization is basically contained in the work of Cox and Cattani-Cox-Dickenstein [7, 9].

Let S be the Cox ring of a complete simplicial toric variety $\mathbb{P}_\Sigma$. This is graded over the effective classes in the class group $\mathrm{Cl}(\mathbb{P}_\Sigma)$ and [8]

$$S^\alpha \simeq H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(\alpha)).$$

As $\mathcal{O}_{\mathbb{P}_\Sigma}(\alpha)$ is coherent and $\mathbb{P}_\Sigma$ is complete, each S^α is finite-dimensional over $\mathbb{C}$; in particular, $S^0 \simeq \mathbb{C}$.

Lemma 3.1 *For every effective $N \in \mathrm{Cl}(\mathbb{P}_\Sigma)$, the set of classes $\alpha \in \mathrm{Cl}(\mathbb{P}_\Sigma)$ such that $N - \alpha$ is effective is finite.*

Proof Since the torsion submodule of $\mathrm{Cl}(\mathbb{P}_\Sigma)$ is finite, we may assume that $\mathrm{Cl}(\mathbb{P}_\Sigma)$ is free. Then the exact sequence

$$0 \to M \to \mathrm{Div}_{\mathbb{T}}(\mathbb{P}_\Sigma) \to \mathrm{Cl}(\mathbb{P}_\Sigma) \to 0$$

splits, and we may identify $\mathrm{Cl}(\mathbb{P}_\Sigma)$ with a free subgroup of $\mathrm{Div}_{\mathbb{T}}(\mathbb{P}_\Sigma)$, generated by a subset $\{D_1, \dots, D_r\}$ of $\mathbb{T}$-invariant divisors. A class in $\mathrm{Cl}(\mathbb{P}_\Sigma)$ is effective if and only its coefficients on this basis are nonnegative, whence the claim follows. □

We shall give a definition of *Cox-Gorenstein ideal* of the Cox rings which generalizes to toric varieties the definition given by Otwinowska in [16] for projective spaces. Let $B \subset S$ be the irrelevant ideal, and for a graded ideal $I \subset B$, denote by $V_{\mathbb{T}}(I)$ the corresponding closed subscheme of $\mathbb{P}_\Sigma$.

Definition 3.2 *A graded ideal I of S contained in B is said to be a Cox-Gorentstein ideal of socle degree* $N \in \mathrm{Cl}(\mathbb{P}_\Sigma)$ if

1. *there exists a* $\mathbb{C}$*-linear form* $\Lambda \in (S^N)^\vee$ *such that for all* $\alpha \in \mathrm{Cl}(\mathbb{P}_\Sigma)$
$$I^\alpha = \{f \in S^\alpha \mid \Lambda(fg) = 0 \text{ for all } g \in S^{N-\alpha}\}; \tag{2}$$
2. $V_{\mathbb{T}}(I) = \emptyset$.

Remark 3.3 Cox-Gorenstein ideals need not be Artinian. Property 2 in this definition replaces that condition.

Proposition 3.4 *Let* $R = S/I$. *If I is Cox-Gorenstein then*

1. $\dim_{\mathbb{C}} R^N = 1$;
2. *the natural bilinear morphism*
$$R^\alpha \times R^{N-\alpha} \to R^N \simeq \mathbb{C} \tag{3}$$
is nondegenerate whenever α and $N - \alpha$ *are effective.*

Proof

1. From eq. (2) we see that the sequence
$$0 \to I^N \to S^N \xrightarrow{\Lambda} \mathbb{C} \to 0$$
is exact.
2. Define $\Phi : R^\alpha \times R^{N-\alpha} \to \mathbb{C}$ as $\Phi(x, y) = \Lambda(\bar{x}\bar{y})$, where $\bar{x}$, $\bar{y}$ are pre-images of x, y in S. One easily checks that this is well defined and that via the isomorphism $R^N \simeq k$ it coincides with the pairing (3). Now if $x \in R^\alpha$ and $\Phi(x, y) = 0$ for all $y \in R^{N-\alpha}$ then $\Lambda(\bar{x}\bar{y}) = 0$ for all $\bar{y} \in S^{N-\alpha}$ so that $\bar{x} \in I^\alpha$, i.e., $x = 0$. □

Let $f_0, \dots, f_d$ be homogeneous polynomials, $f_i \in S^{\alpha_i}$, where $d = \dim \mathbb{P}_\Sigma$ and each α_i is ample, and let $N = \sum_i \alpha_i - \beta_0$, where β_0 is the anticanonical class of $\mathbb{P}_\Sigma$. Assume that the f_i have no common zeroes in $\mathbb{P}_\Sigma$, i.e., $V_{\mathbb{T}}(I) = \emptyset$ if $I = (f_0, \dots, f_d)$.

In [1, 7, 9] it is shown that for each $G \in S^N$ one can define a meromorphic d-form ξ_G on $\mathbb{P}_\Sigma$ by letting

$$\xi_G = \frac{G\,\Omega}{f_0 \cdots f_d}$$

where Ω is a Euler form on $\mathbb{P}_\Sigma$. The form ξ_G determines a class in $H^d(\mathbb{P}_\Sigma, \omega)$, where ω is the canonical sheaf of $\mathbb{P}_\Sigma$ (the sheaf of Zariski d-forms on $\mathbb{P}_\Sigma$), and in turn the trace morphism $\mathrm{Tr}_{\mathbb{P}_\Sigma} : H^d(\mathbb{P}_\Sigma, \omega) \to \mathbb{C}$ associates a complex number to G, so we can define $\Lambda \in (S^N)^\vee$ as

$$\Lambda(G) = \mathrm{Tr}_{\mathbb{P}_\Sigma}([\xi_G]) \in \mathbb{C}. \tag{4}$$

Finally, we can prove a toric version of Macaulay's theorem.

Theorem 3.5 *The linear map defined in Eq. (4) satisfies the condition in Definition 3.2. Therefore, the ideal $I = (f_0, \dots, f_d)$ is a Cox-Gorenstein ideal of socle degree N.*

Proof By Theorem 6 in [7] the map Λ establishes an isomorphism $R^N \simeq \mathbb{C}$. Hence, if $f \in S^\alpha$ is such that $\Lambda(fg) = 0$ for all $g \in S^{N-\alpha}$, then $fg \in I^N$, which implies $f \in I^\alpha$. On the other hand, it is clear that $\Lambda(fg) = 0$ if $f \in I^\alpha$ and $g \in S^{N-\alpha}$. □

Another example is given in terms of *toric Jacobian ideals*. For every ray $\rho \in \Sigma(1)$ we shall denote by v_ρ its rational generator, and by x_ρ the corresponding variable in the Cox ring. Recall that d is the dimension of the toric variety $\mathbb{P}_\Sigma$, while we denote by $r = \#\Sigma(1)$ the number of rays. Given $f \in S^\beta$ one defines its *toric Jacobian ideal* as

$$J_0(f) = \left(x_{\rho_1} \frac{\partial f}{\partial x_{\rho_1}}, \dots, x_{\rho_r} \frac{\partial f}{\partial x_{\rho_r}} \right).$$

We recall from [1] the definition of nondegenerate hypersurface and some properties (Def. 4.13 and Prop. 4.15).

Definition 3.6 Let $f \in S(\Sigma)^\beta$, with β an ample Cartier class. The associated hypersurface X_f is nondegenerate if for all $\sigma \in \Sigma$ the affine hypersurface $X_f \cap O(\sigma)$ is a smooth codimension one subvariety of the orbit $O(\sigma)$ of the action of the torus $\mathbb{T}^d$.

Proposition 3.7

1. *Every nondegenerate hypersurface is quasi-smooth.*
2. *If f is generic then X_f is nondegenerate.*

The following is part of Prop. 5.3 in [9], with some changes in the terminology.

Proposition 3.8 *Let $f \in S(\Sigma)^{\beta}$, and let $\{\rho_1, \dots, \rho_d\} \subset \Sigma(1)$ be such that $v_{\rho_1}, \dots, v_{\rho_d}$ are linearly independent.*

1. *The toric Jacobian ideal of f coincides with the ideal*

$$\left(f, x_{\rho_1}\frac{\partial f}{\partial x_{\rho_1}}, \dots, x_{\rho_d}\frac{\partial f}{\partial x_{\rho_d}}\right).$$

2. *The following conditions are equivalent:*

(a) *f is nondegenerate;*
(b) *the polynomials $x_{\rho_i}\frac{\partial f}{\partial x_{\rho_i}}$, $i = 1, \dots, r$, do not vanish simultaneously on X_f;*
(c) *the polynomials f and $x_{\rho_i}\frac{\partial f}{\partial x_{\rho_i}}$, $i = 1, \dots, d$,do not vanish simultaneously on X_f.*

3. *If moreover $\beta$is ample and f is nondegenerate, then $J_0(f)$ is a Cox-Gorenstein ideal of socle degree $N = (d+1)\beta - \beta_0$, where $\beta_0$is the anticanonical class of $\mathbb{P}^d_\Sigma$.*

4 Asymptotic Hodge conjecture

In this section we prove Theorem 4.2. Let us recall part of the notation and assumptions of [4]. Let $\mathbb{P}^{2k+1}_\Sigma$ be an Oda variety with an ample Cartier class β such that $k\beta - \beta_0 = n\eta$, where β_0 is the anticanonical class, η is a primitive ample class and $n \in \mathbb{N}$.

We need to define a pre-order in the group

$$N^1(\mathbb{P}^{2k+1}_\Sigma) = \mathrm{Pic}(\mathbb{P}^{2k+1}_\Sigma) \otimes \mathbb{Q}/\text{numerical equivalence},$$

by letting $\alpha < \alpha'$ if $\alpha' - \alpha$ is an effective class.

Let $X_f \in |\beta|$ be a quasi-smooth hypersurface in the Noether-Lefschetz locus associated to a nontrival Hodge class $\lambda \in F^k\mathcal{H}^{2k}(U)$. Again, its degree is computed by intersecting with the ample class η, i.e., $\deg X_f = [X_f] \cdot \eta$. Let r be number of rays of Σ, so that $r \geq 2(k+1)$. Assuming that n is big enough, it follows from Proposition 4.7 or Theorem 6.1 in [4] that there exists a k-dimensional subvariety V of X_f satisfying the following conditions:

- $\deg V \leq 2\delta m_\beta$ with $0 < \delta < \frac{1}{4(r-(k+1))}$ (the number m_β was defined in Eq. (1));
- the graded ideals I_V and

$$E = \{g \in S^\bullet \mid \sum_{i=1}^{b} \lambda_i \int_{\mathrm{Tub}\gamma_i} \frac{gh\Omega_0}{f^{k+1}} = 0 \ \text{ for all } \ h \in S^{N-\bullet}\}, \tag{5}$$

coincide in degree less than or equal to $\big(m_\beta - 2 - (r-j)\deg V\big)\eta$ for some j, with $0 < j < r$. Here Tub$(-)$ is the adjoint of the residue map, and $N = (k+1)\beta - \beta_0$ is the socle degre of the *Cox-Gorenstein ideal E*, while

$$\lambda_f = \left(\sum_{i=1}^{b} \lambda_i \gamma_i\right)^{pd}$$

is the Poincaré dual of some rational combination of the homology cycles γ_i generating $H_{2k}(X_f, \mathbb{Q})$. Moreover, via the isomorphism $T_f U \simeq S^\beta$, the degree β summand E^β of E is identified with the tangent space $T_f N^{k,\beta}_{\lambda,U}$ to the Noether-Lefschetz locus, so that E^β contains the degree β part $J(f)^\beta$ the Jacobian ideal of f.

Lemma 4.1 *The toric Jacobian ideal $J_0(f)$ is contained in E.*

Proof $J_0(f) \subset J(f)$, so that $J_0(f)^\beta \subset J(f)^\beta \subset E^\beta$, and since $J_0(f)$ is generated in degree β, one has $J_0(f) \subset E$. □

We denote by λ_V the class of V in $H^{k,k}_{\text{prim}}(X_f, \mathbb{Q})$. In the following theorem we assume that $\text{Pic}(\mathbb{P}^{2k+1}_\Sigma) = 1$, i.e., that $\mathbb{P}^{2k+1}_\Sigma$ is a (possibly fake) weighted projective space [6, 13] (cf. [10] Exer. 5.1.13). This implies that $h^{p,p}(\mathbb{P}^{2k+1}_\Sigma) = 1$ for all p.

Theorem 4.2 *If V is a quasi-smooth intersection subvariety, there exists $c \in \mathbb{Q}^*$ such that $\lambda_f = c\lambda_V$.*

Proof We divide the proof in three steps.

Step I: $\lambda_V \neq 0$. For clarity, for every cohomology class of a subvariety we denote in the cohomology of which ambient variety we consider it (so we write $[V]_{X_f}$ and $[V]_{\mathbb{P}^{2k+1}_\Sigma}$). Since $V \subset X_f$ is a regular embedding we have

$$\begin{aligned} [V]^2_{X_f} &= \textstyle\int_V c_k(N_{V/X_f}) = \int_V \left[c(N_{V/\mathbb{P}^{2k+1}_\Sigma})/c(N_{X_f/\mathbb{P}^{2k+1}_\Sigma}|_V)\right]_k \\ [8pt] &= \textstyle\int_{\mathbb{P}^{2k+1}_\Sigma} [V]_{\mathbb{P}^{2k+1}_\Sigma} \cup \Xi_k \end{aligned} \tag{6}$$

where Ξ_k is the component in $H^{k,k}(\mathbb{P}^{2k+1}_\Sigma)$ of

$$\Xi = \frac{\prod_i (1 + A_i)}{1 + [X_f]_{\mathbb{P}^{2k+1}_\Sigma}};$$

here $A_1, \dots, A_{k+1}$ are the classes in $\text{Cl}(\mathbb{P}^{2k+1}_\Sigma)$ of the hypersurfaces that cut the quasi-smooth intersection subvariety V. The claim is proved by contradiction: if $[V]_{X_f}$ is the restriction of a class in $H^{k,k}(\mathbb{P}^{2k+1}_\Sigma)$, i.e.,

$$[V]_{X_f} = b\,\eta^k_{|X_f}$$

for some b, then comparing this with (6) we obtain

$$\deg V = m_k \deg X_f, \tag{7}$$

where m_k is defined by $\Xi_k = m_k\,\eta^k$. But (7) cannot be true when $\deg X_f$ is big enough.

Step II. Let E_{alg} and E be the Cox-Gorenstein ideals associated to λ_V and λ_f, respectively, as in Eq. (5). To prove the theorem it is enough to show that $E = E_{\text{alg}}$. Note that $I_V + J_0(f)$ is contained in E and E_{alg}. Moreover, since $V \subset X_f$, and f is quasi-smooth, there exist $K_1, \dots K_{k+1} \in B$ such that $f = A_1K_1 + \dots A_{k+1}K_{k+1}$ and $(A_1, \dots, A_{k+1}, K_1, \dots K_{k+1})$ is a Cox-Gorenstein ideal with socle degree N; this will follow from the next step, which concludes the proof.

Step III. It is enough to show that every Cox-Gorenstein ideal $\mathcal{I}$ of socle degree N containing $I_V + J_0(f)$ also contains $(A_1, \dots, A_{k+1}, K_1, \dots K_{k+1})$. By assumption

$$\left(A_j, j \in \{1, \dots, k+1\}, \sum_{j=1}^{k+1} x_i \frac{\partial A_j}{\partial x_i} K_j, i \in 1, \dots, r\right) \subset \mathcal{I}.$$

Let us see that $K_j \in \mathcal{I}$ for every $j \in \{1, \dots, k+1\}$. Let $M \in \text{Mat}(r \times (k+1))$ be the matrix $[x_i \frac{\partial A_j}{\partial x_i}]$ and K the column $(K_j)_{j\in\{1,\dots,k+1\}}$. Let $I \subset \{1, \dots r\}$ with cardinality $k+1$ and let M_I be the matrix obtained extracting the $i \in I$-arrows of M. We have that $\sum_{j=1}^{k+1} x_i \frac{\partial A_j}{\partial x_i} K_j = (MK)_i = (M_I K)_i$; multiplying by the adjoint of M_I we get that $\det(M_I)K_j \in \mathcal{I}$ for all $j \in \{1, \dots k+1\}$. On one hand the ideal $(\mathcal{I} : K_j)$ contains the ideal

$$\mathcal{J} = I_V + \langle \det M_I \mid I \subset \{1, \dots, r\},\ \#I = k+1 \rangle.$$

Since V is a smooth complete intersection subvariety, it follows that $\mathcal{J}$ is base point free, and therefore it contains a complete intersection Cox-Gorenstein ideal $\mathcal{J}'$ by the toric Macaulay theorem, Theorem 3.5. Since $\mathcal{J}$ is generated in degree less than or equal to $(\deg V)\eta$, we can take $\mathcal{J}'$ with the same property. It follows that

$$soc(\mathcal{J}') \leq 2(k+1)(\deg V)\eta - \beta_0 \leq 2rm_\beta\delta\eta - \beta_0.$$

On the other hand if $K_j \notin \mathcal{I}$ then $(\mathcal{I} : K_j)$ contains a Cox-Gorenstein ideal with socle degree

$$N - \deg K_j \geq N - \beta = k\beta - \beta_0;$$

then comparing the above two inequalities and keeping in mind that $r \geq 2(k+1)$, we get

$$\delta \geq \frac{1}{2r} \geq \frac{1}{4(r-(k+1))},$$

which is absurd. □

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