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ON THE HODGE STRUCTURE OF DEGENERATING HYPERSURFACES IN TORIC VARIETIES

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Abstract. We introduce an algebraic method for describing the Hodge filtration of degenerating hypersurfaces in projective toric varieties. For this purpose, we show several fundamental properties of logarithmic differential forms on proper equivariant morphisms of toric varieties.

1. Introduction. There has been a method to describe the Hodge structure of varieties by their defining equations which originated with Griffiths for the case of hypersurfaces in a projective space [7]. Subsequently, the theory was extended to the case of hypersurfaces in simplicial projective toric varieties by Dolgachev [5], Steenbrink [13], Batyrev and Cox [1]. The purpose of this paper is to apply their idea to degenerating families of hypersurfaces in projective toric varieties.

Let $\pi : \boldsymbol{P} \to \boldsymbol{A}$ be a proper surjective equivariant morphism of toric varieties over an algebraically closed field k. Here we assume for simplicity that $\boldsymbol{P}$ is nonsingular, $\boldsymbol{A}$ is an affine space $\boldsymbol{A} = \operatorname{Spec} k[t_1, \dots, t_m]$, and the characteristic of k is 0. Let X be a hypersurface in $\boldsymbol{P}$. When π is flat and geometrically connected, it gives a trivial fibration of a nonsingular complete toric variety over the open torus of $\boldsymbol{A}$, and degenerated fibers appear at the outside of the open torus. Hence we can consider $X \to \boldsymbol{A}$ to be a degenerating family of hypersurfaces in the complete toric variety. We define a Jacobian ring for the family, and describe the Hodge filtration of the family by using the Jacobian ring.

Let $\{D_1, \dots, D_s\}$ be the set of all prime divisors invariant under the torus action on $\boldsymbol{P}$. The homogeneous coordinate ring of $\boldsymbol{P}$ is defined in [2, §1] as a polynomial ring $S_{\boldsymbol{P}} = k[z_1, \dots, z_s]$ which has a grading valued in the divisor class group $\mathrm{Cl}(\boldsymbol{P})$;

$$\deg z_i = [D_i] \in \mathrm{Cl}(\boldsymbol{P}) .$$

We can assume that $\pi(D_i) = \boldsymbol{A}$ for $1 \le i \le r$, and that $\pi(D_j)$ is contained in the divisor $\{t_1 \cdots t_m = 0\}$ for $r+1 \le j \le s$. The hypersurface X is defined by a $\mathrm{Cl}(\boldsymbol{P})$-homogeneous polynomial $F \in S_{\boldsymbol{P}}$. Then we define the Jacobian ring of X over $\boldsymbol{A}$ by

$$R_{X/\boldsymbol{A}} = S_{\boldsymbol{P}} \Big/ \left(\frac{\partial F}{\partial z_1}, \dots, \frac{\partial F}{\partial z_r} \right),$$

which is a $\mathrm{Cl}(\boldsymbol{P})$-graded $k[t_1, \dots, t_m]$-algebra. For $\beta \in \mathrm{Cl}(\boldsymbol{P})$, the degree β part of $R_{X/\boldsymbol{A}}$ is denoted by $R^{\beta}_{X/\boldsymbol{A}}$, which is a finitely generated $k[t_1, \dots, t_m]$-module.

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The Hodge filtration of the degenerating family is defined by using the sheaf of relative logarithmic differential forms, so we consider the situation above with log structure [9]. We define a log structure on $\boldsymbol{P}$ by the divisor $E = \sum_{j=r+1}^{s} D_j$, and define a log structure on $\boldsymbol{A}$ by the divisor $\{t_1 \cdots t_m = 0\}$. Then π is a log smooth morphism, and the log structure of the general fiber is trivial. The sheaf of relative logarithmic differential p-forms is denoted by $\omega^p_{\boldsymbol{P}/\boldsymbol{A}}$, which is a locally free $\mathcal{O}_{\boldsymbol{P}}$-modules. We define a log structure on X by the restriction of the log structure on $\boldsymbol{P}$. The next theorem is our main result, where we need not assume that π is flat and geometrically connected.

THEOREM 1 (Theorem 4.4). *If X is ample and log smooth over an affine open subvariety $U = \operatorname{Spec} A_U$ of $\boldsymbol{A}$, then for $0 \leq p \leq n-1$, there is a natural isomorphism of A_U-modules*

$$H^{n-p-1}(\boldsymbol{P}_U, \omega^{p+1}_{\boldsymbol{P}/\boldsymbol{A}}(\log X)) \simeq A_U \otimes_{k[t_1,\dots,t_m]} R^{[(n-p)X-D]}_{X/\boldsymbol{A}},$$

where $n = \dim \boldsymbol{P} - \dim \boldsymbol{A}$, $\boldsymbol{P}_U = U \times_{\boldsymbol{A}} \boldsymbol{P}$ and $D = \sum_{i=1}^{r} D_i$.

If $m = 0$, then $\boldsymbol{P}$ is a nonsingular complete variety, and the log structure on $\boldsymbol{P}$ is trivial. For an ample smooth hypersurface X in $\boldsymbol{P}$, the isomorphism in Theorem 1 is

$$H^{n-p-1}(\boldsymbol{P}, \Omega^{p+1}_{\boldsymbol{P}}(\log X)) \simeq R^{[(n-p)X-D]}_{X},$$

which was proved in [1, Theorem 10.6]. Namely, Theorem 1 is a generalization of the result of Batyrev and Cox.

When π is the composite of the blowing up $\boldsymbol{P} \to \boldsymbol{A}^1 \times \boldsymbol{P}^n$ at a point and the first projection, then the log smooth family $X_U \to U$ is a semistable degeneration of hypersurfaces in $\boldsymbol{P}^n$. This example was studied by Saito in [12], which is the first work in which the Hodge filtration of degenerating hypersurfaces are described by Jacobian rings.

The key of the proof of Theorem 1 is the following two fundamental properties of the sheaf of relative logarithmic differential forms on $\boldsymbol{P}$. The first property is a generalization of the Bott vanishing theorem:

THEOREM 2 (Corollary 3.8). *If $\mathcal{L}$ is an ample invertible sheaf on $\boldsymbol{P}$, then for $p \geq 0$ and $q \geq 1$,*

$$H^q(\boldsymbol{P}, \omega^p_{\boldsymbol{P}/\boldsymbol{A}} \otimes_{\mathcal{O}_{\boldsymbol{P}}} \mathcal{L}) = 0\,.$$

The second property is a generalization of the Euler exact sequence:

THEOREM 3 (Theorem 3.11). *There is an exact sequence of $\mathcal{O}_{\boldsymbol{P}}$-modules*

$$0 \to \omega^1_{\boldsymbol{P}/\boldsymbol{A}} \to \bigoplus_{i=1}^{r} \mathcal{O}_{\boldsymbol{P}}(-D_i) \to \mathcal{O}_{\boldsymbol{P}} \otimes_{\boldsymbol{Z}} \mathrm{Cl}\,(\boldsymbol{P} \setminus E) \to 0\,.$$

We prove Theorem 2 and Theorem 3 by using the Poincaré residue map for the sheaf of relative logarithmic differential forms, that is the idea of Batyrev and Cox [1].

This paper proceeds as follows. In Section 2, we consider invertible sheaves on a toric variety with a proper equivariant morphism to an affine toric variety, and characterize base point freeness and ampleness of the invertible sheaves in terms of the support functions on the fan. In the case of invertible sheaves on a complete toric variety, this is a well-known fact. In Section 3, we introduce the logarithmic differential forms on a simplicial toric variety with an equivariant morphism to an affine toric variety. Under the assumption that the toric variety is simplicial and satisfies a condition which depends on the characteristic of the base field, we construct the Poincaré residue map for the sheaf of relative logarithmic differential forms. Using the Poincaré residue map, we prove the Bott vanishing theorem and the Euler exact sequence. In Section 4, we consider hypersurfaces in a nonsingular toric variety with a log smooth proper equivariant morphism to an affine toric variety. We define the Jacobian rings for hypersurfaces over the affine toric variety, and prove the main result which describes the cohomology of the sheaf of relative logarithmic differential forms by the Jacobian rings.

2. Invertible sheaves on toric varieties. First we introduce basic notation used in this paper, and then prove some properties of invertible sheaves on a toric variety with a proper equivariant morphism to an affine toric variety. We refer to [6] and [11] for terminology and basic facts in toric geometry.

Let N be a free $\boldsymbol{Z}$-module of finite rank d. We denote by $N_{\boldsymbol{R}}$ the $\boldsymbol{R}$-vector space $\boldsymbol{R} \otimes_{\boldsymbol{Z}} N$, denote by M the dual $\boldsymbol{Z}$-module of N, and denote by $\langle\ ,\ \rangle : M_{\boldsymbol{R}} \times N_{\boldsymbol{R}} \to \boldsymbol{R}$ the canonical bilinear form. Let σ be a strongly convex rational polyhedral cone in $N_{\boldsymbol{R}}$. The dual cone $\sigma^\vee$ is defined by

$$\sigma^\vee = \{u \in M_{\boldsymbol{R}} \mid \langle u, v\rangle \geq 0 \quad \text{for any} \quad v \in \sigma\}\,,$$

and we denote by $\boldsymbol{A}_\sigma$ the affine toric variety $\operatorname{Spec} k[M \cap \sigma^\vee]$ associated to σ over an algebraically closed field k. Let Σ be a finite fan of strongly convex rational polyhedral cones in $N_{\boldsymbol{R}}$. We denote by $|\Sigma|$ the support $\bigcup_{\sigma\in\Sigma}\sigma$ of Σ, and denote by $\boldsymbol{P}_\Sigma$ the toric variety $\bigcup_{\sigma\in\Sigma}\boldsymbol{A}_\sigma$ associated to Σ over k. Then the algebraic torus $\boldsymbol{T}_N = \operatorname{Spec} k[M]$ is contained in $\boldsymbol{P}_\Sigma$ as an open subvariety, and $\boldsymbol{T}_N$ acts on $\boldsymbol{P}_\Sigma$ as an extension of the translations of $\boldsymbol{T}_N$. For $u \in M$, the corresponding character

$$\chi^u : \boldsymbol{T}_N \to \boldsymbol{G}_{\mathrm{m}} = \operatorname{Spec} k[\boldsymbol{Z}]$$

is considered to be a rational function on $\boldsymbol{P}_\Sigma$.

For each $0 \leq r \leq d$, we denote by $\Sigma(r)$ the set of all r-dimensional cones in Σ. For $\tau \in \Sigma(r)$, we denote by N_τ the free $\boldsymbol{Z}$-module $N/(N \cap \tau_{\boldsymbol{R}})$, where $\tau_{\boldsymbol{R}}$ is the subspace of $N_{\boldsymbol{R}}$ generated by τ over $\boldsymbol{R}$, and we define the set Σ_τ of cones in $N_{\tau,\boldsymbol{R}}$ by

$$\Sigma_\tau = \{[\sigma]_\tau \mid \sigma \in \Sigma \text{ and } \sigma \supset \tau\}\,,$$

where $[\sigma]_\tau$ is the image of σ by the natural homomorphism

$$N_{\boldsymbol{R}} \to N_{\tau,\boldsymbol{R}}\,; \quad v \mapsto [v]_\tau\,.$$

Then Σ_τ is a finite fan of strongly convex rational polyhedral cones in $N_{\tau,\boldsymbol{R}}$, and the associated toric variety $\boldsymbol{P}_{\Sigma_\tau}$ can be considered as a $\boldsymbol{T}_N$-invariant closed subvariety of codimension r in

$\boldsymbol{P}_\Sigma$. The closed immersion $\iota_\tau : \boldsymbol{P}_{\Sigma_\tau} \to \boldsymbol{P}_\Sigma$ is induced by

$$\iota_\tau^* : k[M \cap \sigma^\vee] \to k[M_\tau \cap [\sigma]_\tau^\vee]; \quad \chi^u \mapsto \begin{cases} \chi_\tau^u & \text{if } u \in M \cap \tau^\perp \cap \sigma^\vee\,, \\ 0 & \text{if } u \notin M \cap \tau^\perp \cap \sigma^\vee\,, \end{cases}$$

for $\sigma \supset \tau$, where $\chi_\tau^u : \boldsymbol{T}_{N_\tau} \to \boldsymbol{G}_{\mathrm{m}}$ is the character corresponding to $u \in M_\tau = M \cap \tau^\perp$.

Let $B = \sum_{\rho \in \Sigma(1)} b_\rho \boldsymbol{P}_{\Sigma_\rho}$ be a $\boldsymbol{T}_N$-invariant Weil divisor on $\boldsymbol{P}_\Sigma$. We define the convex subset Δ_B of $M_{\boldsymbol{R}}$ by

$$\Delta_B = \{u \in M_{\boldsymbol{R}} \mid \langle u, v_\rho\rangle + b_\rho \geq 0 \ \text{ for any } \rho \in \Sigma(1)\}\,,$$

where v_ρ is the generator of the monoid $\rho \cap N$. Then there is a natural isomorphism

$$H^0(\boldsymbol{P}_\Sigma, \mathcal{O}_{\boldsymbol{P}_\Sigma}(B)) \simeq \bigoplus_{u \in M \cap \Delta_B} k \cdot \chi^u\,.$$

Let $h : |\Sigma| \to \boldsymbol{R}$ be a Σ-linear support function. We define the $\boldsymbol{T}_N$-invariant Cartier divisor D_h on $\boldsymbol{P}_\Sigma$ by $D_h = -\sum_{\rho \in \Sigma(1)} h(v_\rho)\boldsymbol{P}_{\Sigma_\rho}$.

Let N' be another finitely generated free $\boldsymbol{Z}$-module, and let $\pi_* : N \to N'$ be a homomorphism of $\boldsymbol{Z}$-modules. We denote by $\pi_{*\boldsymbol{R}} : N_{\boldsymbol{R}} \to N'_{\boldsymbol{R}}$ the $\boldsymbol{R}$-homomorphism induced by π_*, and denote by $\pi^* : M' \to M$ the dual homomorphism of π_*. Then π^* induces a homomorphism of algebraic tori $\pi_0 : \boldsymbol{T}_N \to \boldsymbol{T}_{N'}$. If a strongly convex rational polyhedral cone σ' in $N'_{\boldsymbol{R}}$ satisfies the condition $|\Sigma| \subset \pi_{*\boldsymbol{R}}^{-1}(\sigma')$, then π^* induces an equivariant morphism of toric varieties $\pi_{\Sigma,\sigma'} : \boldsymbol{P}_\Sigma \to \boldsymbol{A}_{\sigma'}$, which is an extension of π_0.

REMARK 2.1. The morphism $\pi_{\Sigma,\sigma'}$ is proper if and only if $|\Sigma| = \pi_{*\boldsymbol{R}}^{-1}(\sigma')$.

REMARK 2.2. If $\pi_{\Sigma,\sigma'}$ is a proper morphism, then $|\Sigma|$ is a convex subset in $N_{\boldsymbol{R}}$. Conversely, if $|\Sigma|$ is a convex subset in $N_{\boldsymbol{R}}$, then we can find a free $\boldsymbol{Z}$-module N', a surjective homomorphism $\pi_* : N \to N'$, and a strongly convex rational polyhedral cone σ' in $N'_{\boldsymbol{R}}$ satisfying $|\Sigma| = \pi_{*\boldsymbol{R}}^{-1}(\sigma')$.

THEOREM 2.3. *Let $\boldsymbol{P}_\Sigma$ be a toric variety, $\boldsymbol{A}_{\sigma'}$ an affine toric variety, and $\pi : \boldsymbol{P}_\Sigma \to \boldsymbol{A}_{\sigma'}$ a proper equivariant morphism. For a Σ-linear support function h, the following conditions are equivalent*:

(1) *$\mathcal{O}_{\boldsymbol{P}_\Sigma}(D_h)$ is generated by global sections.*
(2) *$\mathcal{O}_{\boldsymbol{P}_\Sigma}(D_h)$ is π-nef.*
(3) *h is upper convex, i.e., for any $v_1, v_2 \in |\Sigma|$,*

$$h(v_1 + v_2) \geq h(v_1) + h(v_2)\,.$$

PROOF. (1) $\Rightarrow$ (2). Let C be a complete integral curve in a fiber of π. We denote by $\tilde{C}$ the normalization of C, and denote by ι the morphism from $\tilde{C}$ to $\boldsymbol{P}_\Sigma$. Since $\mathcal{O}_{\boldsymbol{P}_\Sigma}(D_h)$ is generated by global sections, $\iota^*\mathcal{O}_{\boldsymbol{P}_\Sigma}(D_h)$ has a non-zero global section, so the intersection number is

$$(D_h . C) = \deg \iota^*\mathcal{O}_{\boldsymbol{P}_\Sigma}(D_h) \geq 0\,.$$

(2) $\Rightarrow$ (3). The morphism π is defined as $\pi = \pi_{\Sigma,\sigma'}$ by a homomorphism $\pi_* : N \to N'$. We denote by N'' the image of π_*, and define the strongly convex rational polyhedral

cone σ'' in $N''_{\boldsymbol{R}}$ by $\sigma'' = \sigma' \cap N''_{\boldsymbol{R}}$. Then π is factored to the composite of the proper surjective morphism $\pi_{\Sigma,\sigma''} : \boldsymbol{P}_\Sigma \to \boldsymbol{A}_{\sigma''}$ and the finite morphism $\boldsymbol{A}_{\sigma''} \to \boldsymbol{A}_{\sigma'}$. We denote by s the dimension of the convex subset

$$|\Sigma| = \pi_{*\boldsymbol{R}}^{-1}(\sigma') = \pi_{*\boldsymbol{R}}^{-1}(\sigma'') \subset N_{\boldsymbol{R}},$$

and define the subset $\Sigma(\sigma'', s-1)$ of $\Sigma(s-1)$ by

$$\Sigma(\sigma'', s-1) = \{\tau \in \Sigma(s-1) \mid \pi_{*\boldsymbol{R}}(\tau) \cap \mathrm{Int}(\sigma'') \neq \emptyset\},$$

where $\mathrm{Int}(\sigma'')$ denotes the relative interior of σ''. For $\tau \in \Sigma(\sigma'', s-1)$, there exist exactly two cones $\sigma_+, \sigma_- \in \Sigma(s)$ containing τ. Then $\pi_{\Sigma,\sigma''} \circ \iota_\tau : \boldsymbol{P}_{\Sigma_\tau} \to \boldsymbol{A}_{\sigma''}$ gives a $\boldsymbol{P}^1$-bundle over the closed $\boldsymbol{T}_{N''}$-orbit $\boldsymbol{T}_{N''_{\sigma''}} = \mathrm{Spec}\, k[M'' \cap \sigma''^{\perp}]$ in $\boldsymbol{A}_{\sigma''}$, while $\boldsymbol{P}_{\Sigma_{\sigma_+}}$ and $\boldsymbol{P}_{\Sigma_{\sigma_-}}$ become the 0-section and the ∞-section of the $\boldsymbol{P}^1$-bundle. Let $p \in \boldsymbol{T}_{N''_{\sigma''}}(k)$ be a k-rational point. We denote by $\boldsymbol{P}_{\Sigma_\tau,p}$ the fiber of $\pi_{\Sigma,\sigma''} \circ \iota_\tau$ at p. Then $\boldsymbol{P}_{\Sigma_\tau,p}$ is a nonsingular rational curve in a fiber of π. Since $\rho_- = [\sigma_-]_\tau$ is a 1-dimensional cone in $N_{\tau,\boldsymbol{R}}$, there is a unique generator v_{ρ_-} of the monoid $\rho_- \cap N_\tau$. For $v \in \sigma_- \setminus \tau$, we define the positive real number a_v by

$$[v]_\tau = a_v v_{\rho_-} \in \rho_- \subset N_{\tau,\boldsymbol{R}} = N_{\boldsymbol{R}}/\tau_{\boldsymbol{R}}.$$

For $\sigma \in \Sigma(s)$, we choose an element $u_\sigma \in M$ which coincides with h on σ as a linear function. Then we have an equation

$$(D_h . \boldsymbol{P}_{\Sigma_\tau,p}) = \frac{1}{a_v}(\langle u_{\sigma_+}, v\rangle - h(v))$$

for any $v \in \sigma_- \setminus \tau$. Since $\mathcal{O}_{\boldsymbol{P}_\Sigma}(D_h)$ is π-nef, we have

$$\langle u_{\sigma_+}, v\rangle \geq h(v) \tag{2.1}$$

for any $v \in \sigma_-$.

Now we prove $\langle u_\sigma, v\rangle \geq h(v)$ for any $\sigma \in \Sigma(s)$ and $v \in |\Sigma|$. For $\sigma \in \Sigma(s)$ and $v \in |\Sigma|$, there exists a vector $w \in \mathrm{Int}(\sigma)$ such that

$$L(w, v) = \{(1-t)w + tv \in N_{\boldsymbol{R}} \mid 0 < t < 1\}$$

has no intersection with $\bigcup_{\tau \in \Sigma(s-2)} \tau$. Then $\bigcup_{\tau \in \Sigma(\sigma'', s-1)} \tau$ divides $L(w, v)$ into finite pieces, and we define vectors $w_0, \ldots, w_l \in N_{\boldsymbol{R}}$ by

$$L(w, v) \cap \bigcup_{\tau \in \Sigma(\sigma'', s-1)} \tau = \{w_1, \ldots, w_{l-1}\}$$

and

$$w_i = (1-t_i)w + t_i v, \quad 0 = t_0 < t_1 < \cdots < t_{l-1} < t_l = 1.$$

For $1 \leq i \leq l$, there exists a unique cone $\sigma_i \in \Sigma(s)$ such that $w_{i-1}, w_i \in \sigma_i$. We note that $w = w_0$, $v = w_l$ and $\sigma = \sigma_1$. Since

$$h(w_i) = \langle u_{\sigma_i}, w_i\rangle = \langle u_{\sigma_{i+1}}, w_i\rangle, \quad 1 \leq i \leq l-1,$$

we have

$$\begin{aligned}\langle u_\sigma, v\rangle &= \langle u_{\sigma_l}, v\rangle + \sum_{i=1}^{l-1}\langle u_{\sigma_i} - u_{\sigma_{i+1}}, v\rangle \\ &= h(v) + \sum_{i=1}^{l-1}\left\langle u_{\sigma_i} - u_{\sigma_{i+1}}, \frac{1-t_i}{t_{i+1}-t_i}w_{i+1} + \frac{t_{i+1}-1}{t_{i+1}-t_i}, w_i\right\rangle \\ &= h(v) + \sum_{i=1}^{l-1}\frac{1-t_i}{t_{i+1}-t_i}(\langle u_{\sigma_i}, w_{i+1}\rangle - h(w_{i+1}))\,.\end{aligned}$$

By using (2.1) for $\sigma_+ = \sigma_i$ and $\sigma_- = \sigma_{i+1}$, we have

$$\langle u_{\sigma_i}, w_{i+1}\rangle \geq h(w_{i+1})\,, \quad 1 \leq i \leq l-1\,.$$

Hence

$$\langle u_\sigma, v\rangle \geq h(v) \tag{2.2}$$

for any $\sigma \in \Sigma(s)$ and $v \in |\Sigma|$.

For $v_1, v_2 \in |\Sigma|$, there is a cone $\sigma \in \Sigma(s)$ such that $v_1 + v_2 \in \sigma$. By (2.2),

$$h(v_1+v_2) = \langle u_\sigma, v_1+v_2\rangle = \langle u_\sigma, v_1\rangle + \langle u_\sigma, v_2\rangle \geq h(v_1) + h(v_2)\,.$$

(3) $\Rightarrow$ (1). For $\sigma \in \Sigma(s)$ and $v \in |\Sigma|$, there exists $w \in \sigma$ such that $v + w$ is contained in σ. Because h is upper convex, we have

$$\langle u_\sigma, v\rangle = \langle u_\sigma, v+w\rangle - \langle u_\sigma, w\rangle = h(v+w) - h(w) \geq h(v)\,,$$

so χ^{u_σ} gives a global section of $\mathcal{O}_{\boldsymbol{P}_\Sigma}(D_h)$. Since $\boldsymbol{P}_\Sigma = \bigcup_{\sigma\in\Sigma(s)}\boldsymbol{A}_\sigma$, and χ^{u_σ} generates $\Gamma(\boldsymbol{A}_\sigma, \mathcal{O}_{\boldsymbol{P}_\Sigma}(D_h))$ over $k[M\cap\sigma^\vee]$, the invertible sheaf $\mathcal{O}_{\boldsymbol{P}_\Sigma}(D_h)$ is generated by global sections. □

THEOREM 2.4. *Let $\boldsymbol{P}_\Sigma$ be a toric variety, $\boldsymbol{A}_{\sigma'}$ an affine toric variety, and $\pi : \boldsymbol{P}_\Sigma \to \boldsymbol{A}_{\sigma'}$ a proper equivariant morphism. For a Σ-linear support function h, the following conditions are equivalent*:

(1) *$\mathcal{O}_{\boldsymbol{P}_\Sigma}(D_h)$ is ample.*
(2) *$\mathcal{O}_{\boldsymbol{P}_\Sigma}(D_h)$ is π-ample.*
(3) *h is strictly upper convex, i.e., for any $v_1, v_2 \in |\Sigma|$,*

$$h(v_1+v_2) \geq h(v_1) + h(v_2)\,,$$

and equality holds if and only if there exists a cone $\sigma \in \Sigma$ such that $v_1, v_2 \in \sigma$.

PROOF. The equivalence of (1) and (2) is well-known for any proper morphism to an affine scheme.

(2) $\Rightarrow$ (3). In the proof of Theorem 2.3 (2) $\Rightarrow$ (3), if $\mathcal{O}_{\boldsymbol{P}_\Sigma}(D_h)$ is π-ample, then $(D_h\,.\,\boldsymbol{P}_{\Sigma_\tau,p}) > 0$, and hence $\langle u_{\sigma_+}, v\rangle > h(v)$ for any $v \in \sigma_- \setminus \tau$. This implies that $\langle u_\sigma, v\rangle \geq h(v)$ for any $\sigma \in \Sigma(s)$ and $v \in |\Sigma|$, and equality holds if and only if $v \in \sigma$.

For $v_1, v_2 \in |\Sigma|$, there is a cone $\sigma \in \Sigma(s)$ such that $v_1 + v_2 \in \sigma$. If

$$h(v_1 + v_2) = h(v_1) + h(v_2)\,,$$

then

$$(\langle u_\sigma, v_1\rangle - h(v_1)) + (\langle u_\sigma, v_2\rangle - h(v_2)) = 0\,.$$

So we have $\langle u_\sigma, v_1\rangle = h(v_1)$ and $\langle u_\sigma, v_2\rangle = h(v_2)$, which means that $v_1, v_2 \in \sigma$.

(3) $\Rightarrow$ (1). Let $\mathcal{E}$ be a coherent $\mathcal{O}_{\boldsymbol{P}_\Sigma}$-module. We show that there exists a positive integer m_0 such that $\mathcal{E} \otimes_{\mathcal{O}_{\boldsymbol{P}_\Sigma}} \mathcal{O}_{\boldsymbol{P}_\Sigma}(mD_h)$ is generated by global sections for any integer $m \geq m_0$. We may assume that $\mathcal{E} = \mathcal{O}_{\boldsymbol{P}_\Sigma}(B)$ for a $\boldsymbol{T}_N$-invariant Weil divisor $B = \sum_{\rho\in\Sigma(1)} b_\rho \boldsymbol{P}_{\Sigma_\rho}$, because there exists a surjective homomorphism $\bigoplus_{i=1}^r \mathcal{O}_{\boldsymbol{P}_\Sigma}(B_i) \to \mathcal{E}$ for some $\boldsymbol{T}_N$-invariant Weil divisors $B_1, \ldots, B_r$, by [10, Corollary 1.2].

For $\sigma \in \Sigma(s)$, we fix vectors $u_{\sigma,1}, \ldots, u_{\sigma,c_\sigma} \in M$ such that $\chi^{u_{\sigma,1}}, \ldots, \chi^{u_{\sigma,c_\sigma}}$ generate $\Gamma(\boldsymbol{A}_\sigma, \mathcal{O}_{\boldsymbol{P}_\Sigma}(B))$ over $k[M \cap \sigma^\vee]$. Since h is strictly upper convex, for $\sigma \in \Sigma(s)$ and $\rho \in \Sigma(1)$, we have $\langle u_\sigma, v_\rho\rangle \geq h(v_\rho)$, and if $v_\rho \notin \sigma$, then $\langle u_\sigma, v_\rho\rangle > h(v_\rho)$. Also, if $v_\rho \in \sigma$, then $\langle u_{\sigma,i}, v_\rho\rangle + b_\rho \geq 0$. Hence there exists a positive integer m_0 such that for any $\sigma \in \Sigma(s)$, for any $1 \leq i \leq c_\sigma$ and for any $\rho \in \Sigma(1)$,

$$m_0(\langle u_\sigma, v_\rho\rangle - h(v_\rho)) \geq -b_\rho - \langle u_{\sigma,i}, v_\rho\rangle\,.$$

Then we have

$$\langle u_{\sigma,i} + mu_\sigma, v_\rho\rangle \geq -b_\rho + mh(v_\rho)$$

for any $m \geq m_0$, and this means that $\chi^{u_{\sigma,i}+mu_\sigma}$ is a global section of the coherent sheaf $\mathcal{O}_{\boldsymbol{P}_\Sigma}(B + mD_h)$. Since $\chi^{u_{\sigma,1}+mu_\sigma}, \ldots, \chi^{u_{\sigma,c_\sigma}+mu_\sigma}$ generate $\Gamma(\boldsymbol{A}_\sigma, \mathcal{O}_{\boldsymbol{P}_\Sigma}(B + mD_h))$ over $k[M \cap \sigma^\vee]$, the coherent sheaf $\mathcal{O}_{\boldsymbol{P}_\Sigma}(B + mD_h)$ is generated by global sections. □

3. Log differential forms on toric varieties. We introduce the sheaf of relative logarithmic differential forms on a toric variety with an equivariant morphism to an affine toric variety.

Let $\boldsymbol{P} = \boldsymbol{P}_\Sigma$ be a toric variety, let $\boldsymbol{A} = \boldsymbol{A}_{\sigma'}$ be an affine toric variety, and let $\pi = \pi_{\Sigma,\sigma'} : \boldsymbol{P} \to \boldsymbol{A}$ be an equivariant morphism, which is given by a homomorphism $\pi_* : N \to N'$ with $|\Sigma| \subset \pi_{*\boldsymbol{R}}^{-1}(\sigma')$. For $\tau \in \Sigma$, we denote by $\boldsymbol{P}_\tau$ the corresponding $\boldsymbol{T}_N$-invariant subvariety $\boldsymbol{P}_{\Sigma_\tau}$. We define the subfan Σ_π of Σ by

$$\Sigma_\pi = \{\tau \in \Sigma \mid \tau \subset \pi_{*\boldsymbol{R}}^{-1}(0)\}\,.$$

If $\tau \in \Sigma_\pi$, then $\pi_\tau = \pi \circ \iota_\tau : \boldsymbol{P}_\tau \to \boldsymbol{A}$ is an equivariant morphism induced by

$$N_\tau \to N'; \ [v]_\tau \mapsto \pi_*(v)\,.$$

We denote by Σ^{reg} the set of all nonsingular cones in Σ, and denote by $j : \boldsymbol{P}^{\mathrm{reg}} = \boldsymbol{P}_{\Sigma^{\mathrm{reg}}} \to \boldsymbol{P}$ the natural open immersion. We define the $\boldsymbol{T}_N$-invariant divisors D and E on $\boldsymbol{P}$ by

$$D = \sum_{\rho\in\Sigma_\pi(1)} \boldsymbol{P}_\rho, \quad E = \sum_{\rho\in\Sigma(1)\smallsetminus\Sigma_\pi(1)} \boldsymbol{P}_\rho\,, \tag{3.1}$$

which are divisors with normal crossings on $\boldsymbol{P}^{\mathrm{reg}}$.

Let σ be a cone in $\Sigma^{\mathrm{reg}}(c)$, and let $(v_1, \dots, v_d)$ be a $\mathbf{Z}$-basis of N with

$$\begin{cases} \sigma = \boldsymbol{R}_{\geq 0} v_1 + \cdots + \boldsymbol{R}_{\geq 0} v_c \,, \\ v_1, \dots, v_l \in \pi_*^{-1}(0) \,, \quad v_{l+1}, \dots, v_c \notin \pi_*^{-1}(0) \,. \end{cases} \tag{3.2}$$

Then we have

$$\begin{aligned} \boldsymbol{A}_\sigma &= \operatorname{Spec} k[M \cap \sigma^\vee] = \operatorname{Spec} k[x_1, \dots, x_c, x_{c+1}^{\pm 1}, \dots, x_d^{\pm 1}] \,, \\ \boldsymbol{A}_\sigma \cap D &= \operatorname{Spec} k[x_1, \dots, x_c, x_{c+1}^{\pm 1}, \dots, x_d^{\pm 1}]/(x_1 \cdots x_l) \,, \\ \boldsymbol{A}_\sigma \cap E &= \operatorname{Spec} k[x_1, \dots, x_c, x_{c+1}^{\pm 1}, \dots, x_d^{\pm 1}]/(x_{l+1} \cdots x_c) \,, \end{aligned}$$

where $x_i = \chi^{u_i}$ for the dual basis $(u_1, \dots, u_d)$ of $(v_1, \dots, v_d)$. We define a free $k[M \cap \sigma^\vee]$-module $\omega^1_{\boldsymbol{A}_\sigma}$ by

$$\omega^1_{\boldsymbol{A}_\sigma} = \Omega^1_{\boldsymbol{A}_\sigma}(\log E) = \bigoplus_{j=1}^{l} k[M \cap \sigma^\vee] dx_j \oplus \bigoplus_{j=l+1}^{d} k[M \cap \sigma^\vee] \frac{dx_j}{x_j} \,,$$

which is naturally contained in the free $k[M \cap \sigma^\vee]$-module

$$\Omega^1_{\boldsymbol{A}_\sigma}(\log D \cup E) = \bigoplus_{j=1}^{d} k[M \cap \sigma^\vee] \frac{dx_j}{x_j} \,.$$

We denote by $\omega^1_{\boldsymbol{P}^{\mathrm{reg}}} \subset \Omega^1_{\boldsymbol{P}^{\mathrm{reg}}}(\log D \cup E)$ the sheaves of $\mathcal{O}_{\boldsymbol{P}^{\mathrm{reg}}}$-modules defined by $k[M \cap \sigma^\vee]$-modules $\omega^1_{\boldsymbol{A}_\sigma} \subset \Omega^1_{\boldsymbol{A}_\sigma}(\log D \cup E)$ for $\sigma \in \Sigma^{\mathrm{reg}}$. Then we have an isomorphism $\mathcal{O}_{\boldsymbol{P}^{\mathrm{reg}}} \otimes_{\mathbf{Z}} M \simeq \Omega^1_{\boldsymbol{P}^{\mathrm{reg}}}(\log D \cup E)$ by

$$k[M \cap \sigma^\vee] \otimes_{\mathbf{Z}} M \simeq \Omega^1_{\boldsymbol{A}_\sigma}(\log D \cup E) \,; \quad 1 \otimes u_j \leftrightarrow \frac{dx_j}{x_j} \,.$$

We denote by $\omega^1_{\boldsymbol{P}^{\mathrm{reg}}/\boldsymbol{A}}(\log D)$ the cokernel of the homomorphism

$$\pi^*_{\mathcal{O}_{\boldsymbol{P}^{\mathrm{reg}}}} : \mathcal{O}_{\boldsymbol{P}^{\mathrm{reg}}} \otimes_{\mathbf{Z}} M' \to \mathcal{O}_{\boldsymbol{P}^{\mathrm{reg}}} \otimes_{\mathbf{Z}} M \simeq \Omega^1_{\boldsymbol{P}^{\mathrm{reg}}}(\log D \cup E) \,.$$

Since the image of $\pi^*_{\mathcal{O}_{\boldsymbol{P}^{\mathrm{reg}}}}$ is contained in $\omega^1_{\boldsymbol{P}^{\mathrm{reg}}}$, we denote by $\omega^1_{\boldsymbol{P}^{\mathrm{reg}}/\boldsymbol{A}}$ the cokernel of $\pi^*_{\mathcal{O}_{\boldsymbol{P}^{\mathrm{reg}}}} : \mathcal{O}_{\boldsymbol{P}^{\mathrm{reg}}} \otimes_{\mathbf{Z}} M' \to \omega^1_{\boldsymbol{P}^{\mathrm{reg}}}$. We define the coherent sheaf $\tilde{\omega}^p_{\boldsymbol{P}/\boldsymbol{A}}$ on $\boldsymbol{P}$ by $\tilde{\omega}^p_{\boldsymbol{P}/\boldsymbol{A}} = j_*(\bigwedge^p \omega^1_{\boldsymbol{P}^{\mathrm{reg}}/\boldsymbol{A}})$, which is a submodule of the free $\mathcal{O}_{\boldsymbol{P}}$-module

$$\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D) = j_* \omega^p_{\boldsymbol{P}^{\mathrm{reg}}/\boldsymbol{A}}(\log D) \simeq \mathcal{O}_{\boldsymbol{P}} \otimes_k \bigwedge^p M_{\pi,k} \,,$$

where $M_{\pi,k}$ denotes the cokernel of $\pi^*_k : k \otimes_{\mathbf{Z}} M' \to k \otimes_{\mathbf{Z}} M$. The sheaf $\tilde{\omega}^p_{\boldsymbol{P}/\boldsymbol{A}}$ is the sheaf of relative logarithmic differential p-forms of Zariski. In the paper [1], it is simply denoted by $\Omega^p_{\boldsymbol{P}}$ for the case $\boldsymbol{A} = \operatorname{Spec} k$.

REMARK 3.1. The sheaves $\omega^1_{\boldsymbol{P}^{\mathrm{reg}}/\boldsymbol{A}}$ and $\omega^1_{\boldsymbol{P}/\boldsymbol{A}}(\log D)$ are interpreted as sheaves of relative logarithmic differential forms in the sense of log geometry [9]. If we consider $\boldsymbol{P}^{\mathrm{reg}}$ with the log structure $\mathcal{N}^E_{\boldsymbol{P}^{\mathrm{reg}}}$ defined by the divisor E with normal crossings, and consider $\boldsymbol{A}$ with the canonical log structure $\mathcal{N}^{\mathrm{can}}_{\boldsymbol{A}}$ as a toric variety, then the sheaf $\omega^1_{\boldsymbol{P}^{\mathrm{reg}}/\boldsymbol{A}}$ is the sheaf of relative logarithmic differential forms on $(\boldsymbol{P}^{\mathrm{reg}}, \mathcal{N}^E_{\boldsymbol{P}^{\mathrm{reg}}})$ over $(\boldsymbol{A}, \mathcal{N}^{\mathrm{can}}_{\boldsymbol{A}})$. If we consider $\boldsymbol{P}$ with the canonical log structure $\mathcal{N}_{\boldsymbol{P}}^{\mathrm{can}}$ as a toric variety, then the sheaf $\omega^1_{\boldsymbol{P}/\boldsymbol{A}}(\log D)$ is the sheaf of relative logarithmic differential forms on $(\boldsymbol{P}, \mathcal{N}_{\boldsymbol{P}}^{\mathrm{can}})$ over $(\boldsymbol{A}, \mathcal{N}_{\boldsymbol{A}}^{\mathrm{can}})$.

Sections of the sheaf $\tilde{\omega}^p_{\boldsymbol{P}/\boldsymbol{A}}$ are described explicitly as follows (cf. [3, Proposition 4.3]). For $u \in M$ and $\sigma \in \Sigma$, we define the subset $\sigma_{\pi,u=0}(1)$ of $\Sigma_\pi(1)$ by

$$\sigma_{\pi,u=0}(1) = \{\rho \in \Sigma_\pi(1) \mid \langle u, v_\rho\rangle = 0 \ \text{ and } \ \rho \subset \sigma\}\,,$$

and the k-subspace $H_{\sigma,u}$ of $M_{\pi,k}$ by

$$H_{\sigma,u} = \{w \in M_{\pi,k} \mid \langle w, 1 \otimes v_\rho\rangle = 0 \ \text{ for any } \ \rho \in \sigma_{\pi,u=0}(1)\}\,.$$

LEMMA 3.2. *For $\sigma \in \Sigma$, there is a natural isomorphism*

$$\Gamma(\boldsymbol{A}_\sigma, \tilde{\omega}^p_{\boldsymbol{P}/\boldsymbol{A}}) \simeq \bigoplus_{u \in M \cap \sigma^\vee} \bigwedge^p H_{\sigma,u} \cdot \chi^u\,.$$

PROOF. The isomorphism is induced from the natural isomorphism

$$\Gamma(\boldsymbol{A}_\sigma, \omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D)) \simeq \Gamma(\boldsymbol{A}_\sigma, \mathcal{O}_{\boldsymbol{P}}) \otimes_k \bigwedge^p M_{\pi,k} \simeq \bigoplus_{u \in M \cap \sigma^\vee} \bigwedge^p M_{\pi,k} \cdot \chi^u\,.$$

First we assume that $\sigma \in \Sigma^{\mathrm{reg}}$. We use a $\mathbf{Z}$-basis $(v_1, \dots, v_d)$ of N satisfying (3.2) and the dual basis $(u_1, \dots, u_d)$ of M. There exists a k-basis $(w_1, \dots, w_n)$ of $M_{\pi,k}$ such that for $1 \le i \le l$, the vector w_i is the image of $1 \otimes u_i$ by the natural homomorphism $k \otimes_{\mathbf{Z}} M \to M_{\pi,k}$, and $\Gamma(\boldsymbol{A}_\sigma, \omega^1_{\boldsymbol{P}^{\mathrm{reg}}/\boldsymbol{A}})$ corresponds to the $k[M \cap \sigma^\vee]$-submodule $H_\sigma \subset \bigoplus_{u \in M \cap \sigma^\vee} M_{\pi,k} \cdot \chi^u$ generated by

$$w_1 \chi^{u_1}, \dots, w_l \chi^{u_l}, w_{l+1}\chi^0, \dots, w_n \chi^0\,.$$

For a subset $I = \{i_1, \dots, i_p\} \subset \{1, \dots, n\}$ with $i_1 < \dots < i_p$, we set $u_I = \sum_{i \in I \cap \{1,\dots,l\}} u_i$ in M, and define

$$w_I = w_{i_1} \wedge \dots \wedge w_{i_p} \tag{3.3}$$

in $\bigwedge^p M_{\pi,k}$. Then $(w_I;\ |I| = p)$ is a k-basis of $\bigwedge^p M_{\pi,k}$, and $(w_I \chi^{u_I};\ |I| = p)$ is a $k[M \cap \sigma^\vee]$-basis of $\bigwedge^p H_\sigma$. For $u \in M \cap \sigma^\vee$, we define the subset J_u of $\{1, \dots, l\}$ by

$$J_u = \{j \in \{1, \dots, l\} \mid \langle u, v_j\rangle = 0\}\,.$$

Then $(w_I;\ |I| = p,\ I \cap J_u = \emptyset)$ is a k-basis of $\bigwedge^p H_{\sigma,u}$. If

$$\omega = \sum_{u \in M \cap \sigma^\vee} \sum_{|I|=p} a_{u,I} w_I \chi^u \in \bigoplus_{u \in M \cap \sigma^\vee} \bigwedge^p M_{\pi,k} \cdot \chi^u\,, \quad a_{u,I} \in k\,,$$

is contained in $\bigwedge^p H_\sigma$, then $\omega = \sum_{I=|p|} (\sum_{u \in M \cap \sigma^\vee} b_{I,u} \chi^u) w_I \chi^{u_I}$ for some $b_{I,u} \in k$, so we have

$$a_{u,I} = \begin{cases} b_{I,u-u_I} & \text{if } u - u_I \in \sigma^\vee\,, \\ 0 & \text{if } u - u_I \notin \sigma^\vee\,. \end{cases}$$

If $u - u_I \in \sigma^\vee$, then $I \cap J_u = \emptyset$, and hence $\sum_{|I|=p} a_{u,I} w_I \in \bigwedge^p H_{\sigma,u}$. Conversely, if $I \cap J_u = \emptyset$, then we have $u - u_I \in \sigma^\vee$, and hence

$$w_I \chi^u = \chi^{u-u_I}(w_I \chi^{u_I}) \in \bigwedge^p H_\sigma \,.$$

So $\bigwedge^p H_\sigma$ coincides with $\bigoplus_{u \in M \cap \sigma^\vee} \bigwedge^p H_{\sigma,u} \cdot \chi^u$ for $\sigma \in \Sigma^{\mathrm{reg}}$.

If $\sigma \in \Sigma$ is not a nonsingular cone, then we have

$$\Gamma(\boldsymbol{A}_\sigma, \tilde{\omega}^p_{\boldsymbol{P}/\boldsymbol{A}}) = \Gamma(\boldsymbol{A}_\sigma \cap \boldsymbol{P}^{\mathrm{reg}}, \omega^p_{\boldsymbol{P}^{\mathrm{reg}}/\boldsymbol{A}}) = \bigcap_{\tau \in \sigma^{\mathrm{reg}}} \Gamma(\boldsymbol{A}_\tau, \omega^p_{\boldsymbol{P}^{\mathrm{reg}}/\boldsymbol{A}}) \,,$$

where σ^{reg} denotes the set of all nonsingular faces of σ. Since $\sigma^\vee = \bigcap_{\tau \in \sigma^{\mathrm{reg}}} \tau^\vee$, and $\bigwedge^p H_{\sigma,u} = \bigcap_{\tau \in \sigma^{\mathrm{reg}}} \bigwedge^p H_{\tau,u}$ for $u \in \sigma^\vee$, the isomorphism in Lemma 3.2 is proved for any $\sigma \in \Sigma$. $\square$

Next we define the Poincaré residue map for the sheaf $\tilde{\omega}^p_{\boldsymbol{P}/\boldsymbol{A}}$. For this purpose, we need some assumptions for the fan Σ. For $\sigma \in \Sigma$, we define the positive integer $m(\sigma)$ by

$$m(\sigma) = \operatorname{ord}\left(N \cap \sigma_{\boldsymbol{R}} \Big/ \sum_{\rho \in \sigma(1)} \boldsymbol{Z} v_\rho \right),$$

and set $m(\Sigma) = \prod_{\sigma \in \Sigma} m(\sigma)$. We assume that the fan Σ is simplicial, and $m(\Sigma)$ is prime to the characteristic of k.

We denote by $N_{\pi,k}$ the kernel of $\pi_{*k} : k \otimes_{\boldsymbol{Z}} N \to k \otimes_{\boldsymbol{Z}} N'$. For $\tau \in \Sigma_\pi(r)$, we denote by $v_1, \dots, v_r$ the primitive generators of τ in $N_{\pi,k}$;

$$\{v_1, \dots, v_r\} = \{1 \otimes v_\rho \in N_{\pi,k} \mid \rho \in \Sigma_\pi(1) \ \text{ and } \ \rho \subset \tau\} \,.$$

Then we define the k-homomorphism ϕ_τ by

$$\phi_\tau : \bigwedge^r M_{\pi,k} \to k;\ \ \eta_1 \wedge \cdots \wedge \eta_r \mapsto \det\left(\langle \eta_i, v_j \rangle\right)_{1 \le i \le r, 1 \le j \le r} \,,$$

where $\eta_i \in M_{\pi,k}$, and $\langle\ ,\ \rangle : M_{\pi,k} \times N_{\pi,k} \to k$ denotes the natural bilinear form. We remark that ϕ_τ depends on the ordering of $v_1, \dots, v_r$. We fix the ordering for each $\tau \in \Sigma_\pi(r)$. For $\sigma \in \Sigma$ and $\tau \in \Sigma_\pi(r)$ with $\tau \subset \sigma$, we define the homomorphism $\Phi_{\sigma,\tau}$ by

$$\begin{aligned}
\Phi_{\sigma,\tau} : \bigwedge^r M_{\pi,k} \otimes_k \bigoplus_{u \in M \cap \sigma^\vee} \bigwedge^{p-r} H_{\sigma,u} \cdot \chi^u \ &\to \bigoplus_{u \in M \cap \sigma^\vee \cap \tau^\perp} \bigwedge^{p-r} H_{\sigma,u} \cdot \chi^u_\tau \,; \\
\eta \otimes \sum_{u \in M \cap \sigma^\vee} w_u \chi^u \ &\mapsto \sum_{u \in M \cap \sigma^\vee \cap \tau^\perp} \phi_\tau(\eta) w_u \chi^u_\tau \,,
\end{aligned}$$

where $\eta \in \bigwedge^r M_{\pi,k}$ and $w_u \in \bigwedge^{p-r} H_{\sigma,u}$. Since $m(\Sigma)$ is prime to the characteristic of k, for $u \in M \cap \sigma^\vee \cap \tau^\perp \simeq M_\tau \cap [\sigma]^\vee_\tau$, there is a natural identification

$$H_{\sigma,u} \simeq \{w \in (M_\tau)_{\pi_\tau,k} \mid \langle w, 1 \otimes v_\rho \rangle = 0 \ \text{ for any } \ \rho \in ([\sigma]_\tau)_{\pi_\tau, u=0}(1)\} \,,$$

so $\Phi_{\sigma,\tau}$ gives a homomorphism

$$\Gamma(\boldsymbol{A}_\sigma, \omega^r_{\boldsymbol{P}/\boldsymbol{A}}(\log D) \otimes_{\mathcal{O}_{\boldsymbol{P}}} \tilde{\omega}^{p-r}_{\boldsymbol{P}/\boldsymbol{A}}) \to \Gamma(\boldsymbol{A}_\sigma \cap \boldsymbol{P}_\tau, \tilde{\omega}^{p-r}_{\boldsymbol{P}_\tau/\boldsymbol{A}}) \,.$$

We denote by

$$\Phi_\tau : \omega^r_{\boldsymbol{P}/\boldsymbol{A}}(\log D) \otimes_{\mathcal{O}_{\boldsymbol{P}}} \tilde{\omega}^{p-r}_{\boldsymbol{P}/\boldsymbol{A}} \to \iota_{\tau *}\tilde{\omega}^{p-r}_{\boldsymbol{P}_\tau/\boldsymbol{A}}$$

the homomorphism of sheaves defined by $\Phi_{\sigma,\tau}$ for $\sigma \in \Sigma$, and define the homomorphism Φ by

$$\Phi = \bigoplus_{\tau \in \Sigma_\pi(r)} \Phi_\tau : \omega^r_{\boldsymbol{P}/\boldsymbol{A}}(\log D) \otimes_{\mathcal{O}_{\boldsymbol{P}}} \tilde{\omega}^{p-r}_{\boldsymbol{P}/\boldsymbol{A}} \to \bigoplus_{\tau \in \Sigma_\pi(r)} \iota_{\tau *}\tilde{\omega}^{p-r}_{\boldsymbol{P}_\tau/\boldsymbol{A}} .$$

LEMMA 3.3. *The kernel of the natural homomorphism*

$$\omega^r_{\boldsymbol{P}/\boldsymbol{A}}(\log D) \otimes_{\mathcal{O}_{\boldsymbol{P}}} \tilde{\omega}^{p-r}_{\boldsymbol{P}/\boldsymbol{A}} \to \omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D) ; \quad \eta \otimes w \mapsto \eta \wedge w$$

is contained in the kernel of Φ.

PROOF. Let σ be a cone in Σ, u an element in $M \cap \sigma^\vee$, and $(w_1, \dots, w_n)$ a k-basis of $M_{\pi,k}$ such that $(w_1, \dots, w_s)$ is a k-basis of $H_{\sigma,u}$. For $I \subset \{1, \dots, n\}$ with $|I| = r$, and $J \subset \{1, \dots, s\}$ with $|J| = p - r$, vectors $w_I \in \bigwedge^r M_{\pi,k}$ and $w_J \in \bigwedge^{p-r} H_{\sigma,u}$ are defined by the same way as (3.3). If

$$w = \sum_{|I|=r} \sum_{|J|=p-r} a_{I,J} w_I \otimes w_J , \quad a_{I,J} \in k ,$$

is contained in the kernel of the natural homomorphism

$$\bigwedge^r M_{\pi,k} \otimes_k \bigwedge^{p-r} H_{\sigma,u} \to \bigwedge^p M_{\pi,k} ,$$

then for $L \subset \{1, \dots, n\}$ with $|L| = p$, we have

$$\sum_{I \cup J = L} \operatorname{sgn}(I, J) a_{I,J} = 0 , \tag{3.4}$$

where $\operatorname{sgn}(I, J)$ is defined by $w_I \wedge w_J = \operatorname{sgn}(I, J) w_L$. For $\tau \in \Sigma_\pi(r)$ with $\tau \subset \sigma$, we have to prove $\Phi_{\sigma,\tau}(w\chi^u) = 0$. It is clear for the case $u \notin \tau^\perp$. Hence we assume that $u \in \tau^\perp$. Let $I \subset \{1, \dots, n\}$ be a subset satisfying $|I| = r$ and $\phi_\tau(w_I) \neq 0$. Then we have $I \cap \{1, \dots, s\} = \emptyset$, because $\langle w_i, 1 \otimes v\rangle = 0$ for $1 \leq i \leq s$ and $v \in N \cap \tau_{\boldsymbol{R}}$. By (3.4), we have $a_{I,J} = 0$ for any $J \subset \{1, \dots, s\}$, and hence

$$\Phi_{\sigma,\tau}(w\chi^u) = \sum_{|J|=p-r} \sum_{|I|=r} a_{I,J} \phi_\tau(w_I) w_J \chi^u_\tau = 0 . \qquad \square$$

We denote by $W_r\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D)$ the image of the natural homomorphism

$$\omega^r_{\boldsymbol{P}/\boldsymbol{A}}(\log D) \otimes_{\mathcal{O}_{\boldsymbol{P}}} \tilde{\omega}^{p-r}_{\boldsymbol{P}/\boldsymbol{A}} \to \omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D) ; \quad \eta \otimes w \mapsto \eta \wedge w .$$

By Lemma 3.3, the homomorphism Φ induces a homomorphism

$$\operatorname{Res} : W_r\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D) \to \bigoplus_{\tau \in \Sigma_\pi(r)} \iota_{\tau *}\tilde{\omega}^{p-r}_{\boldsymbol{P}_\tau/\boldsymbol{A}} ,$$

which is called the Poincaré residue map. The Poincaré residue map has the following fundamental property similar to the case of $\Omega^p_{\boldsymbol{P}}$ by [4, II. §3] or [3, §15.7].

THEOREM 3.4. *Let $\boldsymbol{A}$ be an affine toric variety, $\boldsymbol{P} = \boldsymbol{P}_\Sigma$ a simplicial toric variety such that $m(\Sigma)$ is prime to the characteristic of k, and $\pi : \boldsymbol{P} \to \boldsymbol{A}$ an equivariant morphism. Then there is an exact sequence of $\mathcal{O}_{\boldsymbol{P}}$-modules*

$$0 \longrightarrow W_{r-1}\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D) \longrightarrow W_r\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D) \xrightarrow{\mathrm{Res}} \bigoplus_{\tau\in\Sigma_\pi(r)} \iota_{\tau *}\tilde{\omega}^{p-r}_{\boldsymbol{P}_\tau/\boldsymbol{A}} \longrightarrow 0\,.$$

PROOF. We check this on the affine coordinate $\boldsymbol{A}_\sigma$ for $\sigma \in \Sigma$. First we prove that the Poincaré residue map is surjective. Let $\tau \subset \sigma$ be a cone in $\Sigma_\pi(r)$, and let $v_1, \dots, v_c \in N$ be satisfying

$$\begin{aligned}
\{v_1, \dots, v_c\} &= \{v_\rho \mid \rho \in \Sigma(1) \ \text{and} \ \rho \subset \sigma\}\,,\\
\{v_1, \dots, v_l\} &= \{v_\rho \mid \rho \in \Sigma_\pi(1) \ \text{and} \ \rho \subset \sigma\}\,,\\
\{v_1, \dots, v_r\} &= \{v_\rho \mid \rho \in \Sigma_\pi(1) \ \text{and} \ \rho \subset \tau\}\,.
\end{aligned}$$

Since $m(\sigma)$ is prime to the characteristic of k, the vectors $1\otimes v_1, \dots, 1\otimes v_l$ are linearly independent in $N_{\pi,k}$. Then there exists a k-basis $(w_1, \dots, w_n)$ of $M_{\pi,k}$ such that

$$\langle w_i, 1\otimes v_j\rangle = \begin{cases} 1\,, & i=j\,,\\ 0\,, & i\neq j\end{cases}$$

for $1 \le i \le n$ and $1 \le j \le l$. For a subset $I \subset \{1, \dots, l\}$ with $|I| = r$, we set $\tau_I = \sum_{i\in I} \boldsymbol{R}_{\ge 0}v_i \in \Sigma_\pi(r)$. Then we have

$$\phi_{\tau_I}(\eta) = \begin{cases} 1\,, & I = \{1, \dots, r\}\,,\\ 0\,, & I \neq \{1, \dots, r\}\,,\end{cases}$$

where $\eta = w_1 \wedge \cdots \wedge w_r$. For $u \in M \cap \sigma \cap \tau^\perp$ and $w \in \bigwedge^{p-r} H_{\sigma,u}$, we have $w\chi^u_\tau = \mathrm{Res}(\eta \wedge w\chi^u)$, because $\Phi_{\sigma,\tau}(\eta\otimes w\chi^u) = w\chi^u_\tau$ and $\Phi_{\sigma,\tau'}(\eta\otimes w\chi^u) = 0$ for $\tau' \neq \tau$. Hence, for any element

$$\omega = \sum_\tau \omega_\tau \in \bigoplus_{\tau\in\sigma_\pi(r)} \Gamma(\boldsymbol{A}_\sigma, \iota_{\tau *}\tilde{\omega}^{p-r}_{\boldsymbol{P}_\tau/\boldsymbol{A}})\,, \quad \omega_\tau \in \Gamma(\boldsymbol{A}_\sigma, \iota_{\tau *}\tilde{\omega}^{p-r}_{\boldsymbol{P}_\tau/\boldsymbol{A}})\,,$$

there exists $\widetilde{\omega_\tau} \in \Gamma(\boldsymbol{A}_\sigma, W_r\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D))$ such that $\mathrm{Res}(\widetilde{\omega_\tau}) = \omega_\tau$. Since $\mathrm{Res}(\sum_\tau \widetilde{\omega_\tau}) = \sum_\tau \mathrm{Res}(\widetilde{\omega_\tau}) = \omega$, the Poincaré residue map is surjective.

Next we show that $W_{r-1}\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D)$ is contained in the kernel of the Poincaré residue map. For $u \in M \cap \sigma^\vee$, we set

$$J_u = \{j \in \{1, \dots, l\} \mid \langle u, v_j\rangle = 0\}\,.$$

Then $(w_J;\ |J| = q,\ J \cap J_u = \emptyset)$ is a k-basis of $\bigwedge^q H_{\sigma,u}$. For $L \subset \{1, \dots, n\}$ with $|L| = r-1$, and for $J \subset \{1, \dots, n\}$ with $|J| = p-r+1$ and $J \cap J_u = \emptyset$, we have

$$\mathrm{Res}(w_L \wedge w_J\chi^u) = \begin{cases} \Phi(w_L \wedge w_{j_1} \otimes w_{J\smallsetminus\{j_1\}}\chi^u)\,, & L\cap J = \emptyset\,,\\ 0\,, & L \cap J \neq \emptyset\,,\end{cases}$$

where j_1 is the smallest integer in J. For $I \subset \{1, \dots, l\}$ with $|I| = r$, if $u \in \tau_I^\perp$, then $\phi_{\tau_I}(w_L \wedge w_{j_1}) = 0$, so we have $\mathrm{Res}(w_L \wedge w_J\chi^u) = 0$.

Finally, we assume that

$$\sum_{|I|=r}\sum_{|J|=p-r} a_{I,J} w_I \otimes w_J \chi^u\,, \quad a_{I,J}\in k\,,$$

is contained in the kernel of Φ. For $L \subset \{1,\ldots,l\}$ with $|L|=r$, if $L\subset J_u$, then $u\in\tau_L^\perp$, and hence

$$a_{L,J} = \phi_{\tau_L}\Bigl(\sum_{|I|=r} a_{I,J} w_I\Bigr) = 0\,.$$

We have

$$\begin{aligned}\sum_{|I|=r}\sum_{|J|=p-r} a_{I,J} w_I\wedge w_J &= \sum_{I\not\subseteq J_u}\sum_{|J|=p-r} a_{I,J} w_I\wedge w_J\\ &= \sum_{I\not\subseteq J_u}\operatorname{sgn}(I\setminus\{i\},i)\sum_{|J|=p-r} a_{I,J} w_{I\setminus\{i\}}\wedge w_i\wedge w_J\,,\end{aligned}$$

where we take $i\in I\setminus J_u$ for each $I\not\subseteq J_u$. Since $(J\cup\{i\})\cap J_u=\emptyset$, this is contained in $\Gamma(\boldsymbol{A}_\sigma, W_{r-1}\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D))$. □

PROPOSITION 3.5. *If $\pi:\boldsymbol{P}\to\boldsymbol{A}$ is proper, then for any ample invertible sheaf $\mathcal{L}$ on $\boldsymbol{P}$, the global Poincaré residue map*

$$H^0(\boldsymbol{P}, W_r\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D)\otimes_{\mathcal{O}_{\boldsymbol{P}}}\mathcal{L}) \xrightarrow{\mathrm{Res}} \bigoplus_{\tau\in\Sigma_\pi(r)} H^0(\boldsymbol{P}_\tau, \tilde{\omega}^{p-r}_{\boldsymbol{P}_\tau/\boldsymbol{A}}\otimes_{\mathcal{O}_{\boldsymbol{P}_\tau}}\iota_\tau^*\mathcal{L})$$

is surjective.

PROOF. There is a Σ-linear support function h such that $\mathcal{L}\simeq\mathcal{O}_{\boldsymbol{P}}(D_h)$. We define the subset $\Sigma_{\pi,u=h}(1)$ of $\Sigma_\pi(1)$ by

$$\Sigma_{\pi,u=h}(1) = \{\rho\in\Sigma_\pi(1)\mid\langle u, v_\rho\rangle = h(v_\rho)\}\,,$$

and define the k-subspace $H_{u,h}$ of $M_{\pi,k}$ by

$$H_{u,h} = \{w\in M_{\pi,k}\mid\langle w, 1\otimes v_\rho\rangle = 0 \text{ for any } \rho\in\Sigma_{\pi,u=h}(1)\}\,.$$

By Lemma 3.2, there is a natural isomorphism

$$H^0(\boldsymbol{P}, \tilde{\omega}^{p-r}_{\boldsymbol{P}/\boldsymbol{A}}\otimes_{\mathcal{O}_{\boldsymbol{P}}}\mathcal{O}_{\boldsymbol{P}}(D_h)) \simeq \bigoplus_{u\in M\cap\Delta_h}\bigwedge^{p-r} H_{u,h}\cdot\chi^u\,,$$

where

$$\Delta_h = \Delta_{D_h} = \{u\in M_{\boldsymbol{R}}\mid\langle u,v\rangle\geq h(v) \text{ for any } v\in|\Sigma|\}\,.$$

Let τ be a cone in $\Sigma_\pi(r)$. We choose an element $u_\tau\in M$ which coincides with h on τ as a linear function. Then Σ-linear support function $h-u_\tau$ induces a Σ_τ-linear support function $h_\tau:|\Sigma_\tau|\to\boldsymbol{R}$, and there is a natural isomorphism $\iota_\tau^*\mathcal{O}_{\boldsymbol{P}}(D_{h-u_\tau})\simeq\mathcal{O}_{\boldsymbol{P}_\tau}(D_{h_\tau})$. Since π is proper, we have

$$M_\tau\cap\Delta_{h_\tau} = M\cap\tau^\perp\cap\Delta_{h-u_\tau}\,.$$

Since $m(\Sigma)$ is prime to the characteristic of k, for $u \in M \cap \tau^{\perp} \cap \Delta_{h-u_\tau}$, there is a natural identification

$$H_{u,h-u_\tau} \simeq \{w \in (M_\tau)_{\pi_\tau,k} \mid \langle w, 1 \otimes v_\rho \rangle = 0 \ \text{ for any } \ \rho \in (\Sigma_\tau)_{\pi_\tau, u=h_\tau}(1)\} .$$

We denote by T_{u_τ} the isomorphism

$$T_{u_\tau} : \mathcal{O}_{\boldsymbol{P}}(D_h) \to \mathcal{O}_{\boldsymbol{P}}(D_{h-u_\tau}) ; \quad \chi^u \mapsto \chi^{u-u_\tau} .$$

Then the homomorphism

$$\Phi_\tau \circ T_{u_\tau} : \Gamma(\boldsymbol{P}, \omega^r_{\boldsymbol{P}/\boldsymbol{A}}(\log D) \otimes \tilde{\omega}^{p-r}_{\boldsymbol{P}/\boldsymbol{A}} \otimes \mathcal{O}_{\boldsymbol{P}}(D_h)) \to \Gamma(\boldsymbol{P}_\tau, \tilde{\omega}^{p-r}_{\boldsymbol{P}_\tau/\boldsymbol{A}} \otimes \mathcal{O}_{\boldsymbol{P}_\tau}(D_{h_\tau}))$$

is given by

$$\begin{array}{ccc} \bigwedge^r M_{\pi,k} \otimes_k \bigoplus_{u \in M \cap \Delta_h} \bigwedge^{p-r} H_{u,h} \cdot \chi^u & \to & \bigoplus_{u \in M \cap \Delta_{h-u_\tau} \cap \tau^{\perp}} \bigwedge^{p-r} H_{u,h-u_\tau} \cdot \chi^u_\tau ; \\ \eta \otimes w\chi^u & \mapsto & \begin{cases} \phi_\tau(\eta) w \chi_\tau^{u-u_\tau} , & u - u_\tau \in \tau^{\perp} , \\ 0 , & u - u_\tau \notin \tau^{\perp} , \end{cases} \end{array}$$

where we remark that $H_{u,h} = H_{u-u_\tau, h-u_\tau}$. Since h is strictly upper convex, for $u \in M \cap \tau^{\perp} \cap \Delta_{h-u_\tau}$, there exists a cone $\sigma \in \Sigma$ such that $\sigma(1) \supset \Sigma_{\pi, u+u_\tau = h}(1)$.

Using this cone σ, we give a k-basis $(w_1, \dots, w_n)$ of $M_{\pi,\tau}$ in the same way as the proof of Theorem 3.4, and we set $\eta = w_1 \wedge \cdots \wedge w_r$ in $\bigwedge^r M_{\pi,k}$. Then for $w \in \bigwedge^{p-r} H_{u,h-u_\tau}$, we have $w\chi^u = \Phi_\tau \circ T_{u_\tau}(\eta \otimes w \chi^{u+u_\tau})$. If $\tau' \in \Sigma_\pi(r) \setminus \{\tau\}$ is contained in σ, then $\phi_\tau(\eta) = 0$. If $\tau' \in \Sigma_\pi(r) \setminus \{\tau\}$ is not contained in σ, then $u + u_\tau - u_{\tau'} \notin \tau'^{\perp}$. Hence we have $\Phi_{\tau'} \circ T_{u_{\tau'}}(\eta \otimes w\chi^{u+u_\tau}) = 0$ for any $\tau' \in \Sigma_\pi(r) \setminus \{\tau\}$. Since the global Poincaré residue map is induced by $\Phi = \bigoplus_{\tau \in \Sigma_\pi(r)} \Phi_\tau$, for any element

$$\omega = \sum_{\tau \in \Sigma_\pi(r)} \omega_\tau \in \bigoplus_{\tau \in \Sigma_\pi(r)} H^0(\boldsymbol{P}_\tau, \tilde{\omega}^{p-r}_{\boldsymbol{P}_\tau/\boldsymbol{A}} \otimes_{\mathcal{O}_{\boldsymbol{P}_\tau}} \iota_\tau^* \mathcal{O}_{\boldsymbol{P}}(D_h))$$

there exist $\widetilde{\omega_\tau} \in H^0(\boldsymbol{P}, W_r \omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D) \otimes_{\mathcal{O}_{\boldsymbol{P}}} \mathcal{O}_{\boldsymbol{P}}(D_h))$ such that $\mathrm{Res}\,(\widetilde{\omega_\tau}) = \omega_\tau$. Since $\mathrm{Res}\,(\sum_{\tau \in \Sigma_\pi(r)} \widetilde{\omega_\tau}) = \sum_{\tau \in \Sigma_\pi(r)} \mathrm{Res}\,(\widetilde{\omega_\tau}) = \omega$, the global Poincaré residue map is surjective. □

We prove a vanishing theorem of Bott type for the cohomology of the sheaf of relative logarithmic differential forms, that is reduced to the following vanishing theorem for invertible sheaves on toric varieties. The idea is the same as that in [1, Theorem 7.2].

THEOREM 3.6 ([6, p. 74, Corollary]). *Let $\boldsymbol{P}_\Sigma$ be a toric variety, and $\mathcal{L}$ an invertible sheaf on $\boldsymbol{P}_\Sigma$. If the support $|\Sigma|$ is convex, and $\mathcal{L}$ is generated by global sections, then for $q \geq 1$,*

$$H^q(\boldsymbol{P}_\Sigma, \mathcal{L}) = 0 .$$

THEOREM 3.7. *Let $\boldsymbol{A}$ be an affine toric variety, $\boldsymbol{P}$ a simplicial toric variety such that $m(\Sigma)$ is prime to the characteristic of k, and $\pi : \boldsymbol{P} \to \boldsymbol{A}$ a proper equivariant morphism. Let $\mathcal{L}$ be an invertible sheaf on $\boldsymbol{P}$.*

(1) *If $\mathcal{L}$ is generated by global sections, then for $p \geq r \geq 0$ and $q \geq p-r+1$,*

$$H^q(\boldsymbol{P}, W_r\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D) \otimes_{\mathcal{O}_{\boldsymbol{P}}} \mathcal{L}) = 0\,.$$

(2) *If $\mathcal{L}$ is ample, then for $p \geq r \geq 0$ and $q \geq 1$,*

$$H^q(\boldsymbol{P}, W_r\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D) \otimes_{\mathcal{O}_{\boldsymbol{P}}} \mathcal{L}) = 0\,.$$

PROOF. We prove this theorem by induction on $p-r$. If $p-r=0$, then $W_p\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D)$ is isomorphic to the free $\mathcal{O}_{\boldsymbol{P}}$-module $\mathcal{O}_{\boldsymbol{P}} \otimes_k \bigwedge^p M_{\pi,k}$. By Theorem 3.6, we have

$$H^q(W_p\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D) \otimes \mathcal{L}) = \bigwedge^p M_{\pi,k} \otimes H^q(\mathcal{L}) = 0$$

for $q \geq 1$.

For an integer $l \geq 1$, we assume that Theorem 3.7 is true for the case $p-r=l-1$. Then we prove Theorem 3.7 for the case $p-r=l$. By Theorem 3.4, there is an exact sequence

$$\begin{aligned} H^{q-1}(W_{r+1}\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D) \otimes \mathcal{L}) \to \bigoplus_{\tau\in\Sigma_\pi(r+1)} H^{q-1}(\tilde{\omega}^{p-r-1}_{\boldsymbol{P}_\tau/\boldsymbol{A}} \otimes \iota^*_\tau\mathcal{L}) \\ \to H^q(W_r\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D)\otimes\mathcal{L}) \to H^q(W_{r+1}\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D)\otimes\mathcal{L})\,. \end{aligned}$$

Since $p-(r+1)=l-1$, we have $H^q(W_{r+1}\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D) \otimes \mathcal{L}) = 0$ for $q \geq p-(r+1)+1$. Since $\tilde{\omega}^{p-r-1}_{\boldsymbol{P}_\tau/\boldsymbol{A}} = W_0\omega^{p-r-1}_{\boldsymbol{P}_\tau/\boldsymbol{A}}(\log D_\tau)$ and $(p-r-1)-0=l-1$, we have $H^{q-1}(\tilde{\omega}^{p-r-1}_{\boldsymbol{P}_\tau/\boldsymbol{A}} \otimes \iota^*_\tau\mathcal{L}) = 0$ for $q-1 \geq (p-r-1)-0+1$, where D_τ denotes the invariant divisor on $\boldsymbol{P}_\tau$ defined similarly as we defined D on $\boldsymbol{P}$. Hence we have $H^q(W_r\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D) \otimes \mathcal{L}) = 0$ for $q \geq p-r+1$.

Next we consider the case where $\mathcal{L}$ is ample. Since $p-(r+1)=l-1$, we have $H^q(W_{r+1}\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D) \otimes \mathcal{L}) = 0$ for $q \geq 1$. Since $\tilde{\omega}^{p-r-1}_{\boldsymbol{P}_\tau/\boldsymbol{A}} = W_0\omega^{p-r-1}_{\boldsymbol{P}_\tau/\boldsymbol{A}}(\log D_\tau)$ and $(p-r-1)-0=l-1$, we have $H^{q-1}(\tilde{\omega}^{p-r-1}_{\boldsymbol{P}_\tau/\boldsymbol{A}} \otimes \iota^*_\tau\mathcal{L}) = 0$ for $q-1 \geq 1$. Hence we have $H^q(W_r\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D) \otimes \mathcal{L}) = 0$ for $q \geq 2$. By Proposition 3.5, the homomorphism

$$H^0(W_{r+1}\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D) \otimes \mathcal{L}) \to \bigoplus_{\tau\in\Sigma_\pi(r+1)} H^0(\tilde{\omega}^{p-r-1}_{\boldsymbol{P}_\tau/\boldsymbol{A}} \otimes \iota^*_\tau\mathcal{L})$$

is surjective. Hence we also have $H^1(W_r\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D) \otimes \mathcal{L}) = 0$ for by the exact sequence. □

COROLLARY 3.8. (1) *If $\mathcal{L}$ is generated by global sections, then for $q \geq p+1$,*

$$H^q(\boldsymbol{P}, \tilde{\omega}^p_{\boldsymbol{P}/\boldsymbol{A}} \otimes_{\mathcal{O}_{\boldsymbol{P}}} \mathcal{L}) = 0\,.$$

(2) *If $\mathcal{L}$ is ample, then for $q \geq 1$,*

$$H^q(\boldsymbol{P}, \tilde{\omega}^p_{\boldsymbol{P}/\boldsymbol{A}} \otimes_{\mathcal{O}_{\boldsymbol{P}}} \mathcal{L}) = 0\,.$$

PROOF. By the definition, $W_0\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log D)$ is equal to $\tilde{\omega}^p_{\boldsymbol{P}/\boldsymbol{A}}$. The corollary is the case $r=0$ of Theorem 3.7. □

In the rest of this section, we prove the Euler exact sequence for the sheaf of relative logarithmic differential forms. Here we introduce the notion of log smoothness.

DEFINITION 3.9. If the rank of the locally free $\mathcal{O}_{\boldsymbol{P}^{\mathrm{reg}}}$-module $\omega^1_{\boldsymbol{P}^{\mathrm{reg}}/\boldsymbol{A}}$ is equal to $\dim \boldsymbol{P} - \dim \boldsymbol{A}$, then we call that π is *log smooth*.

REMARK 3.10. The following conditions are equivalent:

(1) π is log smooth in the sense of Definition 3.9.

(2) $\pi \circ j : (\boldsymbol{P}^{\mathrm{reg}}, \mathcal{N}^E_{\boldsymbol{P}^{\mathrm{reg}}}) \to (\boldsymbol{A}, \mathcal{N}^{\mathrm{can}}_{\boldsymbol{A}})$ is log smooth in the sense of [9].

(3) $\pi : (\boldsymbol{P}, \mathcal{N}^{\mathrm{can}}_{\boldsymbol{P}}) \to (\boldsymbol{A}, \mathcal{N}^{\mathrm{can}}_{\boldsymbol{A}})$ is log smooth in the sense of [9].

(4) The cokernel of $\pi_* : N \to N'$ is finite, whose order is prime to the characteristic of k.

(5) $\pi^* : M' \to M$ is injective and the order of the torsion part of the cokernel of π^* is prime to the characteristic of k.

We denote by $\hat{N}_\pi$ the free $\boldsymbol{Z}$-module generated by $\Sigma_\pi(1)$, and by P_π the kernel of the homomorphism

$$\psi : \hat{N}_\pi \to N ; \qquad \sum_{\rho \in \Sigma_\pi(1)} b_\rho \rho \mapsto \sum_{\rho \in \Sigma_\pi(1)} b_\rho v_\rho .$$

We identify the group of $\boldsymbol{T}$-invariant Weil divisors on $\boldsymbol{P} \smallsetminus E$ with the dual $\boldsymbol{Z}$-module $\hat{M}_\pi$ of $\hat{N}_\pi$ by the pairing

$$\left\langle \sum_{\rho \in \Sigma_\pi(1)} a_\rho \boldsymbol{P}_\rho, \sum_{\rho \in \Sigma_\pi(1)} b_\rho \rho \right\rangle = \sum_{\rho \in \Sigma_\pi(1)} a_\rho b_\rho .$$

Then the divisor class group $\mathrm{Cl}\,(\boldsymbol{P} \smallsetminus E)$ is naturally isomorphic to the cokernel of the dual homomorphism $\psi^* : M \to \hat{M}_\pi$. Since the image of ψ is contained in N_π, the homomorphism ψ^* induces a morphism of $\mathcal{O}_{\boldsymbol{P}}$-modules

$$\psi^*_{\mathcal{O}_{\boldsymbol{P}}} : \omega^1_{\boldsymbol{P}/\boldsymbol{A}}(\log D) \simeq \mathcal{O}_{\boldsymbol{P}} \otimes_k M_{\pi,k} \to \mathcal{O}_{\boldsymbol{P}} \otimes_{\boldsymbol{Z}} \hat{M}_\pi \simeq \bigoplus_{\rho \in \Sigma_\pi(1)} \mathcal{O}_{\boldsymbol{P}} .$$

THEOREM 3.11. *Let $\boldsymbol{A}$ be an affine toric variety, $\boldsymbol{P}$ a simplicial toric variety such that $m(\Sigma)$ is prime to the characteristic of k, and $\pi : \boldsymbol{P} \to \boldsymbol{A}$ a log smooth proper equivariant morphism. Then there is an exact sequence of $\mathcal{O}_{\boldsymbol{P}}$-modules*

$$0 \longrightarrow \tilde{\omega}^1_{\boldsymbol{P}/\boldsymbol{A}} \xrightarrow{\Psi^*} \bigoplus_{\rho \in \Sigma_\pi(1)} \mathcal{O}_{\boldsymbol{P}}(-\boldsymbol{P}_\rho) \xrightarrow{\gamma} \mathcal{O}_{\boldsymbol{P}} \otimes_{\boldsymbol{Z}} \mathrm{Cl}\,(\boldsymbol{P} \smallsetminus E) \longrightarrow 0 ,$$

where Ψ^* is induced by the morphism $\psi^*_{\mathcal{O}_{\boldsymbol{P}}}$.

PROOF. We denote by N_π the kernel of $\pi_* : N \to N'$. Since π is proper and $m(\Sigma)$ is prime to the characteristic of k, the cokernel of $\psi : \hat{N}_\pi \to N_\pi$ is finite, whose order is prime to the characteristic of k. So we have an exact sequence of k-vector spaces

$$0 \longrightarrow k \otimes_{\boldsymbol{Z}} P_\pi \longrightarrow k \otimes_{\boldsymbol{Z}} \hat{N}_\pi \xrightarrow{\psi_k} k \otimes_{\boldsymbol{Z}} N_\pi \longrightarrow 0 .$$

Since π is log smooth, we have $M_{\pi,k} \simeq \mathrm{Hom}_k(k \otimes_{\mathbf{Z}} N_\pi, k)$. Hence there is an exact sequence of k-vector spaces

$$0 \longrightarrow M_{\pi,k} \xrightarrow{\psi_k^*} k \otimes_{\mathbf{Z}} \hat{M}_\pi \longrightarrow k \otimes_{\mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}}(P_\pi, \mathbf{Z}) \longrightarrow 0\,,$$

which induces an exact sequence of $\mathcal{O}_{\boldsymbol{P}}$-modules

$$0 \longrightarrow \omega^1_{\boldsymbol{P}/\boldsymbol{A}}(\log D) \xrightarrow{\psi^*_{\mathcal{O}_{\boldsymbol{P}}}} \mathcal{O}_{\boldsymbol{P}} \otimes_{\mathbf{Z}} \hat{M}_\pi \longrightarrow \mathcal{O}_{\boldsymbol{P}} \otimes_{\mathbf{Z}} \mathrm{Cl}\,(\boldsymbol{P} \smallsetminus E) \longrightarrow 0\,.$$

The exactness of the sequence in Theorem 3.11 is proved by the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & \tilde{\omega}^1_{\boldsymbol{P}/\boldsymbol{A}} \arrow[d] & \bigoplus_{\rho\in\Sigma_\pi(1)} \mathcal{O}_{\boldsymbol{P}}(-\boldsymbol{P}_\rho) \arrow[d] & \\
0 \arrow[r] & \omega^1_{\boldsymbol{P}/\boldsymbol{A}}(\log D) \arrow[r, "\psi^*_{\mathcal{O}_{\boldsymbol{P}}}"] \arrow[d, "\mathrm{Res}"'] & \mathcal{O}_{\boldsymbol{P}} \otimes_{\mathbf{Z}} \hat{M}_\pi \arrow[r] \arrow[d] & \mathcal{O}_{\boldsymbol{P}} \otimes_{\mathbf{Z}} \mathrm{Cl}\,(\boldsymbol{P} \smallsetminus E) \arrow[r] & 0 \\
 & \bigoplus_{\rho\in\Sigma_\pi(1)} \iota_{\rho *}\mathcal{O}_{\boldsymbol{P}_\rho} \arrow[r, phantom, "="] \arrow[d] & \bigoplus_{\rho\in\Sigma_\pi(1)} \iota_{\rho *}\mathcal{O}_{\boldsymbol{P}_\rho} \arrow[d] & \\
 & 0 & 0\,, &
\end{tikzcd}$$

where the exactness of the left vertical sequence is proved in Theorem 3.4. □

4. Hypersurfaces in toric varieties. Let $\boldsymbol{A} = \mathrm{Spec}\,A$ be an affine toric variety with a torus invariant point 0, $\boldsymbol{P} = \boldsymbol{P}_\Sigma$ a nonsingular toric variety, and $\pi : \boldsymbol{P} \to \boldsymbol{A}$ a log smooth proper equivariant morphism. Then we remark that $\dim \boldsymbol{P} = \dim|\Sigma|$, and $\omega^1_{\boldsymbol{P}/\boldsymbol{A}} = \omega^1_{\boldsymbol{P}^{\mathrm{reg}}/\boldsymbol{A}} = \tilde{\omega}^1_{\boldsymbol{P}/\boldsymbol{A}}$ is a locally free $\mathcal{O}_{\boldsymbol{P}}$-module of rank $n = \dim \boldsymbol{P} - \dim \boldsymbol{A}$.

Since $\dim \boldsymbol{P} = \dim|\Sigma|$, the homogeneous coordinate ring of $\boldsymbol{P}$ is defined in [2, §1] as a $\mathrm{Cl}\,(\boldsymbol{P})$-graded polynomial ring

$$S_{\boldsymbol{P}} = k[z_\rho;\ \rho \in \Sigma(1)] = \bigoplus_{\beta\in\mathrm{Cl}\,(\boldsymbol{P})} S^\beta_{\boldsymbol{P}}$$

with

$$\deg z_\rho = [\boldsymbol{P}_\rho] \in \mathrm{Cl}\,(\boldsymbol{P})\,.$$

For a $\boldsymbol{T}_N$-invariant divisor $B = \sum_{\rho\in\Sigma(1)} b_\rho \boldsymbol{P}_\rho$, there is a natural isomorphism

$$S^{[B]}_{\boldsymbol{P}} \simeq H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(B))\,; \qquad \prod_{\rho\in\Sigma(1)} z_\rho^{\langle u, v_\rho\rangle + b_\rho} \leftrightarrow \chi^u\,. \tag{4.1}$$

By the A-module structure on $H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(B)) = H^0(\boldsymbol{A}, \pi_*\mathcal{O}_{\boldsymbol{P}}(B))$, the homogeneous coordinate ring $S_{\boldsymbol{P}}$ has an A-algebra structure.

Let X be a hypersurface in $\boldsymbol{P}$. Then there is a $\boldsymbol{T}_N$-invariant divisor $B = \sum_{\rho\in\Sigma(1)} b_\rho \boldsymbol{P}_\rho$ such that

$$[X] = [B] \in \mathrm{Cl}\,(\boldsymbol{P})\,.$$

Since the hypersurface X is defined by a global section of $\mathcal{O}_{\boldsymbol{P}}(B)$, using the isomorphism (4.1), it is defined by a $\mathrm{Cl}\,(\boldsymbol{P})$-homogeneous polynomial F. If F' is a $\mathrm{Cl}\,(\boldsymbol{P})$-homogeneous polynomial defined by using another $\boldsymbol{T}_N$-invariant divisor B' with $[X] = [B']$, then there is a non-zero constant $a \in k^\times$ such that $F' = aF$. We define the Jacobian ring $R_{X/\boldsymbol{A}}$ of X over $\boldsymbol{A}$ by

$$R_{X/\boldsymbol{A}} = S_{\boldsymbol{P}} \Big/ \left(\frac{\partial F}{\partial z_\rho};\ \rho \in \Sigma_\pi(1) \right),$$

which is a $\mathrm{Cl}\,(\boldsymbol{P})$-graded A-algebra uniquely determined by X.

REMARK 4.1. If $\boldsymbol{A} = \operatorname{Spec} k$, then $\boldsymbol{P}$ is a complete toric variety, and $\Sigma_\pi(1) = \Sigma(1)$. In this case, our definition of Jacobian ring is same as [1, Definition 10.3].

We denote by $\mathcal{I}_{X/\boldsymbol{P}} \subset \mathcal{O}_{\boldsymbol{P}}$ the ideal defining X in $\boldsymbol{P}$, and define the coherent $\mathcal{O}_X$-module $\omega^p_{X/\boldsymbol{A}}$ by

$$\omega^p_{X/\boldsymbol{A}} = \mathrm{Coker}(\mathcal{I}_{X/\boldsymbol{P}}/\mathcal{I}^2_{X/\boldsymbol{P}} \otimes \omega^{p-1}_{\boldsymbol{P}/\boldsymbol{A}}|_X \to \omega^p_{\boldsymbol{P}/\boldsymbol{A}}|_X;\ [f]\otimes\omega \mapsto (df|_X \wedge \omega))\,,$$

where $d : \mathcal{O}_{\boldsymbol{P}} \to \omega^1_{\boldsymbol{P}/\boldsymbol{A}}$ is the differential operator given by

$$d : \bigoplus_{u\in M\cap\sigma^\vee} k\cdot\chi^u \to \bigoplus_{u\in M\cap\sigma^\vee} H_{\sigma,u}\cdot\chi^u;\ \chi^u \mapsto [1\otimes u]\chi^u\,.$$

DEFINITION 4.2. Let U be an open subset of $\boldsymbol{A}$. If $j_U^*\omega^1_{X/\boldsymbol{A}}$ is a locally free $\mathcal{O}_{X_U}$-module of rank $n-1$, then we call that X is *log smooth* over U, where $j_U : X_U \to X$ denotes the open immersion $U \times_{\boldsymbol{A}} X \to X$.

We define the coherent $\mathcal{O}_{\boldsymbol{P}}$-module $\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log X)$ by

$$\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log X) = \mathrm{Ker}\,(\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(X) \to \omega^p_{X/\boldsymbol{A}} \otimes \mathcal{O}_{\boldsymbol{P}}(X)|_X)\,.$$

If X is ample in $\boldsymbol{P}$, then by the vanishing theorem (Corollary 3.8), we can calculate the cohomology of the sheaf $\omega^p_{\boldsymbol{P}/\boldsymbol{A}}(\log X)$, using the following resolution.

LEMMA 4.3. *If X is log smooth over U, then for $0 \le p \le n-2$, the following sequence is exact on $\boldsymbol{P}_U = U \otimes_{\boldsymbol{A}} \boldsymbol{P}$;*

$$\begin{aligned} 0 \to{} & \omega^{p+1}_{\boldsymbol{P}/\boldsymbol{A}}(\log X) \to \omega^{p+1}_{\boldsymbol{P}/\boldsymbol{A}}(X) \to \omega^{p+2}_{\boldsymbol{P}/\boldsymbol{A}}(2X)|_X \to \cdots \\ & \cdots \to \omega^{n-1}_{\boldsymbol{P}/\boldsymbol{A}}((n-p-1)X)|_X \to \omega^n_{\boldsymbol{P}/\boldsymbol{A}}((n-p)X)|_X \to 0\,. \end{aligned}$$

PROOF. By the short exact sequence

$$0 \to \mathcal{I}_{X/\boldsymbol{P}}/\mathcal{I}^2_{X/\boldsymbol{P}} \otimes \omega^i_{X/\boldsymbol{A}} \to \omega^{i+1}_{\boldsymbol{P}/\boldsymbol{A}}|_X \to \omega^{i+1}_{X/\boldsymbol{A}} \to 0$$

on $\boldsymbol{P}_U$, we have a short exact sequence

$$0 \to \omega^i_{X/\boldsymbol{A}} \otimes \mathcal{O}_{\boldsymbol{P}}((i-p)X)|_X \to \omega^{i+1}_{\boldsymbol{P}/\boldsymbol{A}}((i-p+1)X)|_X \\ \to \omega^{i+1}_{X/\boldsymbol{A}} \otimes \mathcal{O}_{\boldsymbol{P}}((i-p+1)X)|_X \to 0$$

for $p+1 \le i \le n-1$. By the definition of $\omega^{p+1}_{\boldsymbol{P}/\boldsymbol{A}}(\log X)$, we have a short exact sequence

$$0 \to \omega^{p+1}_{\boldsymbol{P}/\boldsymbol{A}}(\log X) \to \omega^{p+1}_{\boldsymbol{P}/\boldsymbol{A}}(X) \to \omega^{p+1}_{X/\boldsymbol{A}} \otimes \mathcal{O}_{\boldsymbol{P}}(X)|_X \to 0\,.$$

By connecting these short exact sequences, we have the long exact sequence. □

The following is the main theorem in this paper, which describes the cohomology of the sheaf of relative logarithmic differential forms by using the Jacobian ring.

THEOREM 4.4. *If X is ample and log smooth over an affine open subvariety $U = \operatorname{Spec} A_U$ of $\boldsymbol{A} = \operatorname{Spec} A$, and the class $[X] \in \mathrm{Cl}\,(\boldsymbol{P} \smallsetminus E)$ is not divisible by the characteristic of k, then for $0 \le p \le n-1$, there is a natural isomorphism of A_U-modules*

$$H^{n-p-1}(\boldsymbol{P}_U, \omega^{p+1}_{\boldsymbol{P}/\boldsymbol{A}}(\log X)) \simeq A_U \otimes_A R^{[(n-p)X-D]}_{X/\boldsymbol{A}}\,.$$

We prove some lemmas for the proof of Theorem 4.4. Let F be a $\mathrm{Cl}\,(\boldsymbol{P})$-homogeneous polynomial which define the hypersurface X. Since $F \in S^{[X]}_{\boldsymbol{P}} \simeq H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(X))$ and $\partial F/\partial z_\rho \in S^{[X-\boldsymbol{P}_\rho]}_{\boldsymbol{P}} \simeq H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(X-\boldsymbol{P}_\rho))$ for $\rho \in \Sigma_\pi(1)$, we denote by $F : \mathcal{O}_{\boldsymbol{P}} \to \mathcal{O}_{\boldsymbol{P}}(X)$ and $\partial F/\partial z_\rho : \mathcal{O}_{\boldsymbol{P}}(\boldsymbol{P}_\rho) \to \mathcal{O}_{\boldsymbol{P}}(X)$ the multiplication by the global sections.

LEMMA 4.5. *The following diagram is commutative*;

$$\begin{array}{ccc} \mathcal{O}_{\boldsymbol{P}} \otimes_{\boldsymbol{Z}} \mathrm{Cl}\,(\boldsymbol{P} \smallsetminus E)^* & \xrightarrow{[X]} & \mathcal{O}_{\boldsymbol{P}} \\ {\scriptstyle \gamma^*}\downarrow & & \downarrow{\scriptstyle F} \\ \bigoplus_{\rho \in \Sigma_\pi(1)} \mathcal{O}_{\boldsymbol{P}}(\boldsymbol{P}_\rho) & \xrightarrow[(\partial F/\partial z_\rho)_{\rho \in \Sigma_\pi(1)}]{} & \mathcal{O}_{\boldsymbol{P}}(X)\,, \end{array}$$

where the map γ^* is the dual of γ defined in Theorem 3.11.

PROOF. We denote by $F = \sum_e a_e z^e$ the $\mathrm{Cl}\,(\boldsymbol{P})$-homogeneous polynomial defining X, where z^e is the monomial $\prod_{\rho \in \Sigma(1)} z^{e_\rho}$ of degree $[X] \in \mathrm{Cl}\,(\boldsymbol{P})$. Let $\varphi : \mathrm{Cl}\,(\boldsymbol{P} \smallsetminus E) \to \boldsymbol{Z}$ be a homomorphism. Then the image of $1 \otimes \varphi$ by the map $(\partial F/\partial z_\rho)_{\rho \in \Sigma_\pi(1)} \circ \gamma^*$ is

$$\sum_{\rho \in \Sigma_\pi(1)} \varphi([\boldsymbol{P}_\rho]) z_\rho \frac{\partial F}{\partial z_\rho} = \sum_{\rho \in \Sigma_\pi(1)} \varphi([\boldsymbol{P}_\rho]) \sum_e a_e e_\rho z^e = \sum_e a_e \varphi\Big(\Big[\sum_{\rho \in \Sigma_\pi(1)} e_\rho \boldsymbol{P}_\rho\Big]\Big) z^e\,.$$

Since $[\sum_{\rho\in\Sigma_\pi(1)} e_\rho \boldsymbol{P}_\rho] = [X] \in \mathrm{Cl}\,(\boldsymbol{P} \setminus E)$, this is equal to

$$\sum_e a_e \varphi([X]) z^e = \varphi([X]) F\,,$$

which is the image of $1 \otimes \varphi$ by the map $F \circ [X]$. □

LEMMA 4.6. *The following diagram is commutative*;

$$\begin{array}{ccc} \bigoplus_{\rho\in\Sigma_\pi(1)} \omega^n_{\boldsymbol{P}/\boldsymbol{A}}(\boldsymbol{P}_\rho) & \xrightarrow{(\partial F/\partial z_\rho)_{\rho\in\Sigma_\pi(1)}} & \omega^n_{\boldsymbol{P}/\boldsymbol{A}}(X) \\ \Psi \big\downarrow & & \big\downarrow \\ \omega^{n-1}_{\boldsymbol{P}/\boldsymbol{A}} & \xrightarrow[\delta]{} & \omega^n_{\boldsymbol{P}/\boldsymbol{A}}(X)|_X , \end{array}$$

where the map Ψ is the dual of the map Ψ^* defined in Theorem 3.11, *and δ is defined by the restriction*

$$\omega^{n-1}_{\boldsymbol{P}/\boldsymbol{A}} \to \omega^{n-1}_{X/\boldsymbol{A}} \simeq \omega^n_{\boldsymbol{P}/\boldsymbol{A}}(X)|_X .$$

PROOF. We check this on a local affine coordinate $\boldsymbol{A}_\sigma$ for $\sigma \in \Sigma$. There is a $\boldsymbol{T}$-invariant divisor $B = \sum_{\rho\in\Sigma(1)} b_\rho \boldsymbol{P}_\rho$ such that $[B] = [X] \in \mathrm{Cl}\,(\boldsymbol{P})$, and $b_\rho = 0$ for $\rho \not\subseteq \sigma$. Let w be a $k[M \cap \sigma^\vee]$-basis of $\Gamma(\boldsymbol{A}_\sigma, \omega^n_{\boldsymbol{P}/\boldsymbol{A}})$, and χ^{u_ρ} a $k[M\cap\sigma^\vee]$-basis of $\Gamma(\boldsymbol{A}_\sigma, \mathcal{O}_{\boldsymbol{P}}(-\boldsymbol{P}_\rho))$. The image of $w\chi^{-u_\rho} \in \Gamma(\boldsymbol{A}, \omega^n_{\boldsymbol{P}/\boldsymbol{A}}(\boldsymbol{P}_\rho))$ by the map $\delta \circ \Psi$ is

$$\frac{w}{f} \sum_{u\in M\cap\sigma^\vee} a_u \langle u, v_\rho\rangle \chi^{u-u_\rho}|_X ,$$

where $f = \sum_{u\in M\cap\sigma^\vee} a_u \chi^u$ is the local equation of X. On the other hand, the image of $w\chi^{-u_\rho} \in \Gamma(\boldsymbol{A}, \omega^n_{\boldsymbol{P}/\boldsymbol{A}}(\boldsymbol{P}_\rho))$ by the map $\partial F/\partial z_\rho$ is

$$\frac{w}{f} \sum_u a_u(\langle u, v_\rho\rangle + b_\rho)\chi^{u-u_\rho} = \frac{w}{f}\left(\sum_u a_u \langle u, v_\rho\rangle \chi^{u-u_\rho} + b_\rho f \chi^{-u_\rho}\right),$$

whose restriction to X is $(\delta\circ\Psi)(w\chi^{-u_\rho}) \in \Gamma(\boldsymbol{A}_\sigma, \omega^n_{\boldsymbol{P}/\boldsymbol{A}}(X)|_X)$. □

PROOF OF THEOREM 4.4. In the case $p = n-1$, we have

$$A_U \otimes_A R^{[X-D]}_{X/\boldsymbol{A}} \simeq A_U \otimes_A S^{[X-D]}_{\boldsymbol{P}} \simeq H^0(\boldsymbol{P}_U, \omega^n_{\boldsymbol{P}/\boldsymbol{A}}(X)) \simeq H^0(\boldsymbol{P}_U, \omega^n_{\boldsymbol{P}/\boldsymbol{A}}(\log X))\,.$$

We assume that $0 \leq p \leq n-2$. By Lemma 4.5 and Lemma 4.6, we have a commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H^0(\boldsymbol{P}_U, \omega^n_{\boldsymbol{P}/\boldsymbol{A}}((n-p-1)X)) \otimes \mathrm{Cl}\,(\boldsymbol{P} \smallsetminus E)^* & \xrightarrow{\alpha_1} & H^0(\boldsymbol{P}_U, \omega^n_{\boldsymbol{P}/\boldsymbol{A}}((n-p-1)X)) \\
\downarrow & & \downarrow \\
\bigoplus_{\rho \in \Sigma_\pi(1)} H^0(\boldsymbol{P}_U, \omega^n_{\boldsymbol{P}/\boldsymbol{A}}((n-p-1)X + \boldsymbol{P}_\rho)) & \xrightarrow{\alpha_2} & H^0(\boldsymbol{P}_U, \omega^n_{\boldsymbol{P}/\boldsymbol{A}}((n-p)X)) \\
\downarrow & & \downarrow \\
H^0(\boldsymbol{P}_U, \omega^{n-1}_{\boldsymbol{P}/\boldsymbol{A}}((n-p-1)X)) & \xrightarrow{\alpha_3} & H^0(\boldsymbol{P}_U, \omega^n_{\boldsymbol{P}/\boldsymbol{A}}((n-p)X)|_X) \\
\downarrow & & \downarrow \\
0 & & 0\,,
\end{array}$$

where the exactness of the left vertical sequence is shown by Theorem 3.11. By Lemma 4.3 and the vanishing theorem (Corollary 3.8), the cohomology group $H^{n-p-1}(\omega^{p+1}_{\boldsymbol{P}/\boldsymbol{A}}(\log X))$ appears in the cokernel of the map α_3. Since the cokernel of the map α_2 is $A_U \otimes_A R^{[(n-p)X-D]}_{X/\boldsymbol{A}}$, we have to show that the map α_1 is surjective. Since $[X] \in \mathrm{Cl}\,(\boldsymbol{P} \smallsetminus E)$ is not divisible by the characteristic of k, the k-linear map

$$[X] : k \otimes_{\boldsymbol{Z}} \mathrm{Cl}\,(\boldsymbol{P} \smallsetminus E)^* \to k;\ a \otimes \gamma \mapsto \gamma([X])a$$

is surjective. Hence

$$\alpha_1 = 1 \otimes [X] : H^0(\omega^n_{\boldsymbol{P}/\boldsymbol{A}}((n-p-1)X)) \otimes \mathrm{Cl}\,(\boldsymbol{P} \smallsetminus E)^* \to H^0(\omega^n_{\boldsymbol{P}/\boldsymbol{A}}((n-p-1)X))$$

is surjective. □

Using Theorem 4.4, we have a description for the cohomology of relative logarithmic differential forms on X.

COROLLARY 4.7. (1) *For* $0 \leq p \leq (n-3)/2$,

$$A_U \otimes_A R^{[(n-p)X-D]}_{X/\boldsymbol{A}} \simeq H^{n-p-1}(X_U, \omega^p_{X/\boldsymbol{A}})\,,$$

and for $p = (n-2)/2$, *there is an exact sequence*

$$H^{n/2}(\boldsymbol{P}_U, \omega^{n/2}_{\boldsymbol{P}/\boldsymbol{A}}) \to A_U \otimes_A R^{[(n/2+1)X-D]}_{X/\boldsymbol{A}} \to H^{n/2}(X_U, \omega^{n/2-1}_{X/\boldsymbol{A}}) \to 0\,.$$

(2) *If* $\boldsymbol{A} = \boldsymbol{A}^m$ *is an affine space, and* $\mathcal{O}_{\boldsymbol{P}}(-E)$ *is generated by global sections, then for* $n/2 \leq p \leq n-1$,

$$A_U \otimes_A R^{[(n-p)X-D]}_{X/\boldsymbol{A}} \simeq H^{n-p-1}(X_U, \omega^p_{X/\boldsymbol{A}})\,,$$

and for $p = (n-1)/2$, *there is an exact sequence*

$$0 \to A_U \otimes_A R^{[((n+1)/2)X-D]}_{X/\boldsymbol{A}} \to H^{(n-1)/2}(X_U, \omega^{(n-1)/2}_{X/\boldsymbol{A}}) \to H^{(n+1)/2}(\boldsymbol{P}_U, \omega^{(n+1)/2}_{\boldsymbol{P}/\boldsymbol{A}})\,.$$

PROOF. By the short exact sequence

$$0 \to \omega^{p+1}_{\boldsymbol{P}/\boldsymbol{A}} \to \omega^{p+1}_{\boldsymbol{P}/\boldsymbol{A}}(\log X) \to \omega^{p}_{X/A} \to 0\,,$$

we have an exact sequence

$$\begin{aligned} &H^{n-p-1}(\boldsymbol{P}_U, \omega^{p+1}_{\boldsymbol{P}/\boldsymbol{A}}) \to H^{n-p-1}(\boldsymbol{P}_U, \omega^{p+1}_{\boldsymbol{P}/\boldsymbol{A}}(\log X)) \\ &\qquad \to H^{n-p-1}(X_U, \omega^{p}_{X/A}) \to H^{n-p}(\boldsymbol{P}_U, \omega^{p+1}_{\boldsymbol{P}/\boldsymbol{A}})\,. \end{aligned}$$

By Corollary 3.8 (1) for $\mathcal{L} = \mathcal{O}_{\boldsymbol{P}}$, we have $H^q(\boldsymbol{P}, \omega^{p+1}_{\boldsymbol{P}/\boldsymbol{A}}) = 0$ for $q \geq p+2$. Hence $H^q(\boldsymbol{P}_U, \omega^{p+1}_{\boldsymbol{P}/\boldsymbol{A}}) = 0$ for $q \geq p+2$. This induces the statements in (1). When $\boldsymbol{A}$ is an affine space, by the next proposition for $\mathcal{L} = \mathcal{O}_{\boldsymbol{P}}$, we have $H^q(\boldsymbol{P}, \omega^{p+1}_{\boldsymbol{P}/\boldsymbol{A}}) = 0$ for $q \leq p$. Hence $H^q(\boldsymbol{P}_U, \omega^{p+1}_{\boldsymbol{P}/\boldsymbol{A}}) = 0$ for $q \leq p$. This induces the statements in (2). □

PROPOSITION 4.8. *Let $\boldsymbol{A} = \boldsymbol{A}^m$ be an affine space, $\boldsymbol{P}$ a nonsingular toric variety, and $\pi : \boldsymbol{P} \to \boldsymbol{A}$ a log smooth proper equivariant morphism. Let $\mathcal{L}$ be an invertible sheaf on $\boldsymbol{P}$. If $\mathcal{H}om_{\mathcal{O}_{\boldsymbol{P}}}(\mathcal{L}, \mathcal{O}_{\boldsymbol{P}}(-E))$ is generated by global sections, then for $0 \leq q \leq p-1$,*

$$H^q(\boldsymbol{P}, \omega^{p}_{\boldsymbol{P}/\boldsymbol{A}} \otimes_{\mathcal{O}_{\boldsymbol{P}}} \mathcal{L}) = 0\,.$$

PROOF. We set $\mathcal{F} = \mathcal{H}om_{\mathcal{O}_{\boldsymbol{P}}}(\mathcal{L}, \omega^{n-p}_{\boldsymbol{P}/\boldsymbol{A}}(-E))$. By the duality theorem [8, III. Theorem 11.1] for the morphism π, there is an isomorphism

$$\mathrm{Ext}^{n+i}_{D^b(\mathcal{O}_{\boldsymbol{P}})}(\mathcal{F}, \pi^{!}\Omega^m_{\boldsymbol{A}}) \simeq \mathrm{Ext}^{i}_{D^b(\mathcal{O}_{\boldsymbol{A}})}(\boldsymbol{R}\pi_*\mathcal{F}, \Omega^m_{\boldsymbol{A}})\,.$$

Since $\pi^{!}\Omega^m_{\boldsymbol{A}} \simeq \mathcal{O}_{\boldsymbol{P}}(-D-E)$ and $\Omega^m_{\boldsymbol{A}} \simeq \mathcal{O}_{\boldsymbol{A}}$, we have

$$\mathrm{Ext}^{n+i}_{\mathcal{O}_{\boldsymbol{P}}}(\mathcal{F}, \omega^{n}_{\boldsymbol{P}/\boldsymbol{A}}(-E)) \simeq \mathrm{Ext}^{i}_{D^b(\mathcal{O}_{\boldsymbol{A}})}(\boldsymbol{R}\pi_*\mathcal{F}, \mathcal{O}_{\boldsymbol{A}})\,.$$

There is a spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^{i}_{\mathcal{O}_{\boldsymbol{A}}}(R^{-j}\pi_*\mathcal{F}, \mathcal{O}_{\boldsymbol{A}}) \Rightarrow \mathrm{Ext}^{i+j}_{D^b(\mathcal{O}_{\boldsymbol{A}})}(\boldsymbol{R}\pi_*\mathcal{F}, \mathcal{O}_{\boldsymbol{A}})\,.$$

By Corollary 3.8, we have $E_2^{i,j} = 0$ for $-j \geq n-p+1$. Hence

$$H^q(\boldsymbol{P}, \omega^{p}_{\boldsymbol{P}/\boldsymbol{A}} \otimes \mathcal{L}) \simeq \mathrm{Ext}^{q}_{\mathcal{O}_{\boldsymbol{P}}}(\mathcal{F}, \omega^{n}_{\boldsymbol{P}/\boldsymbol{A}}(-E)) \simeq \mathrm{Ext}^{q-n}_{D^b(\mathcal{O}_{\boldsymbol{A}})}(\boldsymbol{R}\pi_*\mathcal{F}, \mathcal{O}_{\boldsymbol{A}}) = 0$$

for $n-q \geq n-p+1$. □

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