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ON THE HOLOMORPHIC AUTOMORPHISM GROUP OF A GENERALIZED HARTOGS TRIANGLE

AKIO KODAMA

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Abstract. In this paper, we completely determine the structure of the holomorphic automorphism group of a generalized Hartogs triangle and obtain natural generalizations of some results due to Landucci and Chen-Xu. These give affirmative answers to some open problems posed by Jarnicki and Pflug.

1. Introduction. For any positive integers ℓ_i, m_j and any positive real numbers p_i, q_j with $1 \le i \le I$, $1 \le j \le J$, we set

$$\ell = (\ell_1, \dots, \ell_I)\,, \; m = (m_1, \dots, m_J)\,, \; p = (p_1, \dots, p_I)\,, \; q = (q_1, \dots, q_J)$$

and define a *generalized Hartogs triangle* $\mathcal{H}^{p,q}_{\ell,m}$ in $\mathbf{C}^N$ by

$$\mathcal{H}^{p,q}_{\ell,m} = \left\{ (z, w) \in \mathbf{C}^N \; ; \; \sum_{i=1}^{I} \|z_i\|^{2p_i} < \sum_{j=1}^{J} \|w_j\|^{2q_j} < 1 \right\},$$

where

$$z = (z_1, \dots, z_I) \in \mathbf{C}^{\ell_1} \times \cdots \times \mathbf{C}^{\ell_I} = \mathbf{C}^{|\ell|}, \;\; |\ell| = \ell_1 + \cdots + \ell_I\,,$$
$$w = (w_1, \dots, w_J) \in \mathbf{C}^{m_1} \times \cdots \times \mathbf{C}^{m_J} = \mathbf{C}^{|m|}, \;\; |m| = m_1 + \cdots + m_J\,,$$
$$\text{and } \; \mathbf{C}^N = \mathbf{C}^{|\ell|} \times \mathbf{C}^{|m|}, \;\; N = |\ell| + |m|\,.$$

For convenience and no loss of generality, in this paper we always assume that

$$p_2, \dots, p_I \neq 1\,, \quad q_2, \dots, q_J \neq 1$$

if $I \ge 2$ or $J \ge 2$. Clearly, this domain is not geometrically convex and its boundary is not smooth and contains the origin $0 = (0, 0)$ of $\mathbf{C}^{|\ell|} \times \mathbf{C}^{|m|} = \mathbf{C}^N$. In the special case where all the $\ell_i = m_j = 1$ and all the p_i, q_j are positive integers, the structure of the holomorphic automorphism group $\mathrm{Aut}(\mathcal{H}^{p,q}_{\ell,m})$ of $\mathcal{H}^{p,q}_{\ell,m}$ was already clarified by Landucci [8] and Chen-Xu [3], [4]. Here we would like to remark that these papers contain the following crucial fact: Let $\Phi \in \mathrm{Aut}(\mathcal{H}^{p,q}_{\ell,m})$ and express $\Phi = (f, g)$ with respect to the coordinate system (z, w) in $\mathbf{C}^{|\ell|} \times \mathbf{C}^{|m|} = \mathbf{C}^N$. Then the w-component mapping $g : \mathcal{H}^{p,q}_{\ell,m} \to \mathbf{C}^{|m|}$ does not depend on the variables z; and hence, it has the form $g(z, w) = g(w)$. And, a glance at their proofs of this fact tells us that the assumptions $\ell_i, m_j = 1$ and $p_i, q_j \in \mathbf{N}$ cannot be avoided with their

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techniques. This raises new difficulties to analyze the structure of $\mathrm{Aut}(\mathcal{H}^{p,q}_{\ell,m})$ in our general case.

The purpose of this paper is to overcome these difficulties and obtain more general results for arbitrary generalized Hartogs triangles $\mathcal{H}^{p,q}_{\ell,m}$. In fact, employing some group-theoretic method, we can avoid their hard part and prove that g is always independent on the variables z for every element $\Phi = (f, g) \in \mathrm{Aut}(\mathcal{H}^{p,q}_{\ell,m})$. Once this is accomplished, our previous results in [6] can be applied to establish the following theorems:

THEOREM 1. *Let $\mathcal{H}^{p,q}_{\ell,m}$ be a generalized Hartogs triangle in $\mathbf{C}^{|\ell|} \times \mathbf{C}^{|m|}$ with $|m| = 1$. Then the holomorphic automorphism group $\mathrm{Aut}(\mathcal{H}^{p,q}_{\ell,m})$ consists of all transformations*

$$\Phi : (z_1, \dots, z_I, w) \longmapsto (\tilde{z}_1, \dots, \tilde{z}_I, \tilde{w})$$

of the following form:

(I) $p_1 = 1,\ q \in \mathbf{N}$: *In this case, we have*

$$\tilde{z}_1 = w^q H(z_1/w^q)\,, \quad \tilde{z}_i = \gamma_i(z_1/w^q) A_i z_{\sigma(i)}\ (2 \le i \le I)\,, \quad \tilde{w} = Bw$$

(*think of z_i as column vectors*), *where*

(1) $H \in \mathrm{Aut}(B^{\ell_1})$, *where B^{ℓ_1} denotes the unit ball in $\mathbf{C}^{\ell_1}$*;

(2) γ_i *are nowhere vanishing holomorphic functions on B^{ℓ_1} defined by*

$$\gamma_i(z_1) = \left(\frac{1 - \|a\|^2}{\big(1 - \langle z_1, a\rangle\big)^2} \right)^{1/2p_i}, \quad a = H^{-1}(o) \in B^{\ell_1}\,,$$

where $\langle\cdot,\cdot\rangle$ denotes the standard Hermitian inner product on $\mathbf{C}^{\ell_1}$ and $o \in B^{\ell_1}$ is the origin of $\mathbf{C}^{\ell_1}$;

(3) $A_i \in U(\ell_i)$, *the unitary group of degree ℓ_i, and $B \in \mathbf{C}$ with $|B| = 1$*;

(4) σ *is a permutation of $\{2, \dots, I\}$ satisfying the following*: $\sigma(i) = s$ *can only happen when $(\ell_i, p_i) = (\ell_s, p_s)$*.

(II) $p_1 \neq 1$ *or* $q \notin \mathbf{N}$: *In this case, we have*

$$\tilde{z}_i = A_i z_{\sigma(i)}\ (1 \le i \le I)\,, \quad \tilde{w} = Bw\,,$$

where $A_i \in U(\ell_i)$, $B \in \mathbf{C}$ with $|B| = 1$, and σ is a permutation of $\{1, \dots, I\}$ satisfying the condition: $\sigma(i) = s$ *can only happen when $(\ell_i, p_i) = (\ell_s, p_s)$*.

THEOREM 2. *Let $\mathcal{H}^{p,q}_{\ell,m}$ be a generalized Hartogs triangle in $\mathbf{C}^{|\ell|} \times \mathbf{C}^{|m|}$ with $|m| \ge 2$. Then the holomorphic automorphism group $\mathrm{Aut}(\mathcal{H}^{p,q}_{\ell,m})$ consists of all transformations*

$$\Phi : (z_1, \dots, z_I, w_1, \dots, w_J) \longmapsto (\tilde{z}_1, \dots, \tilde{z}_I, \tilde{w}_1, \dots, \tilde{w}_J)$$

of the form

$$\tilde{z}_i = A_i z_{\sigma(i)}\ (1 \le i \le I)\,, \quad \tilde{w}_j = B_j w_{\tau(j)}\ (1 \le j \le J)$$

(*think of z_i, w_j as column vectors*), *where $A_i \in U(\ell_i)$, $B_j \in U(m_j)$ and σ, τ are permutations of $\{1, \dots, I\}, \{1, \dots, J\}$ respectively, satisfying the condition*: $\sigma(i) = s,\ \tau(j) = t$ *can only happen when $(\ell_i, p_i) = (\ell_s, p_s)$, $(m_j, q_j) = (m_t, q_t)$*.

Considering the special case where all the $\ell_i, m_j = 1$ in this paper, we obtain natural generalizations of some results due to Landucci [8] and Chen-Xu [3], [4]. In particular, our Theorems 1 and 2 give affirmative answers to some open problems posed in Jarnicki and Pflug [5; Remarks 2.5.15 and 2.5.17].

After some preparations in the next Section 2, we prove our Theorems 1 and 2 in Sections 3 and 4, respectively.

2. Preliminaries and several Lemmas. Throughout this paper, we write $\mathcal{H} = \mathcal{H}^{p,q}_{\ell,m}$ for the sake of simplicity. Also, we often use the following notation: For the given points $z = (z_1, \dots, z_I) \in \mathbf{C}^{|\ell|}$, $w = (w_1, \dots, w_J) \in \mathbf{C}^{|m|}$ and $p = (p_1, \dots, p_I)$, $q = (q_1, \dots, q_J)$ as in the Introduction, we set

$$\zeta = (\zeta_1, \dots, \zeta_N) = (z, w) \in \mathbf{C}^{|\ell|} \times \mathbf{C}^{|m|} = \mathbf{C}^N ,$$

$$\rho^p(z) = \sum_{i=1}^{I} \|z_i\|^{2p_i}, \quad \rho^q(w) = \sum_{j=1}^{J} \|w_j\|^{2q_j} , \text{ and} \tag{2.1}$$

$$\mathcal{E}^p = \left\{z \in \mathbf{C}^{|\ell|} \,;\, \rho^p(z) < 1\right\}, \quad \mathcal{E}^q = \left\{w \in \mathbf{C}^{|m|} \,;\, \rho^q(w) < 1\right\}.$$

We denote by $B(\zeta_o, \delta)$ the Euclidean open ball of radius $\delta > 0$ and center $\zeta_o \in \mathbf{C}^N$. For a subset S of $\mathbf{C}^N$, the boundary (resp. closure) of S in $\mathbf{C}^N$ will be denoted by ∂S (resp. $\overline{S}$). Also, we write as usual

$$\zeta^\alpha = \zeta_1^{\alpha_1} \cdots \zeta_N^{\alpha_N} \quad \text{for } \zeta = (\zeta_1, \dots, \zeta_N) \in \mathbf{C}^N,\ \alpha = (\alpha_1, \dots, \alpha_N) \in \mathbf{Z}^N .$$

Let $S_{\mathcal{H}} = \left\{\alpha \in \mathbf{Z}^N \,;\, \zeta^\alpha \in \mathcal{O}(\mathcal{H}),\ \|\zeta^\alpha\|_{A^2(\mathcal{H})} < \infty\right\}$, where $\mathcal{O}(\mathcal{H})$ denotes the set of all holomorphic functions on $\mathcal{H}$ and $A^2(\mathcal{H})$ is the Bergman space of $\mathcal{H}$ with the norm $\|\cdot\|_{A^2(\mathcal{H})}$. Then it is known [1] that the Bergman kernel function $K = K_{\mathcal{H}}$ for $\mathcal{H}$ can be expressed as

$$K(\zeta, \eta) = \sum_{\alpha \in S_{\mathcal{H}}} c_\alpha \zeta^\alpha \bar{\eta}^\alpha , \quad \zeta,\ \eta \in \mathcal{H}, \tag{2.2}$$

with $c_\alpha > 0$ for each $\alpha \in S_{\mathcal{H}}$. Let $r = (r_1, \dots, r_N) \in \mathbf{R}^N_+$, $\zeta = (\zeta_1, \dots, \zeta_N) \in \mathbf{C}^N$ and set

$$r \cdot \zeta := (r_1\zeta_1, \dots, r_N\zeta_N), \quad 1/r := (1/r_1, \dots, 1/r_N).$$

It then follows from (2.2) that, for $r,\ s \in \mathbf{R}^N_+$ and $\zeta,\ \eta \in \mathbf{C}^N$,

$$K\,(r \cdot \zeta,\ (1/r) \cdot \eta) = K\,(s \cdot \zeta,\ (1/s) \cdot \eta) \tag{2.3}$$

whenever $r \cdot \zeta,\ s \cdot \zeta,\ (1/r) \cdot \eta,\ (1/s) \cdot \eta \in \mathcal{H}$; hence, for any points $\zeta,\ \eta \in \mathcal{H}$,

$$K(r \cdot \zeta,\ (1/r) \cdot \eta) = K(\zeta, \eta) \quad \text{if } r \cdot \zeta,\ (1/r) \cdot \eta \in \mathcal{H}. \tag{2.4}$$

Although, in the proofs of Lemmas 1 and 2 below, there are some overlaps with the papers by Barrett [1], Landucci [8] and Chen-Xu [3], we carry out the proofs in details for the sake of completeness and self-containedness.

LEMMA 1. *The Bergman kernel function $K(\zeta, \eta)$ extends holomorphically in ζ and anti-holomorphically in η to an open neighborhood of $\left(\overline{\mathcal{H}} \setminus \{0\}\right) \times \mathcal{H}$ in $\mathbf{C}^{2N}$.*

PROOF. First of all, let us take two points $\zeta_o \in \partial\mathcal{H} \setminus \{0\}$, $\eta_o \in \mathcal{H}$ arbitrarily and represent $\zeta_o = (z_o, w_o)$ by the (z, w)-coordinates in $\mathbf{C}^{|\ell|} \times \mathbf{C}^{|m|} = \mathbf{C}^N$. Since $\zeta_o = (z_o, w_o) \neq (0, 0)$, one can choose two constants r_o, s_o with $0 < r_o < s_o < 1$ in such a way that $\hat{\zeta}_o := (r_o z_o, s_o w_o) \in \mathcal{H}$. Now we fix small balls $B_{\hat{\zeta}_o}$, B_{η_o} in $\mathbf{C}^N$ with centers $\hat{\zeta}_o$, η_o, respectively, such that $\overline{B_{\hat{\zeta}_o}} \cup \overline{B_{\eta_o}} \subset \mathcal{H}$. Set

$$A_{\zeta_o} := \left\{(z, w) \in \mathbf{C}^{|\ell|} \times \mathbf{C}^{|m|}\,;\ (r_o z, s_o w) \in B_{\hat{\zeta}_o}\right\}.$$

Then $O_{\zeta_o\eta_o} := A_{\zeta_o} \times B_{\eta_o}$ is a geometrically convex open neighborhood of (ζ_o, η_o) in $\mathbf{C}^{2N}$. We may assume that r_o, s_o are selected so close to 1 that

$$\left\{(u/r_o, v/s_o) \in \mathbf{C}^{|\ell|} \times \mathbf{C}^{|m|}\,;\ (u, v) \in B_{\eta_o}\right\} \subset \mathcal{H}.$$

Accordingly we can define a real-analytic function $\widehat{K} = \widehat{K}_{\zeta_o\eta_o}$ on $O_{\zeta_o\eta_o}$ by

$$\widehat{K}((z, w), (u, v)) = K((r_o z, s_o w), (u/r_o, v/s_o)),\ ((z, w), (u, v)) \in O_{\zeta_o\eta_o}.$$

In this way, we obtain a collection

$$\mathcal{K} = \left\{(O_{\zeta_o\eta_o}, \widehat{K}_{\zeta_o\eta_o})\,;\ (\zeta_o, \eta_o) \in (\partial\mathcal{H} \setminus \{0\}) \times \mathcal{H}\right\}$$

satisfying the following: For any elements $\left(O_{\zeta\eta}, \widehat{K}_{\zeta\eta}\right)$, $\left(O_{\zeta'\eta'}, \widehat{K}_{\zeta'\eta'}\right) \in \mathcal{K}$, we have that

$$\widehat{K}_{\zeta\eta} = K \ \text{ on } \ O_{\zeta\eta} \cap (\mathcal{H} \times \mathcal{H}) \quad \text{and} \quad \widehat{K}_{\zeta\eta} = \widehat{K}_{\zeta'\eta'} \ \text{ on } \ O_{\zeta\eta} \cap O_{\zeta'\eta'}$$

by (2.4) and (2.3). Therefore these local extensions $\widehat{K}_{\zeta\eta}$ together provide a global extension of K required in Lemma 1. □

Here let us recall the structure of the holomorphic automorphism group $\mathrm{Aut}(\mathcal{H})$ (cf. [9]). Since $\mathcal{H}$ is a bounded domain in $\mathbf{C}^N$, it has the structure of a real Lie group with respect to the compact-open topology by a well-known theorem of H. Cartan. Note that $\mathrm{Aut}(\mathcal{H})$ has a countable basis for the open sets and a sequence $\{\Phi^\nu\}$ in $\mathrm{Aut}(\mathcal{H})$ converges if and only if $\{\Phi^\nu\}$ converges uniformly on compact subsets of $\mathcal{H}$ to an element $\Phi \in \mathrm{Aut}(\mathcal{H})$. From now on, we denote by

$$G(\mathcal{H}) \text{ the identity component of } \mathrm{Aut}(\mathcal{H}) \text{ with Lie algebra } \mathfrak{g}(\mathcal{H}).$$

As is well-known, $\mathfrak{g}(\mathcal{H})$ can be canonically identified with the real Lie algebra of all complete holomorphic vector fields on $\mathcal{H}$. With this notation, we prove the following:

LEMMA 2. *Let ζ_o be an arbitrary point of $\partial\mathcal{H} \setminus \{0\}$. Then there exist a connected open neighborhood U_{ζ_o} of ζ_o in $\mathbf{C}^N \setminus \{0\}$ and a connected open neighborhood W_{ζ_o} of the identity element $\mathrm{id}_{\mathcal{H}}$ in $G(\mathcal{H})$ such that every element $\Phi \in W_{\zeta_o}$ extends to a holomorphic mapping $\widehat{\Phi} : \mathcal{H} \cup U_{\zeta_o} \to \mathbf{C}^N$.*

PROOF. Let $P : L^2(\mathcal{H}) \to A^2(\mathcal{H})$ be the Bergman projection defined by

$$Pf(\zeta) = \int_{\mathcal{H}} K(\zeta, \eta) f(\eta)\, dV_\eta, \quad f \in L^2(\mathcal{H}).$$

It then follows from Lemma 1 that Pf can be extended to a holomorphic function, say $\widehat{P}f$, defined on some domain $\mathcal{H}\cup U_{\zeta_o}$, where U_{ζ_o} is a connected open neighborhood of ζ_o contained in $\mathbf{C}^N \setminus \{0\}$.

Let $\phi \in C_0^\infty(\mathcal{H})$ be a non-negative function such that $\phi(\zeta_1, \dots, \zeta_N) = \phi(|\zeta_1|, \dots, |\zeta_N|)$ and $\int_{\mathcal{H}} \phi(\zeta) dV_\zeta = 1$. For any $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbf{Z}^N$ with $\alpha_j \geq 0,\ 1 \leq j \leq N$, we set

$$\phi_\alpha(\zeta) = (c_\alpha \alpha!)^{-1}(-1)^{|\alpha|}\partial^{|\alpha|}\phi(\zeta)/\partial\bar{\zeta}_1^{\alpha_1}\cdots\partial\bar{\zeta}_N^{\alpha_N}, \quad \zeta \in \mathcal{H},$$

where c_α is the same constant appearing in (2.2) and $\alpha! = \alpha_1!\cdots\alpha_N!$, $|\alpha| = \alpha_1 + \cdots + \alpha_N$. Then, thanks to the concrete description of the expansion of K as in (2.2), we can compute explicitly $P\phi_\alpha$ as $P\phi_\alpha(\zeta) = \zeta^\alpha,\ \zeta \in \mathcal{H}$. Consequently, by analytic continuation

$$\widehat{P}\phi_\alpha(\zeta) = \zeta^\alpha, \quad \zeta \in \mathcal{H} \cup U_{\zeta_o}. \tag{2.5}$$

Now, let us take a sequence $\{\Phi^\nu\}$ in $G(\mathcal{H})$ converging to the identity element $\mathrm{id}_{\mathcal{H}}$ and express $\Phi^\nu = (\Phi_1^\nu, \dots, \Phi_N^\nu)$ with respect to the ζ-coordinate system in $\mathbf{C}^N$. Let $J_{\Phi^\nu}(\zeta)$ be the Jacobian determinant of Φ^ν at $\zeta \in \mathcal{H}$. Then, applying the transformation law by the Bergman projection under proper holomorphic mapping (cf. [2]) and using the fact (2.5), we have that

$$\begin{aligned}\left(J_{\Phi^\nu}\cdot(\Phi_1^\nu)^{\alpha_1}\cdots(\Phi_N^\nu)^{\alpha_N}\right)(\zeta) &= (J_{\Phi^\nu}\cdot P\phi_\alpha\circ\Phi^\nu)(\zeta)\\ &= P(J_{\Phi^\nu}\cdot\phi_\alpha\circ\Phi^\nu)(\zeta) = \int_{\mathcal{H}} K(\zeta,\eta)(J_{\Phi^\nu}\cdot\phi_\alpha\circ\Phi^\nu)(\eta)\,dV_\eta\end{aligned} \tag{2.6}$$

for $\zeta \in \mathcal{H}$. Here, since the last term extends holomorphically to the function $\widehat{P}(J_{\Phi^\nu}\cdot\phi_\alpha\circ\Phi^\nu)$ on $\mathcal{H}\cup U_{\zeta_o}$, we may assume that $J_{\Phi^\nu}\cdot(\Phi_1^\nu)^{\alpha_1}\cdots(\Phi_N^\nu)^{\alpha_N}$ is also a holomorphic function defined on $\mathcal{H}\cup U_{\zeta_o}$ and satisfies the same equalities there. Moreover, since $\{\Phi^\nu\}$ converges to $\mathrm{id}_{\mathcal{H}}$ uniformly on compact subsets of $\mathcal{H}$, we obtain by the Cauchy estimates that

$$\lim_{\nu\to\infty} J_{\Phi^\nu}(\eta) = 1 \quad\text{and}\quad \lim_{\nu\to\infty}(\phi_\alpha\circ\Phi^\nu)(\eta) = \phi_\alpha(\eta)$$

uniformly on compact subsets of $\mathcal{H}$ and $\mathrm{supp}(\phi_\alpha\circ\Phi^\nu)$ are contained in some compact subset of $\mathcal{H}$ for all ν. Hence, the fact (2.5) immediately yields that

$$\lim_{\nu\to\infty}\left(J_{\Phi^\nu}\cdot(\Phi_1^\nu)^{\alpha_1}\cdots(\Phi_N^\nu)^{\alpha_N}\right)(\zeta) = \int_{\mathcal{H}} K(\zeta,\eta)\phi_\alpha(\eta)\,dV_\eta = \zeta^\alpha, \quad \zeta\in\mathcal{H}\cup U_{\zeta_o},$$

uniformly on compact subsets of $\mathcal{H}\cup U_{\zeta_o}$. Thus, considering the special cases where $\alpha = 0$ and $\alpha_j = 1, \alpha_k = 0$ $(1 \leq j, k \leq N,\ j \neq k)$, we obtain that

$$\lim_{\nu\to\infty} J_{\Phi^\nu}(\zeta) = 1 \quad\text{and}\quad \lim_{\nu\to\infty}\left(J_{\Phi^\nu}\cdot\Phi_j^\nu\right)(\zeta) = \zeta_j, \quad 1 \leq j \leq N, \tag{2.7}$$

uniformly on compact subsets of the domain $\mathcal{H}\cup U_{\zeta_o}$. Clearly this says that, after shrinking U_{ζ_o} and passing to a subsequence if necessary, J_{Φ^ν} are nowhere vanishing holomorphic functions on $\mathcal{H}\cup U_{\zeta_o}$ and so $\Phi^\nu : \mathcal{H}\cup U_{\zeta_o} \to \mathbf{C}^N$ are holomorphic mappings for all $\nu = 1, 2, \dots$.

Since the conclusion of the preceding paragraph is valid for any sequence $\{\Phi^\nu\}$ converging to $\mathrm{id}_{\mathcal{H}}$, it is obvious that there exist an open neighborhood U_{ζ_o} of ζ_o and an open neighborhood W_{ζ_o} of $\mathrm{id}_{\mathcal{H}}$ satisfying the requirement of the lemma. □

We now define compact subsets $\partial_r\mathcal{H}$ of $\partial\mathcal{H} \setminus \{0\}$ by setting

$$\partial_r\mathcal{H} = \{\zeta \in \partial\mathcal{H}\,;\ \|\zeta\| \geq r\}\,, \quad 0 < r < 1\,.$$

Then we can prove the following:

LEMMA 3. *For any compact subset $\partial_r\mathcal{H}$ of $\partial\mathcal{H} \setminus \{0\}$ defined as above, there exist a bounded Reinhardt domain D_r in $\mathbf{C}^N \setminus \{0\}$ and a connected open neighborhood O_r of $\mathrm{id}_{\mathcal{H}}$ in $G(\mathcal{H})$ satisfying the following*:

(1) $\mathcal{H} \cup \partial_r\mathcal{H} \subset D_r$;
(2) *every element $\Phi \in O_r$ extends to a holomorphic mapping $\widehat{\Phi} : D_r \to \mathbf{C}^N$.*

PROOF. For each point $\zeta_o \in \partial\mathcal{H} \setminus \{0\}$, we take a connected open neighborhood U_{ζ_o} of ζ_o and a connected open neighborhood W_{ζ_o} of $\mathrm{id}_{\mathcal{H}}$ satisfying the condition in Lemma 2. Then, by the compactness of $\partial_r\mathcal{H}$ there are finitely many points $\zeta^i \in \partial_r\mathcal{H},\ 1 \leq i \leq n_0$, such that $\partial_r\mathcal{H} \subset \bigcup_{i=1}^{n_0} U_{\zeta^i}$. Since $\partial_r\mathcal{H}$ is invariant under the standard action of the N-dimensional torus T^N on $\mathbf{C}^N$ as well as $\mathcal{H}$, we can now find a Reinhardt domain D_r satisying

$$\mathcal{H} \cup \partial_r\mathcal{H} \subset D_r \subset \mathcal{H} \cup \left(\bigcup_{i=1}^{n_0} U_{\zeta^i}\right)\,. \tag{2.8}$$

Let O_r be the connected component of $\bigcap_{i=1}^{n_0} W_{\zeta^i}$ containing the identity $\mathrm{id}_{\mathcal{H}}$. Then it is clear that the pair (D_r, O_r) satisfies the requirement of Lemma 3. □

LEMMA 4. *For any compact subset $\partial_r\mathcal{H}$ of $\partial\mathcal{H} \setminus \{0\}$, there exists a bounded Reinhardt domain $\widehat{D}_r$ in $\mathbf{C}^N \setminus \{0\}$ satisfying the following*:

(1) $\mathcal{H} \cup \partial_r\mathcal{H} \subset \widehat{D}_r$;
(2) *every element $X \in \mathfrak{g}(\mathcal{H})$ extends to a holomorphic vector field $\widehat{X}$ on $\widehat{D}_r$.*

PROOF. By Lemma 3 there exist a bounded Reinhardt domain D_r in $\mathbf{C}^N$ and a connected open neighborhood O_r of $\mathrm{id}_{\mathcal{H}}$ in $G(\mathcal{H})$ such that every element $\Phi \in O_r$ extends to a holomorphic mapping $\widehat{\Phi} : D_r \to \mathbf{C}^N$. Moreover, for any $\varepsilon > 0$ and any compact set $L \subset D_r$, it follows from (2.7) and (2.8) that

$$\|\widehat{\Phi}(\zeta) - \zeta\| < \varepsilon \quad \text{for all}\ \ \zeta \in L,\ \Phi \in O_r\,, \tag{2.9}$$

provided that O_r is sufficiently small.

Now, let $X \in \mathfrak{g}(\mathcal{H})$ and $\{\Phi_t = \exp tX\}_{t\in\mathbf{R}}$ the one-parameter subgroup of $G(\mathcal{H})$ generated by X. Then, thanks to the fact (2.9), one can choose a constant $\varepsilon_o > 0$ satisfying the following conditions: Let $\zeta_o \in \partial_r\mathcal{H}$ and let $B(\zeta_o, \delta(\zeta_o))$ be a small ball such that $B(\zeta_o, 2\delta(\zeta_o)) \subset D_r$. Then

(2.10) Φ_t extends to a holomorphic mapping $\widehat{\Phi}_t : D_r \to \mathbf{C}^N$; and

(2.11) $\widehat{\Phi}_t(B(\zeta_o, \delta(\zeta_o))) \subset B(\zeta_o, 2\delta(\zeta_o))$

for every $t \in \mathbf{R}$ with $|t| < \varepsilon_o$. Under this situation, since $\{\Phi_t\}_{t\in\mathbf{R}}$ is a global one-parameter subgroup of $G(\mathcal{H})$, we obtain by analytic continuation that

$$\widehat{\Phi}_s\big(\widehat{\Phi}_t(\zeta)\big) = \widehat{\Phi}_{s+t}(\zeta)\,, \ \ \zeta \in B(\zeta_o, \delta(\zeta_o))\,, \quad \text{whenever } |s|, |t|, |s+t| < \varepsilon_o\,;$$

accordingly $\{\widehat{\Phi}_t\}_{|t|<\varepsilon_o}$ is a one-parameter local group of local holomorphic transformations. Let $\widehat{X}$ be the holomorphic vector field on $B(\zeta_o, \delta(\zeta_o))$ induced by $\{\widehat{\Phi}_t\}_{|t|<\varepsilon_o}$. Then it is obvious that $\widehat{X}$ is a unique holomorphic extension of X to $B(\zeta_o, \delta(\zeta_o))$. Since $\zeta_o \in \partial_r\mathcal{H}$ is arbitrary and $\partial_r\mathcal{H}$ is compact, by repeating the same argument as in the proof of Lemma 3, we can find a Reinhardt domain $\widehat{D}_r$ satisfying the requirement of Lemma 4. □

Before proceeding, we need to introduce some terminology. Let $T^N = (U(1))^N$ be the N-dimensional torus. Then T^N acts as a group of holomorphic automorphisms on $\mathbf{C}^N$ by the standard rule

$$\alpha \cdot \zeta = (\alpha_1\zeta_1, \dots, \alpha_N\zeta_N) \quad \text{for } \alpha = (\alpha_i) \in T^N\,, \ \ \zeta = (\zeta_i) \in \mathbf{C}^N\,.$$

Let D be an arbitrary Reinhardt domain in $\mathbf{C}^N$. Then, just by the definition, D is invariant under this action of T^N. Each element $\alpha \in T^N$ then induces an automorphism π_α of D given by $\pi_\alpha(\zeta) = \alpha \cdot \zeta$, and the mapping ρ_D sending α to π_α is an injective continuous group homomorphism of T^N into $\mathrm{Aut}(D)$. The subgroup $\rho_D(T^N)$ of $\mathrm{Aut}(D)$ is denoted by $T(D)$. Analogously, the multiplicative group $(\mathbf{C}^*)^N$ acts as a group of automorphisms on $\mathbf{C}^N$. So, denoting by $\Pi(D) = \big\{\alpha \in (\mathbf{C}^*)^N\,;\ \alpha \cdot D \subset D\big\}$, we obtain the topological subgroup $\Pi(D)$ of $\mathrm{Aut}(D)$. We have one more important topological subgroup $\mathrm{Aut}_{\mathrm{alg}}(D)$ of $\mathrm{Aut}(D)$ consisting of all elements Φ of $\mathrm{Aut}(D)$ such that the i-th component function Φ_i of Φ is given by a Laurent monomial having the form

$$\Phi_i(\zeta) = \lambda_i \zeta_1^{a_{i1}} \cdots \zeta_N^{a_{iN}}, \quad 1 \le i \le N\,, \tag{2.12}$$

where $(a_{ij}) \in GL(N, \mathbf{Z})$ and $(\lambda_i) \in (\mathbf{C}^*)^N$. We call $\mathrm{Aut}_{\mathrm{alg}}(D)$ the *algebraic automorphism group of* D and each element of $\mathrm{Aut}_{\mathrm{alg}}(D)$ is called an *algebraic automorphism of* D. It is known [7] that these groups are related in the following manner: The centralizer of the torus $T(D)$ in $\mathrm{Aut}(D)$ is given by $\Pi(D)$, while the normalizer of $T(D)$ in $\mathrm{Aut}(D)$ is given by $\mathrm{Aut}_{\mathrm{alg}}(D)$. Here we consider the mapping $\varpi : \mathrm{Aut}_{\mathrm{alg}}(D) \to GL(N, \mathbf{Z})$ that sends an element Φ of $\mathrm{Aut}_{\mathrm{alg}}(D)$ whose i-th component is given by (2.12) into the element $(a_{ij}) \in GL(N, \mathbf{Z})$. Then it is easy to see that ϖ is a group homomorphism with $\ker\varpi = \Pi(D)$; and hence it induces a group isomorphism

$$\mathrm{Aut}_{\mathrm{alg}}(D)/\Pi(D) \ \xrightarrow{\cong}\ \mathcal{G}(D) := \varpi(\mathrm{Aut}_{\mathrm{alg}}(D)) \ \subset\ GL(N, \mathbf{Z})\,.$$

Let $G(D)$ be the identity component of $\mathrm{Aut}(D)$. Then we know the following fundamental result due to Shimizu [11]:

(2.13) Every element $\Phi \in \mathrm{Aut}(D)$ can be written in the form $\Phi = \Phi'\Phi''$, where $\Phi' \in G(D)$ and $\Phi'' \in \mathrm{Aut}_{\mathrm{alg}}(D)$.

Now let us consider the special case where D is our generalized Hartogs triangle $\mathcal{H}$. Then we have the following:

LEMMA 5. *Every element* $\Phi \in \mathrm{Aut}_{\mathrm{alg}}(\mathcal{H})$ *can be written in the form*

$$\Phi(\zeta) = \big(\lambda_1 \zeta_{\sigma(1)} \zeta_N^{b_1}, \dots, \lambda_{|\ell|} \zeta_{\sigma(|\ell|)} \zeta_N^{b_{|\ell|}}, \lambda_N \zeta_N\big) \ \text{or}$$

$$\Phi(\zeta) = \big(\lambda_1 \zeta_{\sigma(1)}, \dots, \lambda_{|\ell|} \zeta_{\sigma(|\ell|)}, \lambda_{|\ell|+1} \zeta_{\tau(|\ell|+1)}, \dots, \lambda_N \zeta_{\tau(N)}\big)$$

according as $|m| = 1$ *or* $|m| \geq 2$, *where* $(\lambda_i) \in T^N$, $(b_i) \in \mathbf{Z}^{|\ell|}$, *and* σ, τ *are permutations of* $\{1, \dots, |\ell|\}$, $\{|\ell|+1, \dots, N\}$ *respectively.*

PROOF. We assume that the i-th component function Φ_i of Φ is given by (2.12).

We first consider the case $|m| = 1$. Since every point of the form $(0, w) \in \mathbf{C}^{|\ell|} \times \mathbf{C}$ with $w \in \Delta^* = \Delta \setminus \{0\}$, the punctured disc, belongs to $\mathcal{H}$, it is easily seen that Φ_N has the form $\Phi_N(\zeta) = \lambda_N \zeta_N$, $|\lambda_N| = 1$, and the matrix $\varpi(\Phi) \in GL(N, \mathbf{Z})$ can be written as

$$\varpi(\Phi) = \begin{pmatrix} a_{11} & \cdots & a_{1|\ell|} & a_{1N} \\ \vdots & \ddots & \vdots & \vdots \\ a_{|\ell|1} & \cdots & a_{|\ell||\ell|} & a_{|\ell|N} \\ 0 & \cdots & 0 & 1 \end{pmatrix} \quad \text{with } a_{ij} \geq 0 \text{ for } 1 \leq i, j \leq |\ell| .$$

We claim here that the submatrix $A := (a_{ij})_{1 \leq i, j \leq |\ell|}$ is a permutation matrix, that is, there exists a permutation σ of $\{1, \dots, |\ell|\}$ such that $a_{ij} = \delta_{\sigma(i)j}$ for all $1 \leq i, j \leq |\ell|$. Indeed, notice that the mapping $\zeta \mapsto \big(\zeta_1, \dots, \zeta_{|\ell|}, \lambda_N^{-1} \zeta_N\big)$, $\zeta \in \mathcal{H}$, belongs to $\mathrm{Aut}_{\mathrm{alg}}(\mathcal{H})$; and hence one may assume that $\Phi_N(\zeta) = \zeta_N$. Then, for any given point $\zeta_N \in \Delta^*$, the mapping $\widetilde{\Phi}(z) := \big(\Phi_1(z, \zeta_N), \dots, \Phi_{|\ell|}(z, \zeta_N)\big)$ gives rise to a holomorphic automorphism of the bounded Reinhardt domain $\big\{z \in \mathbf{C}^{|\ell|} ;\ \rho^p(z) < |\zeta_N|^{2q}\big\}$ containing the origin of $\mathbf{C}^{|\ell|}$ and, in particular, it maps the complex analytic subset $\mathcal{H} \cap \big\{\zeta \in \mathbf{C}^N ;\ \zeta_i = 0\big\}$ of $\mathcal{H}$ onto some equidimensional complex analytic subset of $\mathcal{H}$ for each $1 \leq i \leq |\ell|$. This yields at once that A is a permutation matrix, as claimed. Therefore, putting $b_i = a_{iN}$, $1 \leq i \leq |\ell|$, we have seen that Φ has the form

$$\Phi(\zeta) = \big(\lambda_1 \zeta_{\sigma(1)} \zeta_N^{b_1}, \dots, \lambda_{|\ell|} \zeta_{\sigma(|\ell|)} \zeta_N^{b_{|\ell|}}, \lambda_N \zeta_N\big) . \tag{2.14}$$

In particular, this says that Φ extends to a holomorphic automorphism of $\mathbf{C}^{|\ell|} \times \mathbf{C}^*$ with $\Phi(\partial\mathcal{H} \setminus \{0\}) \subset \partial\mathcal{H} \setminus \{0\}$. Using this fact, we would like to check that $|\lambda_i| = 1$ for every $1 \leq i \leq |\ell|$. To this end, let $\sigma(i) = s$ and choose an arbitrary element

$$\zeta[s] := (0, \dots, 0, \zeta_s, 0, \dots, 0, \zeta_N) \in \partial\mathcal{H} \quad \text{with } \zeta_N \in \Delta^* .$$

Then, by (2.14), $\Phi(\zeta[s]) = \big(0, \dots, 0, \lambda_i \zeta_s \zeta_N^{b_i}, 0, \dots, 0, \lambda_N \zeta_N\big) \in \partial\mathcal{H}$. Thus we have

$$|\lambda_i \zeta_s \zeta_N^{b_i}|^{2p_a} = |\zeta_N|^{2q} \quad \text{whenever } |\zeta_s|^{2p_b} = |\zeta_N|^{2q} < 1 ,$$

where p_a, p_b are some positive constants appearing in the definition of $\mathcal{H} = \mathcal{H}_{\ell,m}^{p,q}$. Therefore, letting $|\zeta_N| \to 1$, we conclude that $|\lambda_i| = 1$, as desired.

Next we consider the case $|m| \geq 2$. In this case, notice that the Reinhardt domain $\mathcal{H}$ satisfies the condition that $\mathcal{H} \cap \big\{\zeta \in \mathbf{C}^N ;\ \zeta_i = 0\big\} \neq \emptyset$ for each $1 \leq i \leq N$. Hence every component function Φ_i of Φ extends to a holomorphic function on $\mathcal{E}^p \times \mathcal{E}^q$, where $\mathcal{E}^p$ and $\mathcal{E}^q$ are the generalized complex ellipsoids defined in (2.1) (cf. [9; p.15]). Consequently, since $\mathcal{E}^p \times \mathcal{E}^q$ contains the origin $(0, 0) \in \mathbf{C}^{|\ell|} \times \mathbf{C}^{|m|}$, every component a_{ij} of $\varpi(\Phi) =$

$(a_{ij}) \in GL(N, \mathbf{Z})$ has to be non-negative. Hence $\varpi(\Phi)$ reduces to a permutation matrix, because Φ is a holomorphic automorphism of $\mathcal{H}$ and so it maps the complex hypersurface $\mathcal{H} \cap \{\zeta \in \mathbf{C}^N\,;\, \zeta_i = 0\}$ of $\mathcal{H}$ onto another one for every $1 \le i \le N$. This, combined with the fact that $\mathcal{H}$ contains the points having the form $(0, w)$, yields at once that the mapping $g := (\Phi_{|\ell|+1}, \ldots, \Phi_N)$ does not depend on the variables z. From these facts, we deduce that there exist permutations σ of $\{1, \ldots, |\ell|\}$ and τ of $\{|\ell|+1, \ldots, N\}$ with respect to which Φ can be written in the form

$$\Phi(\zeta) = \left(\lambda_1\zeta_{\sigma(1)}, \ldots, \lambda_{|\ell|}\zeta_{\sigma(|\ell|)}, \lambda_{|\ell|+1}\zeta_{\tau(|\ell|+1)}, \ldots, \lambda_N\zeta_{\tau(N)}\right)\,,$$

where $(\lambda_i) \in (\mathbf{C}^*)^N$. In particular, if we express $\Phi = (f, g)$ by coordinates (z, w) in $\mathbf{C}^{|\ell|} \times \mathbf{C}^{|m|} = \mathbf{C}^N$, then f and g may be regarded as the linear automorphisms of $\mathbf{C}^{|\ell|}$ and of $\mathbf{C}^{|m|}$, respectively, such that $f(\partial\mathcal{E}^p) \subset \partial\mathcal{E}^p$ and $g(\partial\mathcal{E}^q) \subset \partial\mathcal{E}^q$. These inclusions immediately yield that $|\lambda_i| = 1$ for every $1 \le i \le N$. Therefore we have completed the proof of Lemma 5. □

LEMMA 6. *Let $\Psi \in \mathrm{Aut}(\mathcal{H})$ and write $\Psi = (h, k)$ with respect to the coordinate system (z, w) in $\mathbf{C}^{|\ell|} \times \mathbf{C}^{|m|} = \mathbf{C}^N$. Then $k : \mathcal{H} \to \mathbf{C}^{|m|}$ does not depend on the variables z; accordingly it has the form $k(z, w) = k(w)$ on $\mathcal{H}$.*

PROOF. Once it is shown that g does not depend on z for every $\Phi = (f, g) \in G(\mathcal{H})$, then our conclusion immediately follows from the fact (2.13) and Lemma 5. Thus we have only to show the lemma when $\Psi \in G(\mathcal{H})$.

To this end, pick a point $\zeta_o = (0, w_o) = (0, \ldots, 0, w_1^o, \ldots, w_J^o) \in \partial\mathcal{H}$ with

$$\|w_1^o\| \cdots \|w_J^o\| \neq 0 \quad \text{and} \quad \rho^q(w_o) = 1\,,$$

where ρ^q is the function appearing in (2.1), and fix an $r \in \mathbf{R}$ with $0 < r < \|\zeta_o\|$. Then $\zeta_o \in \partial_r\mathcal{H}$ and by Lemma 3 there exist a bounded Reinhardt domain $D := D_r$ in $\mathbf{C}^N$ containing $\mathcal{H} \cup \partial_r\mathcal{H}$ and an open neighborhood $O := O_r$ of $\mathrm{id}_{\mathcal{H}}$ in $G(\mathcal{H})$ such that every element $\Phi \in O$ extends to a holomorphic mapping, say again, $\Phi : D \to \mathbf{C}^N$. Here we choose sufficiently small constants δ_1, δ_2 with $0 < \delta_1 < \delta_2 < 1$ and set

$$U_i = \{z \in \mathbf{C}^{|\ell|}\,;\, \rho^p(z) < \delta_i\},$$
$$V_i = \{w \in \mathbf{C}^{|m|}\,;\, 1 - \delta_i < \rho^q(w) < 1 + \delta_i,\ \|w_1\| \cdots \|w_J\| \neq 0\}$$

for $i = 1, 2$. Then $U_i \times V_i$ $(i = 1, 2)$ are bounded Reinhardt domains in $\mathbf{C}^{|\ell|} \times \mathbf{C}^{|m|} = \mathbf{C}^N$ satisfying the condition

$$\zeta_o \in U_1 \times V_1 \subset \overline{U_1 \times V_1} \subset U_2 \times V_2 \subset \overline{U_2 \times V_2} \subset D$$

and the restriction of ρ^q to V_2 gives a C^ω-smooth strictly plurisubharmonic function on V_2. Moreover, after shrinking O if necessary, we may assume by (2.9) that $\Phi(U_1 \times V_1) \subset U_2 \times V_2$ for every $\Phi \in O$.

Now, taking an element $\Phi = (f, g) \in O$ and a point $w \in V_1$ with $\rho^q(w) = 1$ arbitrarily, we set $g_w(z) = g(z, w)$, $z \in U_1$, for a while. Then, since $g_w(U_1) \subset V_2$, we can define a

C^ω-smooth plurisubharmonic function $\hat\rho$ on U_1 by setting $\hat\rho(z) := \rho^q(g_w(z)),\ z \in U_1$. It then follows that $\hat\rho(z) = 1$ on U_1, since

$$\Phi(U_1 \times \{w\}) \subset \partial\mathcal{H} \cap (U_2 \times V_2) \subset \left\{(u,v) \in U_2 \times V_2\,;\ \rho^q(v) = 1\right\}.$$

This combined with the strictly plurisubharmonicity of ρ^q on V_2 implies that $g_w(z)$ is a constant mapping on U_1. As a result, defining the real-analytic hypersurface of V_1 by setting $H := \{w \in V_1\,;\ \rho^q(w) = 1\}$, we have shown that

(2.15) for any $w \in H$, $g_w(z) = g(z,w)$ is constant on U_1.

Now, being a holomorphic mapping on the Reinhardt domain D containing $\mathcal{H} \cup \partial_r\mathcal{H}$, g can be expanded uniquely as

$$g(z,w) = g(\zeta',\zeta'') = \sum_{\nu'} a_{\nu'}(\zeta'')(\zeta')^{\nu'}, \quad \zeta = (\zeta',\zeta'') \in D, \tag{2.16}$$

which converges uniformly on compact subsets of D, where

$$\zeta' = (\zeta_1,\dots,\zeta_{|\ell|}) = z \in \mathbf{C}^{|\ell|},\ \ \zeta'' = (\zeta_{|\ell|+1},\dots,\zeta_N) = w \in \mathbf{C}^{|m|},$$

$a_{\nu'}(\zeta'') = \left(a^1_{\nu'}(\zeta''),\dots,a^{|m|}_{\nu'}(\zeta'')\right)$ are $|m|$-tuples of holomorphic functions, and the summation is taken over all $\nu' = (\nu_1,\dots,\nu_{|\ell|}) \in \mathbf{Z}^{|\ell|}$ with $\nu_1,\dots,\nu_{|\ell|} \ge 0$ (cf. [9]). In particular, the expansion of g in (2.16) converges uniformly on the domain $U_1 \times V_1$ and every $a_{\nu'}(\zeta'')$ is holomorphic on V_1. Then the assertion (2.15) tells us that

$$a_{\nu'}(\zeta'') = 0\,,\ \ \zeta'' \in H, \quad \text{for } \nu' \neq 0\,.$$

Since $a_{\nu'}(\zeta'')$ are holomorphic on V_1 and H is a real-analytic hypersurface of V_1, it is obvious that $a_{\nu'}(\zeta'') = 0$ on V_1 for $\nu' \neq 0$; and hence, by analytic continuation $g(z,w) = a_0(\zeta'')$ does not depend on $z = \zeta'$ globally; proving our lemma for every element $\Phi = (f,g)$ contained in the open neighborhood O of $\mathrm{id}_{\mathcal{H}}$ in $G(\mathcal{H})$.

Finally, recall that a connected topological group is always generated by any neighborhood of the identity id. Hence, replacing O by the open neighborhood $O \cap \{\Phi^{-1}\,;\ \Phi \in O\}$ of $\mathrm{id}_{\mathcal{H}}$ if necessary, we may assume that the given element $\Psi = (h,k) \in G(\mathcal{H})$ can be represented as a finite product $\Psi = \Phi_1 \cdots \Phi_s$ of elements $\Phi_i \in O$. This together with the result of the preceding paragraph guarantees that $k(z,w)$ does not depend on the variables z; completing the proof of Lemma 6. □

We finish this section by the following:

LEMMA 7. *Let Ω be a domain in $\mathbf{C}^n$ and let $A : \Omega \to U(L)$ be a mapping from Ω into the unitary group $U(L)$ of degree L. Assume that all the ij-components a_{ij} of A are holomorphic functions on Ω. Then A is a constant mapping.*

PROOF. By our assumption we have

$$\sum_{j=1}^{L} |a_{ij}(z)|^2 = 1\,,\ \ z \in \Omega\,, \quad \text{for every } 1 \le i \le L\,.$$

Then, since all the a_{ij} are holomorphic on Ω, it is easily seen that $\partial a_{ij}(z)/\partial z_k \equiv 0$ on Ω for all i, j and k. Clearly this implies that A is a constant mapping, as desired. □

3. Proof of Theorem 1. The proof will be carried out in the following two Subsections.

3.1. CASE (I). $p_1 = 1$, $q_1 = q \in \mathbf{N}$: When $I = 1$, that is, for the case $\mathcal{H} = \{(z, w) \in \mathbf{C}^{\ell_1} \times \mathbf{C}\,;\ \|z\|^2 < |w|^{2q} < 1\}$, we consider the mapping $\Lambda_1 : \mathcal{H} \to \mathbf{C}^{\ell_1+1}$ defined by

$$\Lambda_1(z, w) = (z/w^q, w)\,, \quad (z, w) \in \mathcal{H}\,.$$

Then Λ_1 gives rise to a biholomorphic mapping from $\mathcal{H}$ onto $B^{\ell_1} \times \Delta^*$. On the other hand, if we denote by $G(D)$ the identity component of $\mathrm{Aut}(D)$ for a given domain D, we have that $G(B^{\ell_1} \times \Delta^*) = G(B^{\ell_1}) \times G(\Delta^*)$ by a well-known theorem of H. Cartan. Moreover, with exactly the same argument as in the proof of Lemma 5, one can see that every element $\Phi \in \mathrm{Aut}_{\mathrm{alg}}(B^{\ell_1} \times \Delta^*)$ can be written as in (2.14) with $|\ell| = \ell_1$, $\zeta = (\zeta_1, \ldots, \zeta_{\ell_1}, \zeta_N) \in B^{\ell_1} \times \Delta^*$ and $|\lambda_N| = 1$. More precisely, we assert here that $|\lambda_i| = 1$, $b_i = 0$ for every $1 \le i \le \ell_1$. To verify this, notice that Φ is now regarded as a holomorphic automorphism of $\mathbf{C}^{\ell_1} \times \mathbf{C}^*$; accordingly, it leaves the boundary of $B^{\ell_1} \times \Delta^*$ invariant. Thus

$$\sum_{i=1}^{\ell_1} |\lambda_i \zeta_{\sigma(i)} \zeta_N^{b_i}|^2 = 1 \quad \text{whenever} \quad \sum_{i=1}^{\ell_1} |\zeta_i|^2 = 1\,,\ \ \zeta_N \in \Delta^*\,.$$

Clearly, this says that $|\lambda_i| = 1$, $b_i = 0$ for every $1 \le i \le \ell_1$, as asserted. As a result, we have shown that $\mathrm{Aut}_{\mathrm{alg}}(B^{\ell_1} \times \Delta^*) = \mathrm{Aut}_{\mathrm{alg}}(B^{\ell_1}) \times \mathrm{Aut}_{\mathrm{alg}}(\Delta^*)$ and hence $\mathrm{Aut}(B^{\ell_1} \times \Delta^*) = \mathrm{Aut}(B^{\ell_1}) \times \mathrm{Aut}(\Delta^*)$ by (2.13). Therefore we conclude that every element $\Phi \in \mathrm{Aut}(\mathcal{H})$ can be described as

$$\Phi(z, w) = (w^q H(z/w^q), Bw)\,, \quad (z, w) \in \mathcal{H}\,, \tag{3.1}$$

where $H \in \mathrm{Aut}(B^{\ell_1})$ and $B \in \mathbf{C}$ with $|B| = 1$; proving Theorem 1, (I), in the case of $I = 1$.

Next, consider the case where $I \ge 2$. By the identity in [10; Theorem 2.2.5, (2)], it is easy to check that the mapping Φ having the form as in Theorem 1, (I), belongs to $\mathrm{Aut}(\mathcal{H})$. So, taking an arbitrary element $\Phi \in \mathrm{Aut}(\mathcal{H})$, we would like to show that Φ can be described as in the theorem. To this end, write $\Phi = (f, g)$ with respect to the coordinate system (z, w) in $\mathbf{C}^{|\ell|} \times \mathbf{C}$. Then g does not depend on the variables z by Lemma 6. Hence g induces a holomorphic automorphism of Δ^*; so that g has the form $g(w) = Bw$ with $|B| = 1$. Let us define a holomorphic automorphism Φ_B of $\mathcal{H}$ by $\Phi_B(z, w) = (z, B^{-1}w)$. Replacing Φ by $\Phi_B \Phi$ if necessary, we may now assume that Φ has the form $\Phi(z, w) = (f(z, w), w)$ on $\mathcal{H}$. Therefore, if we set

$$\mathcal{E}_w^p = \{z \in \mathbf{C}^{|\ell|}\,;\ \rho^p(z) < |w|^{2q}\}\,, \quad f_w(z) = f(z, w),\ z \in \mathcal{E}_w^p\,, \tag{3.2}$$

for an arbitrarily given point $w \in \Delta^*$, then f_w is a holomorphic automorphism of $\mathcal{E}^p_w$. On the other hand, putting

$$\mathcal{E}^p = \left\{ \xi \in \mathbf{C}^{|\ell|} \, ; \, \sum_{i=1}^{I} \|\xi_i\|^{2p_i} < 1 \right\} \quad \text{and} \quad r_i = \frac{1}{|w|^{q/p_i}} \, , \quad 1 \le i \le I \, , \tag{3.3}$$

where $\xi = (\xi_1, \ldots, \xi_I) \in \mathbf{C}^{\ell_1} \times \cdots \times \mathbf{C}^{\ell_I} = \mathbf{C}^{|\ell|}$, and noting the facts that $p_1 = 1$ and $q \in \mathbf{N}$, we have the biholomorphic mapping $\Lambda : \mathcal{E}^p_w \to \mathcal{E}^p$ defined by

$$\Lambda(z) = \left(z_1/w^q, r_2 z_2, \ldots, r_I z_I\right), \quad z = (z_1, \ldots, z_I) \in \mathcal{E}^p_w \, .$$

Recall here our previous result in [6]: When $p_1 = 1$, every holomorphic automorphism Ψ of $\mathcal{E}^p$ has the form

$$\Psi(\xi) = \left(H(\xi_1), \gamma_2(\xi_1) A_2 \xi_{\sigma(2)}, \ldots, \gamma_I(\xi_1) A_I \xi_{\sigma(I)}\right) \, ,$$

where $H \in \mathrm{Aut}(B^{\ell_1})$, $A_i \in U(\ell_i)$ and γ_i are nowhere vanishing holomorphic functions on B^{ℓ_1} given as in Theorem 1, (I), with $z_1 = \xi_1$, and σ is a permutation of $\{2, \ldots, I\}$ having the property: $\sigma(i) = s$ can only happen when $(\ell_i, p_i) = (\ell_s, p_s)$. Then, applying this result to the holomorphic automorphism $\Lambda \circ f_w \circ \Lambda^{-1}$ of $\mathcal{E}^p$ and noting the fact that $r_i = r_s$ if $\sigma(i) = s$, one can see that f_w has the form

$$f_w(z) = \left(w^q H(z_1/w^q), \gamma_2(z_1/w^q) A_2 z_{\sigma(2)}, \ldots, \gamma_I(z_1/w^q) A_I z_{\sigma(I)}\right), \tag{3.4}$$

where $H \in \mathrm{Aut}(B^{\ell_1})$, $A_i \in U(\ell_i)$ and γ_i are nowhere vanishing holomorphic functions on B^{ℓ_1} determined uniquely by H, and σ is a permutation of $\{2, \ldots, I\}$ having the property: $\sigma(i) = s$ occurs only when $(\ell_i, p_i) = (\ell_s, p_s)$. Of course, all the H, A_i, γ_i and σ are determined by the given point $w \in \Delta^*$; accordingly, expressing them as H^w, A_i^w, γ_i^w and σ^w, we obtain a family $\mathcal{F} = \left\{\left(H^w, A_i^w, \gamma_i^w, \sigma^w\right)\right\}_{w \in \Delta^*}$. The only thing which has to be proved now is that all the members $\left(H^w, A_i^w, \gamma_i^w, \sigma^w\right)$ of $\mathcal{F}$ are independent on the parameter w. To prove this, put

$$\begin{aligned} \mathcal{H}^1 &= \left\{(z_1, w) \in \mathbf{C}^{\ell_1} \times \mathbf{C} \, ; \, \|z_1\|^2 < |w|^{2q} < 1\right\} \quad \text{and} \\ \mathcal{E}^1_w &= \left\{z_1 \in \mathbf{C}^{\ell_1} \, ; \, \|z_1\|^2 < |w|^{2q}\right\}, \quad w \in \Delta^* \, , \end{aligned} \tag{3.5}$$

and regard these as complex submanifolds of $\mathcal{H}$ and of $\mathcal{E}^p_w$, respectively, in the canonical manner. It then follows from (3.4) that $f_w(\mathcal{E}^1_w) = \mathcal{E}^1_w$ and $\Phi(\mathcal{H}^1) = \mathcal{H}^1$. Therefore, denoting by f^1_w, Φ^1 the restrictions of f_w, Φ to $\mathcal{E}^1_w$, $\mathcal{H}^1$, respectively, we see that Φ^1 defines a holomorphic automorphism of $\mathcal{H}^1$ having the form

$$\Phi^1\left(z_1, w\right) = \left(w^q H^w(z_1/w^q), w\right) = \left(f^1_w(z_1), w\right), \quad (z_1, w) \in \mathcal{H}^1 \, ,$$

and the same situation as in the case $I = 1$ above occurs for the domain $\mathcal{H}^1$ and its automorphism Φ^1 of $\mathcal{H}^1$. Consequently, by (3.1) we conclude that the automorphism H^w of B^{ℓ_1} is, in fact, independent on $w \in \Delta^*$; and so is γ_i^w. This combined with the fact that $f_w(z) = f(z, w)$ is holomorphic on $\mathcal{H}$ implies that every component of A_i^w is holomorphic in $w \in \Delta^*$. Thus

A_i^w is a unitary matrix independent on w by Lemma 7. Notice that the mapping Φ_o defined by

$$\Phi_o(z, w) = \left(w^q H(z_1/w^q), \gamma_2(z_1/w^q)A_2 z_2, \dots, \gamma_I(z_1/w^q)A_I z_I, w\right), \quad (z, w) \in \mathcal{H},$$

is now a holomorphic automorphism of $\mathcal{H}$. Then $\Phi_o^{-1}\Phi$ is also a holomorphic automorphism of $\mathcal{H}$ and it has the form

$$\Phi_o^{-1}\Phi(z, w) = \left(z_1, z_{\sigma^w(2)}, \dots, z_{\sigma^w(I)}, w\right), \quad (z, w) \in \mathcal{H},$$

from which it follows at once that σ^w is actually independent on $w \in \Delta^*$. Therefore we have completed the proof of Theorem 1, (I). □

3.2. CASE (II). $p_1 \neq 1$ or $q_1 = q \notin \mathbf{N}$: Clearly we have only to show that every element $\Phi \in \mathrm{Aut}(\mathcal{H})$ can be described as in Theorem 1, (II).

First, consider the case $p_1 \neq 1$. By the same reasoning as in the previous Subsection, we may assume that Φ has the form $\Phi(z, w) = (f(z, w), w)$ on $\mathcal{H}$. Therefore, if we define the domain $\mathcal{E}_w^p$ and the mapping f_w by (3.2) for any given point $w \in \Delta^*$, then f_w is a holomorphic automorphism of $\mathcal{E}_w^p$. Moreover, letting $\mathcal{E}^p$ and r_i be the same objects appearing in (3.3), we obtain the biholomorphic mapping $\Lambda : \mathcal{E}_w^p \to \mathcal{E}^p$ defined by

$$\Lambda(z) = (r_1 z_1, \dots, r_I z_I), \quad z = (z_1, \dots, z_I) \in \mathcal{E}_w^p. \tag{3.6}$$

Then, by recalling the result of [6] in the case $p_1 \neq 1$ and by repeating exactly the same argument as in Subsection 3.1, it can be shown that f_w has the form

$$f_w(z) = \left(A_1 z_{\sigma(1)}, \dots, A_I z_{\sigma(I)}\right), \quad z = (z_1, \dots, z_I) \in \mathcal{E}_w^p, \tag{3.7}$$

where $A_i \in U(\ell_i)$ and σ is a permutation of $\{1, \dots, I\}$ satisfying the following: $\sigma(i) = s$ can only happen when $(\ell_i, p_i) = (\ell_s, p_s)$. Therefore we have completed the proof of Theorem 1, (II), in the case $p_1 \neq 1$.

Next, consider the case $q \notin \mathbf{N}$. Of course, it suffices to consider the case $q \notin \mathbf{N}$ and $p_1 = 1$. Take an element $\Phi \in \mathrm{Aut}(\mathcal{H})$ arbitrarily. Again we may assume that Φ has the form $\Phi(z, w) = (f(z, w), w)$ on $\mathcal{H}$. For an arbitrarily given point $w \in \Delta^*$, let $\mathcal{E}_w^p$, f_w (resp. $\mathcal{E}^p$, r_i) be the same objects appearing in (3.2) (resp. in (3.3)) and let $\Lambda : \mathcal{E}_w^p \to \mathcal{E}^p$ be the biholomorphic mapping defined in (3.6). Then, by the same reasoning as above, f_w is a holomorphic automorphism of $\mathcal{E}_w^p$. Once it is shown that f_w is linear, that is, it is the restriction to $\mathcal{E}_w^p$ of some linear transformation of $\mathbf{C}^{|\ell|}$, then the method used in the preceding paragraph can be applied to prove that f_w is independent on w and, in fact, it has the form as in (3.7). Therefore we have only to verify that f_w is linear. For this purpose, recall the following fact in Lemma 5: Let Ψ be an element of $\mathrm{Aut}_{\mathrm{alg}}(\mathcal{H})$ having the form $\Psi(z, w) = (h(z, w), w)$ on $\mathcal{H}$. Then, for any point $w \in \Delta^*$, $h_w(z) = h(z, w)$ is a linear mapping of z. This together with the fact $\mathrm{Aut}(\mathcal{H}) = G(\mathcal{H})\mathrm{Aut}_{\mathrm{alg}}(\mathcal{H})$ by (2.13) immediately yields that it suffices to show the linearity of f_w for every $\Phi = (f, g) \in G(\mathcal{H})$ with $g(w) = w$.

Now consider again the domain $\mathcal{E}_w^1 \subset \mathbf{C}^{|\ell|}$ defined in (3.5) and the holomorphic automorphism $\Lambda \circ f_w \circ \Lambda^{-1}$ of $\mathcal{E}^p$. Then, in exactly the same way as in Subsection 3.1, one can see that $f_w(\mathcal{E}_w^1) = \mathcal{E}_w^1$ and f_w is a linear automorphism of $\mathcal{E}_w^p$ if and only if the restriction

f_w^1 of f_w to $\mathcal{E}_w^1$ is a linear automorphism of $\mathcal{E}_w^1$. Consequently, the proof is now reduced to showing that f_w^1 is a linear automorphism of $\mathcal{E}_w^1$. Now, assume to the contrary that there exists an element $\Phi = (f, g) \in G(\mathcal{H})$, $g(w) = w$, such that f_w^1 is not a linear automorphism of $\mathcal{E}_w^1$. Then, since Φ leaves all slices $\mathcal{E}_w^p \times \{w\}$, $w \in \Delta^*$, invariant and $f_w(\mathcal{E}_w^1) = \mathcal{E}_w^1$, one can find a complete holomorphic vector field X on $\mathcal{H}$ satisfying the following two conditions: For any point $w \in \Delta^*$,

(3.8) X is tangent to the complex submanifold $\mathcal{E}_w^1 \times \{w\}$ of $\mathcal{H}$; and

(3.9) the restriction of X to $\mathcal{E}_w^1 \times \{w\}$, say again X, is a non-zero complete holomorphic vector field having the form

$$X = \sum_{k=1}^{\ell_1} \left(\alpha_k(w) + \sum_{\mu,\nu=1}^{\ell_1} \beta_{\mu\nu}^k(w) \zeta_\mu \zeta_\nu \right) \frac{\partial}{\partial \zeta_k},$$

where α_k, $\beta_{\mu\nu}^k$ are holomorphic functions on Δ^* (cf. [12; Proposition 2]).

Here we know that $X \neq 0$ if and only if $\alpha_k(w) \neq 0$ for some k. Moreover, we may assume by Lemma 4 that X extends holomorphically across the set $\partial\mathcal{H} \setminus \{0\}$.

From now on, for any given point $w \in \Delta^*$, we identify naturally $\mathcal{E}_w^1 \times \{w\}$ with $\mathcal{E}_w^1$; so that X is regarded as a complete holomorphic vector field on $\mathcal{E}_w^1$ and

$$\rho_w(z_1) = \rho_w(\zeta_1, \ldots, \zeta_{\ell_1}) := \sum_{j=1}^{\ell_1} |\zeta_j|^2 - |w|^{2q}$$

is a defining function of $\mathcal{E}_w^1$ in $\mathbf{C}^{\ell_1}$. Note that X is now defined on some domain in $\mathbf{C}^{\ell_1}$ containing the closure $\overline{\mathcal{E}_w^1}$ of $\mathcal{E}_w^1$. It then follows from the tangency condition $\mathrm{Re}(X\rho_w) = 0$ on the boundary $\partial\mathcal{E}_w^1$ that

$$\text{(3.10)}\quad \mathrm{Re}\left\{ \sum_{k=1}^{\ell_1} \left(\alpha_k(w) + \sum_{\mu,\nu=1}^{\ell_1} \beta_{\mu\nu}^k(w) \zeta_\mu \zeta_\nu \right) \bar{\zeta}_k \right\} = 0 \quad \text{whenever} \quad \rho_w(\zeta_1, \ldots, \zeta_{\ell_1}) = 0.$$

Fix an index k with $\alpha_k(w) \neq 0$ and consider the points $(0, \ldots, 0, \zeta_k, 0, \ldots, 0) \in \mathbf{C}^{\ell_1}$ with $|\zeta_k|^2 = |w|^{2q}$. Then, by routine computations it follows from (3.10) that

$$\alpha_k(w) + \overline{\beta_{kk}^k(w)} |w|^{2q} = 0, \quad w \in \Delta^*.$$

Hence we have

$$\overline{\left(\frac{d\beta_{kk}^k(w)}{dw} \right)} \cdot |w|^{2q} + \overline{\beta_{kk}^k(w)} \cdot q w |w|^{2(q-1)} = 0$$

or equivalently

$$\frac{d\beta_{kk}^k(w)}{dw} w + q\beta_{kk}^k(w) = 0.$$

Let $\beta^k_{kk}(w) = \sum_\nu A_\nu w^\nu$ be the Laurent expansion of β^k_{kk} on Δ^*, where $\nu \in \mathbf{Z}$. Inserting this into the equation above, we then obtain that

$$(q+\nu)A_\nu = 0 \quad \text{for all} \quad \nu \in \mathbf{Z}\,.$$

Since $0 < q \notin \mathbf{N}$ by our assumption, this implies that $A_\nu = 0$ for all $\nu \in \mathbf{Z}$. Thus $\beta^k_{kk}(w) = 0$ and so $\alpha_k(w) = 0$ on Δ^*, a contradiction. Eventually we have shown that every automorphism f_w is linear; and accordingly, $\mathrm{Aut}(\mathcal{H})$ consists only of linear automorphisms having the description as in Theorem 1, (II), as desired. □

4. Proof of Theorem 2. Clearly the mapping Φ having the form as in Theorem 2 belongs to $\mathrm{Aut}(\mathcal{H})$. Conversely, take an arbitrary element $\Phi \in \mathrm{Aut}(\mathcal{H})$ and write $\Phi = (\Phi_1, \dots, \Phi_N)$ with respect to the coordinate system $\zeta = (\zeta_1, \dots, \zeta_N)$ in $\mathbf{C}^N$. Then, since $|m| \geq 2$, by the same reasoning as in the proof of Lemma 5 every component function Φ_i extends to a unique holomorphic function $\widehat{\Phi}_i$ defined on $\mathcal{E}^p \times \mathcal{E}^q$. Accordingly, we obtain a holomorphic extension $\widehat{\Phi} := (\widehat{\Phi}_1, \dots, \widehat{\Phi}_N) : \mathcal{E}^p \times \mathcal{E}^q \to \mathbf{C}^N$ of Φ. We first assert that $\widehat{\Phi}(\mathcal{E}^p \times \mathcal{E}^q) \subset \mathcal{E}^p \times \mathcal{E}^q$. To prove this, represent again $\Phi = (f, g)$ and $f = (f_1, \dots, f_I)$, $g = (g_1, \dots, g_J)$ by coordinates $(z, w) = (z_1, \dots, z_I, w_1, \dots, w_J)$ in $\mathbf{C}^{|\ell|} \times \mathbf{C}^{|m|} = \mathbf{C}^N$. Let $\hat{f}$, $\hat{g}$ be the holomorphic extensions of f, g to $\mathcal{E}^p \times \mathcal{E}^q$, respectively. Since $g(z, w)$ does not depend on the variables z by Lemma 6, $\hat{g}$ gives now a holomorphic automorphism of $\mathcal{E}^q$ with $\hat{g}(0) = 0$; consequently it follows from our result of [6] that $\hat{g}$ can be written in the form

$$\hat{g}(w) = \big(B_1 w_{\tau(1)}, \dots, B_J w_{\tau(J)}\big)\,, \quad w = (w_1, \dots, w_J) \in \mathcal{E}^q\,, \tag{4.1}$$

where $B_j \in U(m_j)$, $1 \leq j \leq J$, and τ is a permutation of $\{1, \dots, J\}$ such that $\tau(j) = t$ if and only if $(m_j, q_j) = (m_t, q_t)$. On the other hand, picking a point $z_o \in \mathcal{E}^p$ arbitrarily, we have $(z_o, w) \in \mathcal{H}$ for all points $w \in \mathbf{C}^{|m|}$ with $\rho^p(z_o) < \rho^q(w) < 1$; and hence $\rho^p(f(z_o, w)) < \rho^q(g(w)) < 1$ for such points. So, taking account of the maximum principle for the continuous plurisubharmonic function $\rho^p(\hat{f}(z_o, w))$ on $\mathcal{E}^q$, we obtain that $\rho^p(\hat{f}(z_o, w)) < 1$ for all $w \in \mathcal{E}^q$. Thus $\hat{f}(\mathcal{E}^p \times \mathcal{E}^q) \subset \mathcal{E}^p$ and so $\widehat{\Phi}(\mathcal{E}^p \times \mathcal{E}^q) \subset \mathcal{E}^p \times \mathcal{E}^q$. Also, repeating exactly the same argument for the holomorphic extension $\widehat{\Psi}$ of the inverse $\Psi := \Phi^{-1}$ of Φ, we obtain the same conclusion $\widehat{\Psi}(\mathcal{E}^p \times \mathcal{E}^q) \subset \mathcal{E}^p \times \mathcal{E}^q$. Then

$$\widehat{\Phi} \circ \widehat{\Psi}(z, w) = \widehat{\Psi} \circ \widehat{\Phi}(z, w) = (z, w)\,, \quad (z, w) \in \mathcal{E}^p \times \mathcal{E}^q\,,$$

by analytic continuation. Hence $\widehat{\Phi}$ is a holomorphic automorphism of the bounded Reinhardt domain $\mathcal{E}^p \times \mathcal{E}^q$. Moreover, since

$$\sum_{i=1}^{I} \|f_i(z, w)\|^{2p_i} < \sum_{j=1}^{J} \|g_j(w)\|^{2q_j} = \sum_{j=1}^{J} \|w_j\|^{2q_j}\,, \quad (z, w) \in \mathcal{H}\,,$$

by (4.1), it follows that $\widehat{\Phi}(0, 0) = (0, 0)$ by taking the limit $(z, w) \to (0, 0)$ through $\mathcal{H}$. Then, as an immediate consequence of a well-known theorem of H. Cartan, it follows that $\widehat{\Phi}$ is a linear automorphism of $\mathcal{E}^p \times \mathcal{E}^q$.

Let us define the mapping $\hat{f}_o : \mathcal{E}^p \to \mathbf{C}^{|\ell|}$ by setting $\hat{f}_o(z) := \hat{f}(z, 0)$, $z \in \mathcal{E}^p$. Then it is easily seen that $\hat{f}_o$ is a holomorphic automorphism of $\mathcal{E}^p$. So, our previous result [6] implies that it can be expressed as

$$\hat{f}_o(z) = \big(A_1 z_{\sigma(1)}, \ldots, A_I z_{\sigma(I)}\big)\,, \quad z = (z_1, \ldots, z_I) \in \mathcal{E}^p\,, \tag{4.2}$$

where $A_i \in U(\ell_i)$, $1 \le i \le I$, and σ is a permutation of $\{1, \ldots, I\}$ such that $\sigma(i) = s$ occurs only when $(\ell_i, p_i) = (\ell_s, p_s)$. Now define the linear automorphism $\widehat{\Phi}_o$ of $\mathcal{E}^p \times \mathcal{E}^q$ by

$$\widehat{\Phi}_o(z, w) = \big(\hat{f}_o(z), \hat{g}(w)\big)\,, \quad (z, w) \in \mathcal{E}^p \times \mathcal{E}^q\,,$$

and consider the holomorphic automorphism

$$\Gamma(z, w) = \widehat{\Phi}_o^{-1} \circ \widehat{\Phi}(z, w)\,, \quad (z, w) \in \mathcal{E}^p \times \mathcal{E}^q\,, \tag{4.3}$$

of $\mathcal{E}^p \times \mathcal{E}^q$. Then Γ can be written in the form

$$\Gamma(z, w) = (z + Mw, w)\,, \quad (z, w) \in \mathcal{E}^p \times \mathcal{E}^q\,,$$

(think of z, w as column vectors), where M is a certain $|\ell| \times |m|$ matrix. Thus, denoting by Γ^n the n-th iteration of Γ, we have

$$\Gamma^n(z, w) = (z + nMw, w)\,, \quad (z, w) \in \mathcal{E}^p \times \mathcal{E}^q, \ n = 1, 2, \ldots .$$

Hence M has to be the zero matrix, that is, Γ is the identity transformation of $\mathcal{E}^p \times \mathcal{E}^q$, since $\{\Gamma^n\}_{n=1}^{\infty}$ is contained in the isotropy subgroup K_0 of $\mathrm{Aut}(\mathcal{E}^p \times \mathcal{E}^q)$ at the origin $0 = (0, 0) \in \mathcal{E}^p \times \mathcal{E}^q$ and K_0 is compact, as is well-known. Therefore we have shown that $\widehat{\Phi} = \widehat{\Phi}_o$ has the form described in Theorem 2; thereby completing the proof. □

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FACULTY OF MATHEMATICS AND PHYSICS
INSTITUTE OF SCIENCE AND ENGINEERING
KANAZAWA UNIVERSITY
KANAZAWA 920–1192
JAPAN

E-mail address: kodama@staff.kanazawa-u.ac.jp