ON THE HOLOMORPHISMS OF A GROUP*

BY

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Introduction.

If every operator of an abelian group is put into correspondence with its ath power, an isomorphism of the group with itself or with one of its subgroups is obtained for any integral value of a.† If a is prime to the order of every operator in the group, the resulting isomorphism is simple; otherwise it is multiple. To avoid an unnecessarily cumbrous phrase, let us denote by *a-isomorphism* any isomorphism obtained by putting each operator of a group into correspondence with its ath power; and let us say *a-holomorphism* whenever the resulting isomorphism is simple.

It has been shown that the a-holomorphisms of an abelian group G constitute the totality of invariant operators in the group of isomorphisms of G, and that their number is equal to the number of integers less than and prime to the highest order occurring among the operators of G.‡

Every group admits an a-holomorphism, when $a \equiv 1 \pmod{m}$, where m denotes the lowest common multiple of the orders occurring among the operators of the group. The questions naturally arise: (1) Under what conditions do non-abelian groups admit a-holomorphisms other than the identical? (2) What are the properties of the corresponding operators in the group of isomorphisms? The present paper concerns itself with these questions. The writer is indebted to Professor G. A. Miller for suggestions and criticisms during the preparation of this paper.

Conditions for the existence of a-holomorphisms. §§ 1–7.

1. Let s, t be any operators of a group G and suppose that G admits an a-holomorphism.§ This requires that the relation

$$s^a t^a = (st)^a$$

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† Miller, Transactions of the American Mathematical Society, vol. 1 (1900), p. 396.

‡ Miller, Comptes Rendus, vol. 132 (1901), p. 912.

§ When we speak of a group as "admitting an a-holomorphism" we shall always mean, unless otherwise specified, a holomorphism other than the identical.

be satisfied for every pair of operators s, t of G, and that a be prime to the order of every operator of G.

Multiplying both members by st we obtain

$$s^a t^a st = (st)^{a+1} = s(ts)^a t = st^a s^a t,$$

and, therefore,

$$s^{a-1} t^a = t^a s^{a-1}.$$

Since a is prime to the order of every operator, t^a represents any operator. The last equality may then be written

$$(A) \qquad s^{a-1} t = t s^{a-1},$$

which must be satisfied for every pair s, t.

Hence, *if a group admits an a-holomorphism, the* $(a-1)$*th power of every operator is invariant.*

2. From the relation

$$s^a t^a = (st)^a = s(ts)^{a-1} t,$$

we obtain at once

$$s^{a-1} t^{a-1} = (ts)^{a-1}.$$

By § 1, interchanging s and t, we have

$$(B) \qquad s^{a-1} t^{a-1} = (st)^{a-1}.$$

Hence, *if a group admits an a-holomorphism, it admits also an* $(a-1)$*-isomorphism.*

3. Multiplying (B) by st and making use of (A), we obtain

$$s^a t^a = (st)^a.$$

Hence, *the conditions* (A) *and* (B), *which are necessary for the existence of an a-holomorphism, are also sufficient, provided* a *is prime to the order of every operator.*

4. If we write δ for $a-1$, (A) and (B) respectively become

$$s^\delta t = t s^\delta, \qquad s^\delta t^\delta = (st)^\delta.$$

If these are satisfied, so also are

$$s^{\kappa\delta} t = t s^{\kappa\delta}, \qquad s^{\kappa\delta} t^{\kappa\delta} = (st)^{\kappa\delta},$$

where κ is any integer. Multiplying the last relation by st, we have, since $s^{\kappa\delta}$ is invariant,

$$s^{\kappa\delta+1} t^{\kappa\delta+1} = (st)^{\kappa\delta+1},$$

which insures a $(\kappa\delta + 1)$-holomorphism, provided $\kappa\delta + 1$ is prime to the order of every operator of G. Collecting results, we have the following theorem:

If there is a number δ such that the relations $s^\delta t = ts^\delta$ and $s^\delta t^\delta = (st)^\delta$ are satisfied for every pair of operators s, t of a group G, and if $\delta + 1$ is prime to the order of every operator of G, then, and only then, does G admit an a-holomorphism; and a may have any value of the form $\kappa\delta + 1$ which is prime to the order of every operator of G.

5. The relation $s^\delta t^\delta = (st)^\delta$ leads at once (by the reasoning of §1) to

$$s^{\delta-1} t^\delta = t^\delta s^{\delta-1}.$$

If $\delta - 1$ is prime to the order of every non-invariant operator of G, the last relation states that t^δ is invariant; i. e., with this restriction on δ, the first condition in the theorem of §4 is contained in the second. Hence,

If $\delta - 1$ is prime to the order of every non-invariant operator of G, and if $\delta + 1$ is prime to the order of every operator, then, if G admits a δ-isomorphism, it admits a $(\kappa\delta + 1)$-holomorphism for every value of $\kappa\delta + 1$ which is prime to the order of every operator.

It may be noted that all numbers $\kappa\delta + 1$ are prime to the order of every operator, provided G contains no invariant operators whose orders are prime to δ. For δ must in this case contain as a factor every prime which divides the order of any operator of G.

6. If G is of order p^m (p a prime), δ may be assumed to be of the form p^β. For the assumption $\delta = kp^\beta$ (k prime to p), leads at once to the possibility $\delta = k'kp^\beta$; and k' may be so chosen that $k'k \equiv 1 \pmod{p^\mu}$, where p^μ is the highest order occurring among the operators of G. In this case the theorem of §4 becomes:

If a group G of order p^m (p a prime) admits a p^β-isomorphism, then, and only then, does it admit an a-holomorphism, and a may have any value of the form $\kappa p^\beta + 1$; moreover, if G is non-abelian, a must be of this form.

7. It follows immediately that *if a group G of order p^m admits an a-holomorphism, the operators of G whose orders are not greater than $p^{\beta+\lambda}$ ($\lambda = 0, 1, 2, \cdots$) form an invariant subgroup G_λ, and G/G_λ is abelian.*

In particular, *if a group G of order p^m admits an a-holomorphism, the operators of G of order less than the highest form an invariant subgroup H, and G/H is abelian of type* $(1, 1, 1, \cdots)$.

Furthermore, it follows from the last part of the theorem of §6 that *a non-abelian group of order p^m, which contains no operators of order higher than p, does not admit an a-holomorphism.*

Properties of a-holomorphisms considered as operators in the group of isomorphisms. §§ 8–12.

8. In what follows we regard a holomorphism as defining an operator in the group of isomorphisms and shall frequently use the word "holomorphism" in the sense of "operator defined by the holomorphism."

If two holomorphisms transform each operator of a group into its a_1th and a_2th power respectively, their product transforms each operator into its $a_1 a_2$th power. Hence, the totality of a-holomorphisms of a group G forms a subgroup in the group of isomorphisms of G.

Moreover, *the a-holomorphisms of a group G are invariant in the group of isomorphisms of G.*

For if T is any holomorphism of G, such that* $T^{-1} s T = s'$, where s is any operator of G, and T_a is an a-holomorphism ($T_a^{-1} s T_a = s^a$), then $T_a T$ and $T T_a$ both transform s into s'^a.†

9. Before proceeding to the proof of another theorem concerning the invariant operators in the group of isomorphisms, the following may be noted:

If G contains a subgroup H consisting solely of invariant operators such that G can be made isomorphic with H without making any operator correspond to its inverse, then a holomorphism of G results from making G isomorphic with H in the manner specified and making every operator of G correspond to itself multiplied by the corresponding operator of H.‡

This follows from the fact that, if s, t be any two operators of G, and s_1, t_1 the corresponding operators of H, then st and $s_1 t_1$ correspond. To obtain a holomorphism, we make s, t, st correspond respectively to ss_1, tt_1, $st s_1 t_1$; and since s_1, t_1 are invariant, we have $ss_1 tt_1 = st s_1 t_1$. This establishes the holomorphism, since under the assumed conditions the correspondence is simple.

10. *If a holomorphism of a group G can be obtained by making G isomorphic with a subgroup H, all of whose operators are invariant, and multiplying corresponding operators, then any invariant operator in the group of isomorphisms of G must, in general, establish an a-holomorphism among the operators of H.*

This theorem applies to any group admitting an a-holomorphism. For (§§ 1, 2, 9) a holomorphism of the group results by making it isomorphic with the subgroup obtained by raising each operator to the $(a - 1)$th power and multiplying corresponding operators.

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* As to the propriety of this transformation notation, cf. e. g., BURNSIDE, *Theory of Groups*, p. 227.

† This theorem was proved for abelian groups by MILLER, loc. cit., footnote 3, p. —.

‡ Every holomorphism of an abelian group can be obtained in this way: MILLER, Bulletin, May, 1900, p. 337.

To prove the theorem, let T be any invariant operator in the group of isomorphisms I of G; and let $T^{-1}sT = s'$, where s is any operator of G. By hypothesis I contains an operator T_1, such that $T_1^{-1}sT_1 = ss_1$, and $T_1^{-1}s'T = s's_1'$, where s_1, s_1' are operators of H. Then, from the fact that TT_1 and T_1T must transform s into the same operator, it follows that

$$T^{-1}s_1T = s_1'.$$

If s_1' is not a power of s_1, then, by establishing a new isomorphism of G with H, another holomorphism of G, say T_2, may, in general, be obtained, such that

$$T_2^{-1}sT_2 = ss_1, \qquad T_2^{-1}s'T_2 = s's_2' \qquad (s_2' \neq s_1').$$

But then we should have $T^{-1}s_1T = s_2'$, which contradicts the former result. Hence, s_1' is a power of s_1; therefore T transforms every subgroup of H into itself. But H is abelian, and all the operators in the group of isomorphisms of any abelian group A, which transform every subgroup of A into itself, transform every operator into the same power.* This proves the theorem.

11. It has been shown (§ 6) that if a non-abelian group of order p^m (p a prime) admits an a-holomorphism, then $a \equiv 1 \pmod{p^\beta}$ $(\beta > 0)$. Returning to the notation adopted in the previous articles, let p^β be the smallest value of δ available for a non-abelian group G of order p^m; then all the a-holomorphisms are obtained by giving to a the values $1 + \kappa p^\beta$ $(\kappa = 1, 2, \cdots, p^{\mu-\beta})$, where p^μ is the highest order occurring among the operators of G. Hence, *a non-abelian group of order p^m admits just $p^{\mu-\beta}$ a-holomorphisms, including identity.*

12. We turn now to the consideration of the orders of these $p^{\mu-\beta}$ operators. Let T be one of them; and let s be an operator of G of highest order p^μ. Then

$$T^{-1}sT = s^{1+\kappa p^\beta}.$$

By repetition

$$T^{-n}sT^n = s^{(1+\kappa p^\beta)^n}.$$

The order of T is the smallest value of n, for which

$$(1 + \kappa p^\beta)^n \equiv 1 \pmod{p^\mu}.$$

By expanding, if κ is prime to p, the order ν of T is found to be

$$\nu = p^{\mu-\beta}, \text{ if } p \text{ is odd, or if } p = 2, \beta > 1;$$

$$\nu = 2^{\mu-2}, \text{ if } p = 2, \beta = 1 \text{ and } \mu > 3 \qquad (\kappa = 1, 2);$$

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* MILLER, Transactions, vol. 2 (1901), p. 260.

also in the latter case, there are no operators of order higher than $2^{\mu-2}$, since $n = 2^{\mu-2}$ will always satisfy $(1 + 2\kappa)^n \equiv 1 \pmod{2^\mu}$. In connection with the result of § 11 we have, therefore, the following:

The a-holomorphisms of a non-abelian group G of order p^m form a sub-group H of order $p^{\mu-\beta}$ in the group of isomorphisms I of G; all the operators of H are invariant in I; and H is cyclic, except when $p = 2$, $\beta = 1$; in the latter case, if $\mu > 3$, H is abelian of type $(\mu - 2, 1)$.

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