On the Homogenization of Quasilinear Divergence Structure Operators (*).

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Summary. – *We study the homogenization of second order quasilinear operators of the form*

$$A_\varepsilon u = -\operatorname{div} a\left(\frac{x}{\varepsilon}, u, Du\right)$$

in Sobolev spaces $H^{1,p}$ $(p > 1)$. *An explicit formula of the homogenized operator is given.*

1. – Introduction...

In this paper we study the homogenization of a family of quasilinear operators

$$\text{(1.1)} \qquad \begin{cases} A_\varepsilon u = -\operatorname{div} a\left(\dfrac{x}{\varepsilon}, u, Du\right) \\ u \in H_0^{1,p}(\Omega) \qquad p > 1\,, \end{cases}$$

where $a(x, u, \xi)$ is periodic in x and verifies suitable growth conditions, $\varepsilon > 0$.

Indeed we prove that the solutions u_ε of the problems

$$\text{(1.2)} \qquad \begin{cases} A_\varepsilon u = f \\ u \in H_0^{1,p}(\Omega) \end{cases}$$

converge in the weak topology of $H_0^{1,p}(\Omega)$ to a function u_0 which is the solution of the problem:

$$\begin{cases} -\operatorname{div} b(u, Du) = f \\ u \in H_0^{1,p}(\Omega) \end{cases}$$

and $a(x/\varepsilon, u_\varepsilon, Du_\varepsilon)$ converge to $b(u_0, Du_0)$ in the weak topology of $L^{p'}$, with $p' = p/(p-1)$. Moreover the matrix $b(u, \xi)$ is given by an explicit formula.

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The study of the homogenization of a family of linear elliptic operators

$$A_\varepsilon u = -\operatorname{div}\left(a\left(\frac{x}{\varepsilon}\right), Du\right)$$

goes back to DE GIORGI-SPAGNOLO ([16]), but many different proofs of the main results have been given by several authors ([1], [2], [3], [4], [10], [13], [17], [20]). On the other hand, the homogenization of a family of variational integrals

$$F_\varepsilon(u) = \int_\Omega f\left(\frac{x}{\varepsilon}, Du\right) dx \tag{1.3}$$

has been studied by MARCELLINI in [9] and by CARBONE-SBORDONE in [5], using the techniques of Γ-convergence introduced by De Giorgi. However a direct proof of the homogenization of the functionals (1.3) has been recently given in [7].

For non linear equations of the type (1.2) some homogenization results, in the case $p = 2$, were first given by BABUSKA ([2]) and recently extended to the case of systems by SUQUET ([14]) and TARTAR [18].

Some general results of G-convergence for non linear operators of the form (1.1), in the case $p \geqslant 2$, have been recently announced by RAITUM ([12]), although in this paper he is not concerned with the problem of giving a representation formula for the limit operator. In any case, he shows that if $p > 2$ the limit operator may not verify the same structure conditions of the A_ε. However, we show here that certain conditions are preserved, passing to the limit, and that in some cases one may have stability results.

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2. – Preliminary lemmas.

In the following Y will denote the unit cube in $\boldsymbol{R}^n$, y an element of Y and x an element of $\boldsymbol{R}^n$. If $f \in L^1_{n,\mathrm{loc}}(\boldsymbol{R}^n)$ and $E \subset \boldsymbol{R}^n$ is a bounded set of positive measure, we define:

$$\langle f \rangle_E = \frac{1}{\operatorname{meas}(E)} \int_E f(x)\, dx .$$

However we shall omit the subscript E, when it is clear to which set we refer.

If $p > 1$ and $p' = p/(p-1)$ we shall consider the following spaces:

$$H^{1,p}_{\text{per}}(Y) = \{u(y) \in H^{1,p}(Y) \colon u \text{ has the same trace on the opposite faces of } Y\};$$

$$L^{p'}_{n,\text{per}}(Y) = \Big\{q(y) \in L^{p'}_n(Y) \colon \int_Y q\cdot Du\, dy = 0,\ \forall u \in H^{1,p}_{\text{per}}(Y)\Big\}.$$

We recall the following lemmas about the above spaces (for a proof see e.g. [14]).

LEMMA 2.1. – *If $u(y)$ is an element of $H^{1,p}_{\text{per}}(Y)$, then it can be extended by periodicity to an element of $H^{1,p}_{\text{loc}}(\boldsymbol{R}^n)$.*

LEMMA 2.2. – *If $q(y)$ is an element of $L^{p'}_{n,\text{per}}(Y)$, then it can be extended by periodicity to an element of $L^{p'}_{n,\text{loc}}(\boldsymbol{R}^n)$, still denoted by q, such that*

$$\operatorname{div}_x q(x) = 0\,.$$

In the following we shall consider the Dirichlet problems:

$$(\mathcal{P}_\varepsilon)\qquad \begin{cases} -\operatorname{div} a\left(\dfrac{x}{\varepsilon}, u, Du\right) = f & \text{in } \Omega, \\ u \in H^{1,p}_0(\Omega) \end{cases}$$

where Ω is a bounded open set in $\boldsymbol{R}^n$, $f \in L^q$ with $q > \max\{n/p, p/(p-1)\}$, $\varepsilon > 0$, and $a(x, u, \xi)$ verifies the following structure conditions:

H_1) a is Y-periodic and measurable with respect to x.

H_2) For any $x \in \boldsymbol{R}^n$ a.e., any $u, u_1, u_2 \in \boldsymbol{R}$ and $\xi_1, \xi_2 \in \boldsymbol{R}^n$ then

if $p \geqslant 2$

i) $|a(x, u_1, \xi_1) - a(x, u_2, \xi_2)| \leqslant \beta(1 + |u_1| + |u_2| + |\xi_1| + |\xi_2|)^{p-2}(|\xi_1 - \xi_2| + |u_1 - u_2|)$

ii) $(a(x, u, \xi_1) - a(x, u, \xi_2), \xi_1 - \xi_2) \geqslant \alpha|\xi_1 - \xi_2|^p$

or, *if* $1 < p \leqslant 2$

j) $|a(x, u_1, \xi_1) - a(x, u_2, \xi_2)| \leqslant \beta(|u_1 - u_2| + |\xi_1 - \xi_2|)^{p-1}$

jj) $(a(x, u, \xi_1) - a(x, u, \xi_2), \xi_1 - \xi_2) \geqslant \alpha|\xi_1 - \xi_2|^2(|\xi_1| + |\xi_2|)^{p-2}$

H_3) $a(x, 0, 0) \in L^{p'}_n$ if $p > n$ or $a(x, 0, 0) \in L^q_n$ with $q > \dfrac{n}{p-1}$ if $p \leqslant n$.

REMARK 2.3. – We remark that under the above hypothesis, using the same argument of [8] (theorems 8.15-8.16), one can prove uniform (i.e. not dependent on ε) a priori bounds for the L^∞ norm of the solutions of the problems $(\mathcal{T}_\varepsilon)$. From this then one can easily deduce the existence of solutions for $(\mathcal{T}_\varepsilon)$.

In the next section we will prove the convergence of the solutions of $(\mathcal{T}_\varepsilon)$ to the solutions of the homogenized problem

$$(\mathcal{T}_0) \qquad \begin{cases} -\operatorname{div} b(u, Du) = f & \text{in } \Omega \\ u \in H_0^{1,p}(\Omega) \end{cases}$$

where b is given by:

$$b(u, \xi) = \int_Y a(y, u, Dv(y))\, dy \tag{2.1}$$

and $v(y)$ is the solution of the problem:

$$\begin{cases} \int_Y a(y, u, Dv(y)) \cdot D\varphi(y)\, dy = 0\,, & \forall \varphi \in H^{1,p}_{\mathrm{per}}(Y) \\ v(y) \in \xi \cdot y + H^{1,p}_{\mathrm{per}}(Y)\,. \end{cases} \tag{2.2}$$

Using the above assumptions on a it is straightforward to prove that problem (2.2) (in which u is fixed) has a unique solution. So $b(u, \xi)$ is well defined.

We state now some lemma about the structure properties of b.

LEMMA 2.4.

$$|b(u, \xi)| \leq c[1 + |u| + |\xi|]^{p-1}$$

where $c \equiv c(\alpha, \beta, p, \|a(y, 0, 0)\|_{p'})$.

PROOF. – Let us fix (u, ξ) and denote by $v(y)$ the corresponding solution of (2.2). Then by i) or j) we get:

$$\left|b(u, \xi) - \int_Y a(y, 0, 0)\, dy\right| \leq \beta \int_Y (1 + |u| + |Dv(y)|)^{p-1}\, dy\,. \tag{2.3}$$

On the other hand, using ii) or jj) and the fact that $v(y)$ is a solution of (2.2) we have:

$$\alpha \int_Y |Dv(y)|^p\, dy \leq \int_Y \big(a(y, u, Dv(y)) - a(y, u, 0), Dv(y)\big)\, dy =$$
$$= b(u, \xi) \cdot \xi + \int_Y (a(y, 0, 0) - a(y, u, 0)) \cdot Dv(y)\, dy - \int_Y (a(y, 0, 0), Dv(y))\, dy\,.$$

Then, applying Young inequality to the two integrals on the left side, i) or j) and (2.3):

$$\int_Y |Dv(y)|^p \leqslant c(1 + |u| + |\xi|)^p \tag{2.4}$$

where $c \equiv c(\alpha, \beta, p, \|a(y, 0, 0)\|_{p'})$. Then, applying again (2.3), we get the proof. □

LEMMA 2.5. – *$b(u, \xi)$ is locally Hölder (Lipschitz if $p = 2$) with respect to (u, ξ).*

PROOF. – Let us denote by v_1 and v_2 the solutions of (2.2) defining respectively $b(u_1, \xi_1)$ and $b(u_2, \xi_2)$. We shall put

$$H = 1 + |u_1| + |u_2| + |\xi_1| + |\xi_2| . \tag{2.5}$$

Case $p \geqslant 2$. – By ii) we have

$$\begin{aligned}\int_Y |Dv_1 - Dv_2|^p \, dy &\leqslant \int_Y (a(y, u_1, Dv_1) - a(y, u_1, Dv_2), Dv_1 - Dv_2) \, dy \leqslant \\ &\leqslant (b(u_1, \xi_1) - b(u_2, \xi_2), \xi_1 - \xi_2) + \\ &+ |u_1 - u_2| \int_Y (1 + |u_1| + |u_2| + |Dv_2|)^{p-2} |Dv_1 - Dv_2| \, dy .\end{aligned}$$

Then, by the Young inequality and the estimate (2.4), we get:

$$\int_Y |Dv_1 - Dv_2|^p \, dy \leqslant c\{(1 + H)^{p'(p-2)}|u_1 - u_2|^{p'} + (b(u_2, \xi_2) - b(u_1, \xi_1), \xi_2 - \xi_1)\} . \tag{2.6}$$

On the other hand by i) and (2.4):

$$\begin{aligned}|b(u_2, \xi_2) - b(u_1, \xi_1)| &\leqslant \int_Y |a(y, u_1, Dv_1) - a(y, u_2, Dv_2)| \, dy \leqslant \\ &\leqslant c(1 + H)^{p-2}\Big(\int_Y (|u_1 - u_2|^p + |Dv_1 - Dv_2|^p) \, dy\Big)^{1/p} .\end{aligned}$$

Then, using (2.6) and again Young inequality to separate the term $(b(u_2, \xi_2) - b(u_1, \xi_1), \xi_2 - \xi_1)$ we have:

$$|b(u_1, \xi_1) - b(u_2, \xi_2)| \leqslant c(1 + H)^{p(p-2)/(p-1)}(|u_1 - u_2| + |\xi_1 - \xi_2|)^{1/(p-1)} . \tag{2.7}$$

Case $1 < p \leqslant 2$. – With the same argument used to prove (2.6) one can prove:

$$\begin{aligned}\int_Y |Dv_1 - Dv_2|^2(|Dv_1| + |Dv_2|)^{p-2} &\leqslant c\{(1 + H)^{(2-p)}|u_1 - u_2|^{2(p-1)} + \\ &+ (b(u_2, \xi_2) - b(u_1, \xi_1), \xi_2 - \xi_1)\} .\end{aligned} \tag{2.8}$$

But on the other hand, from j) we have:

$$|b(u_1, \xi_1) - b(u_2, \xi_2)| \leqslant \beta \int_Y (|u_1 - u_2| + |Dv_1 - Dv_2|)^{p-1}\, dy\,.$$

And so from this and from (2.8) the following estimate comes:

$$|b(u_2, \xi_2) - b(u_1, \xi_1)| \leqslant c\{(1+H)^{(2-p)(p-1)}|u_1 - u_2|^{(p-1)^2} + \\ + (1+H)^{(2-p)(p-1)/(3-p)}|\xi_1 - \xi_2|^{(p-1)/(3-p)}\}\,. \qquad \square \tag{2.9}$$

We remark that both (2.7) and (2.9) show that if $p = 2$ then b is Lipschitz with respect to its arguments. Moreover a counter-example given in [12] shows that, at least if $p > 2$, the Hölder exponents appearing in the above estimates in general cannot be improved.

Lemma 2.6. – *For any* $u \in \boldsymbol{R}$, $\xi_1, \xi_2 \in \boldsymbol{R}^n$ *we have*

$$(b(u, \xi_1) - b(u, \xi_2), \xi_1 - \xi_2) \geqslant \alpha|\xi_1 - \xi_2|^p\,, \quad \textit{if } p \geqslant 2\,, \tag{2.10$_1$}$$

$$(b(u, \xi_1) - b(u, \xi_2), \xi_1 - \xi_2) \geqslant \alpha'|\xi_1 - \xi_2|^2(1 + |u| + |\xi_1| + |\xi_2|)^{p-2}\,, \quad \textit{if } 1 < p \leqslant 2\,. \tag{2.10$_2$}$$

Proof. – Let us denote by v_1 and v_2 the solutions of (2.2) defining respectively $b(u, \xi_1)$ and $b(u, \xi_2)$. Let us consider $u_i = v_i(y) - \xi_i \cdot y$, $i = 1, 2$. Then $u_i(y)$ is in $H^{1,p}_{\text{per}}(Y)$ and so, if we extend it by periodicity, the resulting function (still denoted by u_i) is in $H^{1,p}_{\text{loc}}(\boldsymbol{R}^n)$ (see Lemma 2.1). So, if we define:

$$w_i^\varepsilon(x) = \varepsilon u_i\left(\frac{x}{\varepsilon}\right) + \xi_i \cdot x\,, \qquad i = 1, 2$$

it is easy to check that:

$$\begin{cases} w_i^\varepsilon(x) \to \xi_i \cdot x & \text{in } w - H^{1,p}_{\text{loc}}(\boldsymbol{R}^n)\,, \\ a\left(\dfrac{x}{\varepsilon}, u, Dw_i^\varepsilon\right) \to b(u, \xi_i) & \text{in } w - L^{p'}_{n,\text{loc}}(\boldsymbol{R}^n)\,, \\ \operatorname{div} a\left(\dfrac{x}{\varepsilon}, u, Dw_i^\varepsilon(x)\right) = 0 \end{cases} \tag{2.11}$$

where the last relation is proved using Lemma 2.2.

If $p \geqslant 2$ from ii) we get:

$$\alpha \int_Y \eta |Dw_1^\varepsilon - Dw_2^\varepsilon|^p\, dx \leqslant \int_Y \eta \left(a\left(\frac{x}{\varepsilon}, u, Dw_1^\varepsilon\right) - a\left(\frac{x}{\varepsilon}, u, Dw_2^\varepsilon\right), Dw_1^\varepsilon - Dw_2^\varepsilon \right) dx\,,$$

where η is a $C_0^1(Y)$ function, $0 \leqslant \eta \leqslant 1$. Then, passing to the limit as $\varepsilon \to 0$, and using (2.11), by the compensated compactness results of [11] we get $(2.10)_1$. In the case $1 < p \leqslant 2$ we can argue in a similar way. In fact from jj) we have

$$\sqrt{\alpha}\int_Y \eta|Dw_1^\varepsilon - Dw_2^\varepsilon|\,dx \leqslant \left(\int_Y \eta\left(a\left(\frac{x}{\varepsilon}, u, Dw_1^\varepsilon\right) - a\left(\frac{x}{\varepsilon}, u, Dw_2^\varepsilon\right), Dw_1^\varepsilon - Dw_2^\varepsilon\right)dx\right)^{\frac12} \cdot \left(\int_Y \eta(|Dw_1^\varepsilon| + |Dw_2^\varepsilon|)^{2-p}\,dx\right)^{\frac12}.$$

Then, passing to the limit as before, and remarking that $\int_Y |Dw_i^\varepsilon|^p\,dx \leqslant c\int_Y |Dv_i|^p\,dy$, and using (2.4) we get soon $(2.10)_2$. □

Finally we want to show how in some special case the Hölder estimate on b, provided by the Lemma 2.5 can be improved. In fact let us suppose that $a = a(x, \xi)$ and that verifies the following assumption:

K_2) i) $|a(x, \xi_1) - a(x, \xi_2)| \leqslant \beta(|\xi_1| + |\xi_2|)^{p-2}|\xi_1 - \xi_2|$;

ii) $(a(x, \xi_1) - a(x, \xi_2), \xi_1 - \xi_2) \geqslant \alpha|\xi_1 - \xi_2|^2(|\xi_1| + |\xi_2|)^{p-2}$.

Then we have:

Proposition 2.7. – *If* $p \geqslant 2$ *and* $a(x, \xi)$ *verifies* K_2) *then*

$$|b(\xi_1) - b(\xi_2)| \leqslant c(1 + |\xi_1| + |\xi_2|)^{p-2}|\xi_1 - \xi_2|. \tag{2.12}$$

Proof. – Let us denote by v_1 and v_2 the solutions of (2.2) defining respectively $b(\xi_1)$ and $b(\xi_2)$. Then from K_2) we get:

$$\begin{aligned}|b(\xi_1) - b(\xi_2)| &\leqslant \beta\int_Y (|Dv_1| + |Dv_2|)^{p-2}|Dv_1 - Dv_2|\,dy \leqslant \\ &\leqslant \beta\alpha^{-\frac12}\left(\int_Y |(Dv_1| + |Dv_2|)^{p-2}\right)^{\frac12}\left(\int_Y (a(y, Dv_1) - a(y, Dv_2), Dv_1 - Dv_2)\right)^{\frac12} \leqslant \\ &\leqslant \beta\alpha^{-\frac12}\left(\int_Y (|Dv_1| + |Dv_2|)^{p-2}\right)^{\frac12}|b(\xi_1) - b(\xi_2)|^{\frac12}|\xi_1 - \xi_2|^{\frac12}.\end{aligned}$$

Then using (2.4), we easily deduce (2.12). □

We remark also that if $a(x, 0) = 0$, then we have in particular

$$|b(\xi_1) - b(\xi_2)| \leqslant \frac{\beta^p}{\alpha^{p-1}}(|\xi_1| + |\xi_2|)^{p-2}|\xi_1 - \xi_2| \tag{2.13}$$

since it is easy to check that in this case (2.4) reduces to

$$\alpha\|Dv\|_p \leqslant \beta|\xi| .$$

However the following example shows that if $a(x, 0) \neq 0$, in general (2.12) cannot be improved in order to have an estimate of the type of (2.13). The same example shows also that if $p \neq 2$ and $a(x, \xi) = \xi|\xi|^{p-2} + d(x)$, then $b(\xi)$ is not equal to $\xi|\xi|^{p-2} +$ constant, as it happens if $p = 2$. Let us take, for instance, $p = 3$, $n = 1$ and $a(x, \xi) = \xi|\xi| + d(x)$, where $d(y) = 1$ if $0 < y < \frac{1}{2}$, $d(y) = 2$ if $\frac{1}{2} < y < 1$. Of course $a(x, \xi)$ verifies the condition K_2), while an easy calculation shows that:

$$b(\xi) = \begin{cases} 1 + \dfrac{(4\xi|\xi| + 1)^2}{16\xi|\xi|} & \text{if } |\xi| \geqslant \dfrac{1}{2}, \\ \dfrac{3}{2} + \xi\sqrt{2 - 4\xi^2} & \text{if } |\xi| \leqslant \dfrac{1}{2}. \end{cases}$$

3. - Homogenization.

Let us prove the following homogenization result:

THEOREM 3.1. – *If $a(x, u, \xi)$ verifies the structure conditions H_1, H_2 and H_3, then for any $f \in L^q$, with $q > n/p$, and any sequence (u_{ε_h}) of solutions of $(\mathcal{T}_{\varepsilon_h})$, with $\varepsilon_h \to 0$, there exist a subsequence (u_{ε_r}) and a function u_0, solution of $(\mathcal{T}_0)$ such that:*

$$u_{\varepsilon_r} \to u_0 \qquad \textit{weakly in } H^{1,p}(\Omega), \tag{3.1}$$

$$a\left(\frac{x}{\varepsilon_r}, u_{\varepsilon_r}, Du_{\varepsilon_r}\right) \to b(u_0, Du_0) \qquad \textit{weakly in } L^{p'}_n(\Omega). \tag{3.2}$$

PROOF. – Let us denote by u_ε a solution of $(\mathcal{T}_\varepsilon)$. By Remark 2.3 we have that $\|Du_\varepsilon\|_{L^p} \leqslant C$ (with C independent of ε). Then by i) or j) we get that also $\|a(x/\varepsilon, u_\varepsilon, Du_\varepsilon)\|_{L^{p'}_n}$ is uniformly bounded. So passing eventually to a subsequence, we may suppose that:

$$\begin{cases} u_\varepsilon \to u_0 & \text{in } w - H^{1,p}_0(\Omega), \\ a\left(\dfrac{x}{\varepsilon}, u_\varepsilon, Du_\varepsilon\right) \to a_0(x) & \text{in } w - L^{p'}_n(\Omega). \end{cases}$$

The theorem will be proved if we show that

$$a_0(x) = b(u_0, Du_0) \quad \text{a.e. in } \Omega . \tag{3.3}$$

Let us fix $\nu \in N$ and denote by $\{Q_{i\nu}\}_i$ a partition of R^n in cubes with the edges equal to $2^{-\nu}$. Then we define: $I_\nu = \{i\colon Q_{i\nu} \subset \Omega\}$, $\Omega_\nu = \bigcup_{i\in I_\nu} Q_{i\nu}$. For any i let us consider $\langle u_0\rangle_{i\nu} = \langle u_0\rangle_{Q_{i\nu}}$ and $\langle Du_0\rangle_{i\nu} = \langle Du_0\rangle_{Q_{i\nu}}$. Then, if $\chi_{i\nu}(x)$ is the characteristic function of $Q_{i\nu}$, by the continuity of b (see Lemma 2.5), we have if $\nu \to +\infty$ then

$$\sum_{i\in I_\nu} \chi_{i\nu}(x)\, b(\langle u_0\rangle_{i\nu}, \langle Du_0\rangle_{i\nu}) \to b\big(u_0(x), Du_0(x)\big) \quad \text{a.e.} \tag{3.4}$$

Moreover, from Lemma 2.4, we have that for any measurable set $E \subset \Omega$

$$\int_E \Big|\sum_{i\in I_\nu} \chi_{i\nu}(x)\, b(\langle u_0\rangle_{i\nu}, \langle Du_0\rangle_{i\nu})\Big|^{p'} dx \leqq c \int_E \Big(1 + \Big|\sum_i \chi_{i\nu}(x) \langle u_0\rangle_{i\nu}\Big| + \Big|\sum_i \chi_{i\nu}(x) \langle Du_0\rangle_{i\nu}\Big|\Big)^p dx\,.$$

So, from the equi-absolute continuity of the integral on the left and from (3.4) we deduce that:

$$\sum_i \chi_{i\nu}(x)\, b(\langle u_0\rangle_{i\nu}, \langle Du_0\rangle_{i\nu}) \to b\big(u_0(x), Du_0(x)\big) \quad \text{in } L^{p'}_n(\Omega)\,, \tag{3.5}$$

as $\nu \to +\infty$. If $v_{i\nu} \in \langle Du_0\rangle_{i\nu}\cdot y + H^{1,p}_{\mathrm{per}}(Y)$ is the solution of (2.2) corresponding to $(\langle u_0\rangle_{i\nu}, \langle Du_0\rangle_{i\nu})$, then $u_{i\nu}(y) = v_{i\nu}(y) - \langle Du_0\rangle_{i\nu}\cdot y$ may be extended by periodicity to a function in $H^{1,p}_{\mathrm{loc}}(\mathbf{R}^n)$ (see Lemma 2.1). So we can define

$$w^\varepsilon_{i\nu}(x) = \varepsilon u_{i\nu}\Big(\frac{x}{\varepsilon}\Big) + \langle Du_0\rangle_{i\nu}\cdot x\,.$$

Hence by the above definitions and Lemma 2.2 we have that for any fixed i and ν

$$\begin{cases} w^\varepsilon_{i\nu} \to \langle Du_0\rangle_{i\nu}\cdot x & w - H^{1,p}_{\mathrm{loc}}(\mathbf{R}^n)\,, \\ a\Big(\dfrac{x}{\varepsilon}, \langle u_0\rangle_{i\nu}, Dw^\varepsilon_{i\nu}\Big) \to b\big(\langle u_0\rangle_{i\nu}, \langle Du_0\rangle_{i\nu}\big) & w - L^{p'}_{n,\mathrm{loc}}(\mathbf{R}^n)\,, \\ \operatorname{div}_x a\Big(\dfrac{x}{\varepsilon}, \langle u_0\rangle_{i\nu}, Dw^\varepsilon_{i\nu}\Big) = 0\,. \end{cases} \tag{3.6}$$

Using the periodicity of $u_{i\nu}$ and the (2.4), we have also the following estimate:

$$\sum_{i\in I_\nu} \int_{Q_{i\nu}} |Dw^\varepsilon_{i\nu}|^p\, dx \leqslant \sum_i 2^{-\nu n} \varepsilon^n \Big(\frac{1}{\varepsilon} + 2^\nu\Big)^n \int_Y |Dv_{i\nu}(y)|^p\, dy \leqslant$$
$$\leqslant C \sum_i 2^{-\nu n}(1 + \varepsilon^n 2^{\nu n})\,(1 + |\langle u_0\rangle_{i\nu}| + |\langle Du_0\rangle_{i\nu}|)^p$$

where C is independent of ε and ν. Then, writing the last term as an integral over Ω_ν we have:

$$\sum_{i\in I_\nu} \int_{Q_{i\nu}} |Dw^\varepsilon_{i\nu}|^p\, dx \leqslant C(1 + \varepsilon^n 2^{n\nu}) \int_\Omega (1 + |u_0| + |Du_0|)^p\, dx\,. \tag{3.7}$$

Finally, let us consider $\eta \in C_0^1(Q_{i\nu})$, $0 \leqslant \eta \leqslant 1$ and extend it by periodicity to the whole $\boldsymbol{R}^n$.

Case $p > 2$. – If φ is any $C_n^0(\bar\Omega)$ function and $M_\varphi = \sup\limits_\Omega |\varphi|$, then from i) we get:

$$\left|\int_\Omega a\left(\frac{x}{\varepsilon}, u_\varepsilon, Du_\varepsilon\right)\cdot\varphi\eta\,dx - \sum_{i\in I_\nu}\int_{Q_{i\nu}} a\left(\frac{x}{\varepsilon}, \langle u_0\rangle_{i\nu}, Dw_{i\nu}\right)\cdot\varphi\eta\,dx\right| \leqslant \tag{3.8}$$

$$\leqslant CM_\varphi|\Omega - \Omega_\nu|^{1/p} + \sum_i \int_{Q_{i\nu}} M_\varphi\eta\{(|u_\varepsilon| + |\langle u_0\rangle_{i\nu}| + |Du_\varepsilon| + |Dw_{i\nu}^\varepsilon|)^{p-2}\cdot$$

$$\cdot(|u_\varepsilon - \langle u_0\rangle_{i\nu}| + |Du_\varepsilon - Dw_{i\nu}^\varepsilon|)\}\,dx \leqq CM_\varphi|\Omega - \Omega_\nu| + CM_\varphi^{p/(p-1)}\delta^{p/(p-1)}(1 + \varepsilon^n 2^{\nu n}) +$$

$$+ \delta^{-p}\sum_i\int_{Q_{i\nu}} |u_\varepsilon - \langle u_0\rangle_{i\nu}|^p\,dx + \delta^{-p}\sum_i\int_{Q_{i\nu}} |Du_\varepsilon - Dw_{i\nu}^\varepsilon|^p\,\eta\,dx$$

where the last inequality is obtained by applying Young inequality with $\delta > 0$ and the estimate (3.7). Then by the same argument used in Lemma 2.5 to prove (2.6) we have:

$$\sum_i\int_{Q_{i\nu}} |Du_\varepsilon - Dw_{i\nu}^\varepsilon|^p\,\eta\,dx \leqslant \tag{3.9}$$

$$\leqslant C\sum_i\int_{Q_{i\nu}} |u_\varepsilon - \langle u_0\rangle_{i\nu}|^{p'}(1 + |u_\varepsilon| + |u_{0i\nu}| + |Du_\varepsilon| + |\langle Dw_{i\nu}^\varepsilon\rangle|)^{p'(p-2)}\,dx +$$

$$+ \sum_i\int_{Q_{i\nu}}\left(a\left(\frac{x}{\varepsilon}, u_\varepsilon, Du_\varepsilon\right) - a\left(\frac{x}{\varepsilon}, \langle u_0\rangle_{i\nu}, Dw_{i\nu}^\varepsilon\right), Du_\varepsilon - Dw_{i\nu}^\varepsilon\right)\eta\,dx = a_\nu^\varepsilon + b_\nu^\varepsilon\,.$$

But applying again Young inequality with δ^{-p} and (3.7) we get:

$$a_\nu^\varepsilon \leqslant C\delta^{-p(p-1)}\sum_i\int_{Q_{i\nu}} |u_\varepsilon - \langle u_0\rangle_{i\nu}|^p\,dx + C\delta^{p(p-1)/(p-2)}(1 + \varepsilon^n 2^{n\nu})\,. \tag{3.10}$$

On the other hand, using the fact that $\eta \in C_0^1(Q_{i\nu})$ for any i, we may write, integrating by parts:

$$b_\nu^\varepsilon = \sum_i\int_{Q_{i\nu}}\left(a\left(\frac{x}{\varepsilon}, u_\varepsilon, Du_\varepsilon\right) - a\left(\frac{x}{\varepsilon}, \langle u_0\rangle_{i\nu}, Dw_{i\nu}^\varepsilon\right), Du_0 - \langle Du_0\rangle_{i\nu}\right)\eta\,dx +$$

$$+ \sum_i\int_{Q_{i\nu}}\left\{\left[\eta f - D\eta\cdot\left(a\left(\frac{x}{\varepsilon}, u_\varepsilon, Du_\varepsilon\right) - a\left(\frac{x}{\varepsilon}, \langle u_0\rangle_{i\nu}, Dw_{i\nu}^\varepsilon\right)\right)\right]\cdot\right.$$

$$\left.\cdot[(u_\varepsilon - u_0) - (w_{i\nu}^\varepsilon - \langle Du_0\rangle_{i\nu}\cdot x)]\right\}dx\,,$$

where we used also the fact that u_ε is a solution of $(\mathcal{P}_\varepsilon)$ and the (3.6). The passing to the limit as $\varepsilon \to 0$, by (3.6) we get:

$$\lim_{\varepsilon\to 0} b_\nu^\varepsilon \leqslant \sum_i \int_{Q_{i\nu}} |(a_0(x) - b(\langle u_0\rangle_{i\nu}, \langle Du_0\rangle_{i\nu}), Du_0 - \langle Du_0\rangle_{i\nu})|\, dx\,.$$

So if we first pass to the limit as $\varepsilon \to 0$, then let η converge to 1 in L^p, and then take the limit as $\nu \to +\infty$, from the above formula and from (3.8), (3.9) and (3.10), using (3.5) we obtain:

$$\left|\int_\Omega a_0(x)\cdot\varphi\, dx - \int_\Omega b(u_0, Du_0)\cdot\varphi\, dx\right| \leqslant C\,(M_\varphi^{p/(p-1)}\delta^{p/(p-1)} + \delta^{p(p-1)/(p-2)})\,.$$

So, letting δ go to zero, from the arbitrariety of φ we get (3.3).

Case $1 < p \leqslant 2$. – In this case the proof is, with minor changes, essentially the same as in the previous case. So, instead of (3.8), now we have, using j):

$$\left|\int_\Omega a\left(\frac{x}{\varepsilon}, u_\varepsilon, Du_\varepsilon\right)\cdot\varphi\eta\, dx - \sum_{i\in I_\nu}\int_{Q_{i\nu}} a\left(\frac{x}{\varepsilon}, \langle u_0\rangle_{i\nu}, Dw_{i\nu}^\varepsilon\right)\cdot\varphi\eta\, dx\right| \leqslant$$
$$\leqslant CM_\varphi|\Omega - \Omega_\nu| + \beta\sum_i\int_{Q_{i\nu}} (|u_\varepsilon - \langle u_0\rangle_{i\nu}|^{p-1} + |Du_\varepsilon - Dw_{i\nu}^\varepsilon|^{p-1})\,|\varphi|\eta\, dx\,.$$

Then, using jj) we can control the last term:

$$\sum_i\int_{Q_{i\nu}} |Du_\varepsilon - Dw_{i\nu}^\varepsilon|^{p-1}\eta\, dx \leqslant c\delta^{2/(3-p)}\sum_i\int_{Q_{i\nu}} (|Du_\varepsilon| + |Dw_{i\nu}^\varepsilon|)^{(2-p)(p-1)/(3-p)} +$$
$$+\,\delta^{-2/(p-1)}\sum_i\int_{Q_{i\nu}}\left(a\left(\frac{x}{\varepsilon}, \langle u_0\rangle_{i\nu}, Dw_{i\nu}^\varepsilon\right) - a\left(\frac{x}{\varepsilon}, u_\varepsilon, Dw_{i\nu}^\varepsilon\right), Du_\varepsilon - Dw_{i\nu}^\varepsilon\right)\eta\, dx +$$
$$+\,\delta^{-2/(p-1)}\sum_i\int_{Q_{i\nu}}\left(a\left(\frac{x}{\varepsilon}, u_\varepsilon, Du_\varepsilon\right) - a\left(\frac{x}{\varepsilon}, \langle u_0\rangle_{i\nu}, Dw_{i\nu}^\varepsilon\right), Du_\varepsilon - Dw_{i\nu}^\varepsilon\right)\eta\, dx$$

and each of these terms is treated as in the previous case: The first using (3.7), the second using j) and the Young inequality, the third as b_i^ε before. □

We observe that if $p = 2$, then the structure condition H_2 implies that for any $u_1, u_2 \in \boldsymbol{R}$, $\xi_1, \xi_1 \in \boldsymbol{R}^n$

$$(a(x, u_1, \xi_1) - a(x, u_2, \xi_2), \xi_1 - \xi_2) \geqslant c_1|\xi_1 - \xi_2|^2 - c_2|u_1 - u_2|^2$$

and so, with the same argument used in [18], one can prove that the problem $(\mathcal{P}_\varepsilon)$ has a unique solution. Moreover Lemma 2.5 and 2.6 show that also $b(u, \xi)$ verifies the same structure condition. So, also problem $(\mathcal{T}_0)$ has a unique solution. Hence we may state the following

Corollary 3.2. – *If $p = 2$, under the same hypothesis of Theorem 3.1, for any $f \in L^q$ with $q > n/2$ we have*

$$u_\varepsilon \to u_0 \qquad \textit{weakly in } H^{1,2}(\Omega) \quad \textit{as } \varepsilon \to 0,$$

$$a\left(\frac{x}{\varepsilon}, u_\varepsilon, Du_\varepsilon\right) \to b(u_0, Du_0) \qquad \textit{weakly in } L^2_n(\Omega) \qquad \textit{as } \varepsilon \to 0$$

where u_0 and u_ε are the unique solutions of $(\mathcal{T}_0)$ and $(\mathcal{T}_\varepsilon)$.

Another case in which $(\mathcal{T}_\varepsilon)$ has a unique solution, even under weaker hypothesis of f and $a(x, 0, 0)$, is where $a(x, u, \xi)$ does not depend on u. In this case Lemma 2.6 shows that also $(\mathcal{T}_0)$ has a unique solution. Hence by theorem 3.1 we have again

Corollary 3.3. – *If $a(x, u, \xi)$ does not depend on u and verifies the structure conditions H_1 and H_2 and if $a(x, 0) \in L^{p'}(\Omega)$, then for any $f \in H^{-1,p'}$, if u_ε is the solution of $(\mathcal{T}_\varepsilon)$, and u_0 of $(\mathcal{T}_0)$:*

$$u_\varepsilon \to u_0 \qquad \textit{weakly in } H^{1,p}(\Omega),$$

$$a\left(\frac{x}{\varepsilon}, Du_\varepsilon\right) \to b(Du_0) \qquad \textit{weakly in } L^{p'}_n(\Omega).$$

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