ON THE HOMOLOGY OF THE HILBERT SCHEME OF POINTS IN THE PLANE

Geir Ellingsrud[1] and Stein Arild Strømme[2]

[1] Matematisk institutt, Universitetet i Oslo, Blindern, Oslo 3, Norway

[2] Matematisk institutt, Universitetet i Bergen, 5014 Bergen, Norway

Although several authors have been interested in the Hilbert scheme $\mathrm{Hilb}^d(\mathbb{P}^2)$ parametrizing finite subschemes of length d in the projective plane ([I1], [I2], [F1], [F2], [Br] among others) not much is known about the topological properties of this space. The Picard group has been calculated ([F2]), and the homology groups of $\mathrm{Hilb}^3(\mathbb{P}^2)$ have been computed ([H]). In this paper we give a precise description of the additive structure of the homology of $\mathrm{Hilb}^d(\mathbb{P}^2)$, applying the results of Birula-Bialynicki ([B1], [B2]) on the cellullar decompositions defined by a torus action to the natural action of a maximal torus of SL(3) on $\mathrm{Hilb}^d(\mathbb{P}^2)$. A rather easy consequence of the fact that this action has finitely many fixpoints is that the cycle maps between the Chow groups and the homology groups are isomorphisms. In particular there is no odd homology, and the homology groups are all free. The main objective of this work is to compute their ranks: the Betti numbers of $\mathrm{Hilb}^d(\mathbb{P}^2)$.

As a byproduct of our method we get similar results on the homology of the punctual Hilbert scheme and of the Hilbert scheme of points in the affine plane.

It seems natural to generalize our results to any toric smooth surface. However, we give the results only for the rational ruled surfaces $\mathbb{F}_n$ with an indication of the necessary changes in the proofs.

For simplicity we work over the field of complex numbers, but with an appropriate interpretation of the word "homology" our results remain valid over any base field.

§1

Let $\mathbb{P}^2$ be the projective plane over $\mathbb{C}$. For any positive integer d, let $\mathrm{Hilb}^d(\mathbb{P}^2)$ denote the Hilbert scheme parametrizing finite subschemes of $\mathbb{P}^2$ of length d. If $\mathbb{A}^2$ denotes the complement of a line in $\mathbb{P}^2$, let $\mathrm{Hilb}^d(\mathbb{A}^2)$ denote the open subscheme of $\mathrm{Hilb}^d(\mathbb{P}^2)$ corresponding to subschemes with support in $\mathbb{A}^2$. Furthermore let $\mathrm{Hilb}^d(\mathbb{A}^2,0)$ be the closed subscheme of $\mathrm{Hilb}^d(\mathbb{A}^2)$ parametrizing subschemes supported in the origin.

For any complex variety X, let $H_*(X)$ be the Borel-Moore homology of X (homology with locally finite supports). By the i-th Betti number $b_i(X)$ we shall mean the rank of the finitely generated abelian group $H_i(X)$. Let $\chi(X) = \sum(-1)^i b_i(X)$ be the Euler-Poincaré characteristic of X. As usual, $A_*(X)$ is the Chow group of X, and $cl: A_*(X) \to H_*(X)$ is the cycle map (see [Fu] ch. 19.1).

If m and n are non-negative integers, let $P(m,n)$ denote the number of sequences $n > b_0 > b_1 > \ldots > b_m = 0$ such that $\sum b_i = m$. If $n > m$, then $P(m,n) = P(m)$, the number of partitions of m. Let $P(m,n) = 0$ if m or n is negative.

(1.1) Theorem. (i) *Let X denote one of the schemes $\mathrm{Hilb}^d(\mathbb{P}^2)$, $\mathrm{Hilb}^d(\mathbb{A}^2)$, or $\mathrm{Hilb}^d(\mathbb{A}^2,0)$. Then the cycle map $cl: A_*(X) \to H_*(X)$ is an isomorphism, and in particular the odd homology vanishes. Furthermore, both groups are free abelian groups.*

(ii) $$b_{2k}(\mathrm{Hilb}^d(\mathbb{P}^2)) = \sum_{d_0+d_1+d_2=d} \ \sum_{p+r=k-d_1} P(p,d_0-p)P(d_1)P(2d_2-r,r-d_2)$$

and $$\chi(\mathrm{Hilb}^d(\mathbb{P}^2)) = \sum_{d_0+d_1+d_2=d} P(d_0)P(d_1)P(d_2).$$

(iii) $b_{2k}(\mathrm{Hilb}^d(\mathbb{A}^2)) = P(2d-k,k-d)$ and $\chi(\mathrm{Hilb}^d(\mathbb{A}^2)) = P(d)$.

(iv) $b_{2k}(\mathrm{Hilb}^d(\mathbb{A}^2,0)) = P(k,d-k)$ and $\chi(\mathrm{Hilb}^d(\mathbb{A}^2,0)) = P(d)$.

Remark. The Betti numbers of $\mathrm{Hilb}^3(\mathbb{P}^2)$ were determined by A. Hirschowitz ([H]). In table 1 we have listed the Betti numbers of $\mathrm{Hilb}^d(\mathbb{P}^2)$ for $1 \le d \le 10$.

d \ k	0	1	2	3	4	5	6	7	8	9	10
1	1	1									
2	1	2	3								
3	1	2	5	6							
4	1	2	6	10	13						
5	1	2	6	12	21	24					
6	1	2	6	13	26	39	47				
7	1	2	6	13	28	49	74	83			
8	1	2	6	13	29	54	94	131	150		
9	1	2	6	13	29	56	105	167	232	257	
10	1	2	6	13	29	57	110	189	298	395	440

Table 1.

The Betti numbers $b_{2k}(\mathrm{Hilb}^d(\mathbb{P}^2))$ are listed for $1 \le d \le 10$ and $0 \le k \le d$. For $d \le k \le 2d = \dim \mathrm{Hilb}^d(\mathbb{P}^2)$ the number $b_{2k}(\mathrm{Hilb}^d(\mathbb{P}^2))$ is given by Poincaré duality.

(1.2) Corollary. (Briancon; [Br] V.3.3.) $\mathrm{Hilb}^d(\mathbb{A}^2,0)$ is irreducible.

Proof. By a result of Gaffney-Lazarsfeld (see [Ga] or [I2] theorem 2), any irreducible component of $\mathrm{Hilb}^d(\mathbb{A}^2,0)$ has dimension at least $d-1$. From (iv) of theorem (1.1) it follows that $b_{2k}(\mathrm{Hilb}^d(\mathbb{A}^2,0)) = 1$ if $k = d-1$ and 0 if $k>d-1$. The corollary follows from [Fu] lemma 19.1.1. □

Let S denote the graded $\mathbb{Z}$-algebra freely generated by $c_1,\dots,c_d$, $c_1',\dots,c_d'$ and $c_2'',\dots,c_{d-1}''$ where the degree of c_i, c_i' and c_i'' is i. Denote by S_k the graded part of S of degree k.

(1.3) Corollary. If $2k<d$, then $b_{2k}(\mathrm{Hilb}^d(\mathbb{P}^2)) = \mathrm{rk}_{\mathbb{Z}} S_k$.

Proof. Assume $2k<d$. Let d_0, d_1, d_2, p and r be indices such that the corresponding term in the expression for $b_{2k}(\mathrm{Hilb}^d(\mathbb{P}^2))$ in (1.1) part (ii) is non-zero. Then $P(2d_2-r,r-d_2)\neq 0$ and $r-d_2\geq 0$. Therefore $p = k-d_1-r\leq k-d_1-d_2$ and hence $2p\leq 2k-2d_1-2d_2<d-2d_1-2d_2\leq d_0$. Thus $p<d_0-p$ and $P(p,d_0-p) = P(p)$. We may therefore write

$$b_{2k}(\mathrm{Hilb}^d(\mathbb{P}^2)) = \sum_{p,d_1} P(p)P(d_1)B(k-d_1-p)$$

where $B(j) = \sum_m P(2m-j,j-m)$. This completes the proof since the Hilbert function of $\mathbb{Z}[c_1,c_2,\dots]$ is $P(j)$ and that of $\mathbb{Z}[c_2,c_3,\dots]$ is $B(j)$. □

The reason for giving this corollary is the following. Let $\pi: Z\longrightarrow \mathrm{Hilb}^d(\mathbb{P}^2)$ be the universal family and let $\phi: Z\longrightarrow \mathbb{P}^2$ be the natural map. Then $E_i = \pi_* \psi^* \mathcal{O}_{\mathbb{P}^2}(i)$ are vectorbundles of rank d on $\mathrm{Hilb}^d(\mathbb{P}^2)$. The Chern classes of E_0, E_1 and E_2 are natural candidates for algebra generators of the Chow ring of $\mathrm{Hilb}^d(\mathbb{P}^2)$. One verifies that $c_1(E_2) = 2c_1(E_1) - c_1(E_0)$. The algebra S therefore maps surjectively onto the subalgebra of $A^*(\mathrm{Hilb}^d(\mathbb{P}^2))$ generated by the Chern classes of the E_i's. The corollary can thus be regarded as evidence for the following conjecture

(1.4) Conjecture. $A^*(\mathrm{Hilb}^d(\mathbb{P}^2))$ is generated as a $\mathbb{Z}$-algebra by the Chern-classes of E_0, E_1 and E_2.

We end this section by recalling two results which are fundamental for this work.

Following Fulton ([Fu] example 1.9.1) we say that a scheme X has a cellullar decomposition if there is a filtration $X = X_n \supset X_{n-1} \supset \ldots \supset X_0 \supset X_{-1} = \emptyset$ by closed subschemes with each $X_i - X_{i-1}$ a disjoint union of schemes U_{ij} isomorphic to affine spaces $\mathbb{A}^{n_{ij}}$. The U_{ij}'s will be called the cells of the decomposition.

(1.5) Proposition. Let X be a scheme with a cellullar decomposition. Then for $0 \leq i \leq \dim X$

(i) $H_{2i+1}(X) = 0$

(ii) $H_{2i}(X)$ is a $\mathbb{Z}$-module freely generated by the classes of the closures of the i-dimensional cells.

(iii) The cycle map $cl: A_*(X) \to H_*(X)$ is an isomorphism.

For a proof of this proposition see [Fu] chapter 19.1.

Let X be a variety with an action of $\mathbb{G}_m$ and let x be a fixpoint. Then there is an induced action of $\mathbb{G}_m$ on the tangent space $T_{X,x}$. The part of $T_{X,x}$ where the weights of $\mathbb{G}_m$ are positive is denoted by $(T_{X,x})^+$. The following theorem is proved in [B1] and [B2].

(1.6) Theorem. (Birula-Bialynicki). Let X be a smooth projective variety with an action of $\mathbb{G}_m$. Suppose that the fixpoint set $\{x_1,\dots,x_n\}$ is finite, and let $X_i = \{x \in X \mid \lim_{t\to 0} tx = x_i\}$. Then

(i) X has a cellullar decomposition with cells X_i.

(ii) $T_{X_i,x_i} = (T_{X,x_i})^+$.

§2.

From now on we fix a system of homogeneous coordinates T_0, T_1, T_2 of $\mathbb{P}^2$. Let $G \subseteq SL(3,\mathbb{C})$ be the maximal torus consisting of all diagonal matrices. We denote by $\lambda_0, \lambda_1, \lambda_2$ the complex characters of G such that for any $g \in G$ we have $g = \mathrm{diag}(\lambda_0(g), \lambda_1(g), \lambda_2(g))$. Then G acts on $\mathbb{P}^2$ via $gT_i = \lambda_i(g)T_i$, and on points (a_0, a_1, a_2), this action is given by $g(a_0, a_1, a_2) = (\lambda_0(g)^{-1}a_0, \lambda_1(g)^{-1}a_1, \lambda_2(g)^{-1}a_2)$. The fixpoints are clearly $P_0 = (1,0,0)$, $P_1 = (0,1,0)$ and $P_2 = (0,0,1)$.

Let L be the line $T_2 = 0$, and put $F_0 = \{P_0\}$, $F_1 = L - P_0$, and $F_2 = \mathbb{P}^2 - L$. Then $F_i \simeq \mathbb{A}^i$, and they define a cellullar decomposition of $\mathbb{P}^2$. The one-parameter subgroups $\phi: \mathbb{G}_m \to G$ inducing this cellullar decomposition are those of the type $\phi(t) = \mathrm{diag}(t^{w_0}, t^{w_1}, t^{w_2})$ where $w_0 < w_1 < w_2$ and $w_0 + w_1 + w_2 = 0$.

The action of G on $\mathbb{P}^2$ induces in a natural way an action of G on $\mathrm{Hilb}^d(\mathbb{P}^2)$. If $Z \subseteq \mathbb{P}^2$ corresponds to a fixpoint of this action, clearly the support of Z is contained in the fixpoint set $\{P_0, P_1, P_2\}$ of G. Hence we may write $Z = Z_0 \cup Z_1 \cup Z_2$ where Z_i is supported in P_i and corresponds to a fixpoint in $\mathrm{Hilb}^{d_i}(\mathbb{P}^2)$, where $d_i = \mathrm{length}(\mathcal{O}_{Z_i})$.

(2.1) Lemma. The action of G on $\mathrm{Hilb}^d(\mathbb{P}^2)$ has only finitely many fixpoints.

Proof. A point of $\mathrm{Hilb}^d(\mathbb{P}^2)$ is a fixpoint if and only if the corresponding ideal I in $\mathbb{C}[T_0, T_1, T_2]$ is invariant under G, which is the case if and only if I is generated by monomials. These ideals obviously form a finite family. □

It is well known that $\mathrm{Hilb}^d(\mathbb{P}^2)$ is smooth and projective ([Gr], [F1]). Hence (1.5) and (1.6) apply to the action of any sufficiently general one-parameter subgroup of G on $\mathrm{Hilb}^d(\mathbb{P}^2)$, and we have proved the statements in (1.1) part (i) concerning $\mathrm{Hilb}^d(\mathbb{P}^2)$. To prove the rest of (1.1) it remains to count the cells of a given dimension. For this purpose we use a decomposition of the Hilbert scheme which we now proceed to describe.

For any $Z \subseteq \mathbb{P}^2$ of finite length d we can write Z uniquely as a disjoint union $Z = Z_0 \cup Z_1 \cup Z_2$ where each Z_i is a closed subscheme of $\mathbb{P}^2$ supported in F_i. Put $d_i(Z) = \mathrm{length}(\mathcal{O}_{Z_i})$. For any triple (d_0, d_1, d_2) of non-negative integers with $d = d_0+d_1+d_2$, we define $W(d_0,d_1,d_2)$ to be the (locally closed) subset of $\mathrm{Hilb}^d(\mathbb{P}^2)$ corresponding to subschemes Z with $d_i(Z) = d_i$ for $i = 0,1,2$. Clearly $\mathrm{Hilb}^d(\mathbb{P}^2) = \bigcup_{d_0+d_1+d_2=d} W(d_0,d_1,d_2)$.

Let ϕ be any one-parameter subgroup of G respecting the cellullar decomposition $\{F_0,F_1,F_2\}$ of $\mathbb{P}^2$. Then ϕ induces a cellullar decomposition of $\mathrm{Hilb}^d(\mathbb{P}^2)$, and $W(d_0,d_1,d_2)$ is a union of cells from this decomposition. In fact, let Z be in $W(d_0,d_1,d_2)$ and write $Z = Z_0 \cup Z_1 \cup Z_2$. Then, as $t \to 0$, $\phi(t)(Z_i)$ approaches a subscheme supported in P_i. Thus $W(d_0,d_1,d_2)$ has a cellullar decomposition and (1.5) applies to it.

Since $W(d_0,d_1,d_2) \simeq W(d_0,0,0) \times W(0,d_1,0) \times W(0,0,d_2)$ we get

(2.2) <u>Lemma</u>. $b_{2k}(\mathrm{Hilb}^d(\mathbb{P}^2)) =$

$$\sum_{d_0+d_1+d_2=d} \ \sum_{p+q+r=k} b_{2p}(W(d_0,0,0))\, b_{2q}(W(0,d_1,0))\, b_{2r}(W(0,0,d_2)).$$

This reduces our problem to the calculation of the Betti numbers of $W(d_0,0,0)$, $W(0,d_1,0)$ and $W(0,0,d_2)$.

§3.

The spaces $W(d,0,0)$, $W(0,d,0)$ and $W(0,0,d)$ are all contained in $\mathrm{Hilb}^d(\mathbb{P}^2)$. In the previous section we saw that they are unions of cells from a cellullar decomposition of $\mathrm{Hilb}^d(\mathbb{P}^2)$. The cells contained in $W(d,0,0)$ (resp. $W(0,d,0)$, $W(0,0,d)$) are exactly those corresponding to fixpoints supported in P_0 (resp. P_1, P_2). We are thus reduced to the study of G-invariant subschemes of $\mathbb{P}^2$ concentrated in one fixpoint of G. Any such subschemes is contained in a G-invariant affine plane. Hence we are interested in ideals of $R = \mathbb{C}[x,y]$ of finite colength, invariant under the action of a two-dimensional torus T given by $t.x = \lambda(t)x$ and $t.y = \mu(t)y$, where λ and μ are two linearly independent characters of T. We shall also denote by λ and μ the elements in the representation ring of T induced by the corresponding one-dimensional representations.

Let I be such an ideal. Then since I is T-invariant, it is generated by monomials in x and y. Hence the number $b_j = \inf\{k \mid x^j y^k \in I\}$ exists for each integer $j \geq 0$. Clearly $b_j = 0$ if $j >> 0$. Let r be the least integer such that $b_r = 0$. The b_j form a non-increasing sequence and $\sum_{j=0}^{r} b_j = \mathrm{length}(R/I) = d$. Furthermore $y^{b_0}, xy^{b_1}, \ldots, x^j y^{b_j}, \ldots, x^r$ is a (not necessarily minimal) set of generators for I. Note that this sets up a one-one correspondence between T-invariant ideals of colength d in R and partitions of d.

For any ordered pair $\underline{a} = (\alpha,\beta)$ of integers, let $R[\alpha,\beta]$, also denoted $R[\underline{a}]$, be the R-module R with the action of T given by $t.x^m y^n = \lambda(t)^{m-\alpha}\mu(t)^{n-\beta}x^m y^n$. In the representation ring of T we may write $R[\alpha,\beta] = \sum_{\substack{p \geq -\alpha \\ q \geq -\beta}} \lambda^p \mu^q$.

(3.1) Lemma. *There is a T-equivariant resolution*

$$0 \longrightarrow \bigoplus_{i=1}^{r} R[-\underline{n}_i] \xrightarrow{\ M\ } \bigoplus_{i=0}^{r} R[-\underline{d}_i] \longrightarrow I \longrightarrow 0$$

where $\underline{n}_i = (i, b_{i-1})$ *and* $\underline{d}_i = (i, b_i)$. *If* $e_i = b_{i-1} - b_i$ *for* $1 \leq i \leq r$ *then*

$$M = \begin{pmatrix} x & 0 & \cdots & \cdots & 0 \\ y^{e_1} & x & 0 & & \vdots \\ 0 & y^{e_2} & x & & \vdots \\ \vdots & 0 & & & 0 \\ \vdots & & & & x \\ 0 & \cdots & \cdots & \cdots & y^{e_r} \end{pmatrix}.$$

Proof. This amounts to checking that M is equivariant and that the maximal minors of M are $y^{b_0}, xy^{b_1}, \ldots, x^j y^{b_j}, \ldots, x^r$, which is straightforward. □

(3.2) Lemma. *In the representation ring of* T *we have the identity*

$$\mathrm{Hom}_R(I, R/I) = \sum_{1 \leq i \leq j \leq r} \ \sum_{s=b_j}^{b_{j+1}-1} \left(\lambda^{i-j-1}\mu^{b_{i-1}-s-1} + \lambda^{j-i}\mu^{b_{i-1}-s}\right).$$

Proof. First we prove that $\mathrm{Hom}_R(I, R/I) \cong \mathrm{Ext}^1_R(I, I)$ in a T-equivariant way. The T-equivariant exact sequence

$$0 \to I \to R \to R/I \to 0$$

induces a T-equivariant sequence

$$0 \to \mathrm{Hom}_R(I, R/I) \to \mathrm{Ext}^1_R(I, I) \to \mathrm{Ext}^1(I, R) \to \mathrm{Ext}^1(I, R/I) \to 0$$

The last map of this sequence is an isomorphism because

$$\mathrm{Ext}_R^1(I,R) \cong \mathrm{Ext}_R^2(R/I,R) \cong \mathrm{Ext}_R^2(R/I,R/I) \cong \mathrm{Ext}_R^1(I,R/I).$$

To compute $\mathrm{Ext}_R^1(I,I)$ we use the T equivariant complex

$$E_0^\vee \otimes E_1 \xrightarrow{A} (E_0^\vee \otimes E_0) \oplus (E_1^\vee \otimes E_1) \xrightarrow{B} E_1^\vee \otimes E_0$$

where $E_0 = \bigoplus_{i=0}^{r} R[-\underline{d}_i]$ and $E_1 = \bigoplus_{i=1}^{r} R[-\underline{n}_i]$. The maps A and B are given by $A = (\mathrm{id}_{E_0^\vee} \otimes M, M^\vee \otimes \mathrm{id}_{E_1})$ and $B = (M^\vee \otimes \mathrm{id}_{E_0}, -\mathrm{id}_{E_1^\vee} \otimes M)$. The cokernel of B is $\mathrm{Ext}_R^1(I,I)$, the middle homology is $\mathrm{Hom}_R(I,I) = R$, and A is injective. Hence in the representation ring we get the formula

$\mathrm{Ext}_R^1(I,I) =$

$$R + \sum_{\substack{1\le i\le r\\ 0\le j\le r}} R[\underline{n}_i-\underline{d}_j] - \sum_{1\le i,j\le r} R[\underline{n}_i-\underline{n}_j] - \sum_{0\le i,j\le r} R[\underline{d}_i-\underline{d}_j] + \sum_{\substack{1\le i\le r\\ 0\le j\le r}} R[\underline{d}_j-\underline{n}_i].$$

For $1\le i\le j\le r$ define $K_{ij} = R[\underline{n}_j-\underline{d}_{i-1}]-R[\underline{n}_i-\underline{n}_j]-R[\underline{d}_j-\underline{d}_{i-1}]+R[\underline{d}_j-\underline{n}_i]$ and $L_{ij} = R[\underline{n}_i-\underline{d}_j]-R[\underline{n}_i-\underline{n}_j]-R[\underline{d}_{i-1}-\underline{d}_j]+R[\underline{d}_{i-1}-\underline{n}_j]$. Then, regrouping the terms in the formula above, it is easily verified that $\mathrm{Ext}_R^1(I,I) = \sum_{1\le i\le j\le r} K_{ij}+L_{ij}$. Now using that $\underline{d}_j = (j,b_j)$ and that $\underline{n}_i = (i,b_{i-1})$ we get

$$K_{ij} = \sum_{\substack{p\ge i-j-1\\ q\ge b_{i-1}-b_{j-1}}} \lambda^p\mu^p - \sum_{\substack{p\ge i-j\\ q\ge b_{i-1}-b_{j-1}}} \lambda^p\mu^q - \sum_{\substack{p\ge i-j-1\\ q\ge b_{i-1}-b_j}} \lambda^p\mu^q + \sum_{\substack{p\ge i-j\\ q\ge b_{i-1}-b_j}} \lambda^p\mu^q$$

$$= \sum_{q\ge b_{i-1}-b_{j-1}} \lambda^{i-j-1}\mu^q - \sum_{q\ge b_{i-1}-b_j} \lambda^{i-j-1}\mu^q$$

$$= \sum_{s=b_j}^{b_{j-1}-1} \lambda^{i-j-1}\mu^{b_{i-1}-s-1}.$$

In a similar way one checks that $L_{ij} = \sum_{s=b_j}^{b_{j-1}-1} \lambda^{j-1}\mu^{s-b_{i-1}}$. □

§4.

We now proceed to compute the Betti numbers of $W(0,0,d)$, $W(0,d,0)$ and $W(d,0,0)$. We start with $W(0,0,d)$.

As all the subschemes of $\mathbb{P}^2$ corresponding to points in $W(0,0,d)$ are contained in the affine plane $\operatorname{Spec} \mathbb{C}[\frac{T_0}{T_2},\frac{T_1}{T_2}]$ we put $x = \frac{T_0}{T_2}$ and $y = \frac{T_1}{T_2}$. In the computation in §3 we may take $T = G$; then $\lambda = \lambda_0\lambda_2^{-1}$ and $\mu = \lambda_1\lambda_2^{-1}$.

Choose a one-parameter subgroup $\phi:\mathbb{G}_m \to G$ given by $\phi(t) = \operatorname{diag}(t^{w_0},t^{w_1},t^{w_2})$ where $w_0<w_1<w_2$ and $w_0+w_1+w_2 = 0$. Then $\lambda\circ\phi(t) = t^{w_0-w_2}$ and $\mu\circ\phi(t) = t^{w_1-w_2}$. More generally, for any character $\lambda^\alpha\mu^\beta$ of G we have $\lambda^\alpha\mu^\beta\circ\phi(t) = t^{\alpha(w_0-w_2)+\beta(w_1-w_2)}$.

Pick a cell U from the cellullar decomposition of $\operatorname{Hilb}^d(\mathbb{P}^2)$ defined by ϕ, contained in $W(0,0,d)$. We want to compute its dimension. The cell U corresponds to a fixpoint of G in $\operatorname{Hilb}^d(\mathbb{P}^2)$, contained in $\operatorname{Spec} \mathbb{C}[\frac{T_1}{T_2},\frac{T_1}{T_2}] = \operatorname{Spec} \mathbb{C}[x,y]$, hence to an invariant ideal I in $\mathbb{C}[x,y]$. According to (1.2), $\dim U = \dim T^+$ where T is the tangent space of $\operatorname{Hilb}^d(\mathbb{P}^2)$ at the fixpoint. There is a canonical G-equivariant identification $T=\operatorname{Hom}_R(I,R/I)$ where $R = \mathbb{C}[x,y]$ (see [Gr]). We may assume that $\frac{w_2-w_0}{w_1-w_1}>>0$. Then any one dimensional representation $\lambda^\alpha\mu^\beta$ occurring in $\operatorname{Hom}_R(I,R/I)$ has a positive weight with respect to ϕ if and only if $\alpha<0$, or $\alpha = 0$ and $\beta<0$. It follows from (3.2) that

$$T^+ = \sum_{1\leq i<j\leq r} \sum_{s=b_j}^{b_{j-1}-1} \lambda^{i-j-1}\mu^{b_{i-1}-s-1} + \sum_{j=1}^{r}\sum_{s=b_j}^{b_{j-1}-1} \mu^{s-b_{j-1}}.$$

The number of summands in the first sum is $\sum_{i=1}^{r}\sum_{j=i}^{r}(b_{j-1}-b_j) = \sum_{i=1}^{r} b_{i-1} = d$ and in the second sum there are $\sum_{j=1}^{r}(b_{j-1}-b_j) = b_0$ summands. Therefore $\dim U = \dim T^+ = d+b_0$.

In order to compute one of the Betti numbers of $W(0,0,d)$, say $b_{2k}(W(0,0,d))$, we have to count the number of cells of dimension k. Since there is a one-one correspondence between invariant ideals of $\mathbb{C}[x,y]$ of colength d and partitions $b_0 \geq b_1 \geq \ldots \geq b_r = 0$ of d, $b_{2k}(W(0,0,d))$ is the number of partitions of $2d-k$ in parts bounded by $k-d$. We have proved

(4.1) Proposition. $b_{2k}(W(0,0,d)) = P(2d-k,k-d)$.

Remark. This concludes the proof of theorem (1.1) part (iii) since $W(0,0,d) \cong \mathrm{Hilb}^d(\mathbb{A}^2)$.

Next we turn to $W(d,0,0)$. Subschemes of $\mathbb{P}^2$ corresponding to points in $W(d,0,0)$ are supported in P_0. In particular they are contained in Spec $\mathbb{C}[\frac{T_1}{T_0},\frac{T_2}{T_0}]$. Put $x = \frac{T_1}{T_0}$ and $y = \frac{T_2}{T_0}$. In the computation in §2 we may take $T = G$, $\lambda = \lambda_1\lambda_0^{-1}$, and $\mu = \lambda_2\lambda_0^{-1}$.

Choosing a one-parameter subgroup ϕ with $w_0 < w_1 < w_2$ and $\frac{w_1-w_0}{w_2-w_0} \gg 0$, and reasoning as above, we get

$$T^+ = \sum_{1 \leq i < j \leq r} \sum_{s=b_j}^{b_{j-1}-1} \lambda^{j-i}\mu^{s-b_{i-1}}$$

where T is the tangent space to $\mathrm{Hilb}^d(\mathbb{P}^2)$ at the fixpoint corresponding to the partition $b_0 \geq b_1 \geq \ldots \geq b_r = 0$ of d. Hence the dimension of the corresponding cell is $\sum_{i=1}^{r} \sum_{j=i+1}^{r} (b_{j-1}-b_j) = \sum_{i=1}^{r} b_i = d-b_0$. This gives

(4.2) Proposition. $b_{2k}(W(d,0,0)) = P(k,d-k)$.

Remark. This proves theorem (1.1) part (ii) since $W(d,0,0) \simeq \mathrm{Hilb}^d(\mathbb{A}^2,0)$.

The last case to treat is $W(0,d,0)$. This time we put $x = \frac{T_0}{T_1}$, $y = \frac{T_2}{T_1}$, $\lambda = \lambda_0\lambda_1^{-1}$, and $\mu = \lambda_2\lambda_1^{-1}$.

As usual, let ϕ be a one-parameter subgroup of G with $w_0 < w_1 < w_2$. Let $\lambda^\alpha\mu^\beta$ be a one-dimensional representation of G with $\alpha\beta < 0$. Since $w_0 - w_1 < 0$ and $w_2 - w_1 > 0$ the weight of $\lambda^\alpha\mu^\beta$ with respect to ϕ is positive if and only if $\alpha < 0$ and $\beta > 0$. Using this and (3.2) it is easily verified that

$$T^+ = \sum_{1 \le i \le j \le r} \sum_{s=b_j}^{b_{j-1}-1} \lambda^{i-j-1}\mu^{b_{i-1}-s-1}$$

where T is the tangent space of $\mathrm{Hilb}^d(\mathbb{P}^2)$ at the fixpoint corresponding to the partition $b_0 > b_1 > \ldots > b_r = 0$ of d. Hence all the cells in $W(0,d,0)$ are of dimension d, and we get

(4.3) Proposition. $b_{2k}(W(0,d,0)) = \begin{cases} 0 & \text{if } k \neq d \\ P(d) & \text{if } k = d \end{cases}$.

Substituting the expressions of (4.1), (4.2) and (4.3) in the formula in lemma (2.2) we get theorem (1.1) part (ii). This concludes the proof of (1.1).

§5.

Denote by $\mathbb{F}_n$ the rational, ruled surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(-n))$. A maximal torus T of the automorphism group of $\mathbb{F}_n$ is of dimension two and has four fixpoints on $\mathbb{F}_n$. It is easily checked that for an appropriate class of one-parameter subgroups of T, the weights on the tangent space of $\mathbb{F}_n$ at two of these fixpoints are of opposite sign, and at the two remaining fixpoints, the two weights are respectively positive and negative. Thus the corresponding cellullar decomposition of $\mathbb{F}_n$ contains a point, two copies of $\mathbb{A}^1$, and an $\mathbb{A}^2$. Adapting the proof of (1.1) to this situation we get

(5.1) Theorem. The cycle map $cl: A_*(\mathrm{Hilb}^d(\mathbb{F}_n)) \longrightarrow H_*(\mathrm{Hilb}^d(\mathbb{F}_n))$ is an isomorphism, and in particular the odd homology vanishes. The homology groups are free abelian groups. Furthermore,

$$b_{2k}(\mathrm{Hilb}^d(\mathbb{F}_n)) =$$

$$\sum_{d_0+d_1+d_2+d_3=d}\ \sum_{p+r=k-d_1-d_2} P(p,d_0-p)P(d_1)P(d_2)P(2d_3-r,r-d_3)$$

and

$$\chi(\mathrm{Hilb}^d(\mathbb{F}_n)) = \sum_{d_0+d_1+d_2+d_2=d} P(d_0)P(d_1)P(d_2)P(d_3).$$

References.

[B1] Bialynicki-Birula, A.: Some theorems on actions of algebraic groups. Annals of Mathematics, Vol. 98, No. 3 (1973) pp 480-497.

[B2] Bialynicki-Birula, A.: Some Properties of the Decompositions of Algebraic Varieties Determined by Actions of a Torus. Bulletin de l'Academie Polonaise des Sciences. Série des sciens math. astr. et phys. Vol 24, No. 9 (1976) p. 667-674.

[F1] Fogarty, J.: Algebraic families on an algebraic surface. Amer. J. Math. 10 (1968) 511-521.

[F2] Fogarty, J.: Algebraic families on an algebraic surface II: Picard scheme of the punctual Hilbert scheme. Amer. J. Math. 96 (1979) 660-687.

[Br] Briancon, J.: Description de $\mathrm{Hilb}^n\mathbb{C}\{x,y\}$. Inventiones math. 41 (1977) 45-89.

[Fu] Fulton, W.: Intersection theory. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag 1984.

[Ga] Gaffney, T.: Multiple points and associated ramification loci. In Singularities, Proceedings of symposia in pure mathematics of the AMS. Volume 40, part 1.

[Gr] Grothendieck, A.: Techniques de construction et théorèmes d'existence en gèometrie algébrique IV: Les schémas de Hilbert. Sem. Bourbaki 221 (1960/61).

[H] Hirschowitz, A.: Le group de Chow équivariant. C. R. Acad. Sc. Paris t 298 (1984) 87.

[I1] Iarrobino, A.: Punctual Hilbert Schemes. Memoirs of the AMS, No 188 (1977).

[I2] Iarrobino, A.: Deforming complete intersection Artin algebras. In Singularities, Proceedings of symposia in pure mathematics of the AMS. Volume 40, part 1.