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On the Homotopical Significance of the Type of von Neumann Algebra Factors

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Abstract. The set of all projections and the set of all unitaries in a von Neumann algebra factor $\mathscr{A}$ are studied from the homotopical point of view relative to the operator norm topology.

Two projections E and F can be deformed continuously to each other if and only if $E \sim F$ and $1-E \sim 1-F$ where $\sim$ denotes the equivalence of projections in $\mathscr{A}$ in the sense of von Neumann. In other words, the relative dimension and co-dimension are a complete homotopical invariants of projections in $\mathscr{A}$ and label pathwise connected components of the set of projections.

The first homotopy group $\pi_1(\mathscr{U}(\mathscr{A}))$ of unitaries in $\mathscr{A}$ is shown to be 0 for $\mathscr{A}$ of infinite type. For type II_1 and type I_n factors, $\pi_1(\mathscr{U}(\mathscr{A}))$ are isomorphic to additive groups of reals $\boldsymbol{R}$ and integers $\boldsymbol{Z}$, respectively, in which the first homotopy group $\pi_1(\mathscr{Z}\mathscr{U}(\mathscr{A}))$ of the center of $\mathscr{U}(\mathscr{A})$ is imbedded as $\boldsymbol{Z}$ and $n\boldsymbol{Z}$, respectively.

§ 0. Introduction

In [5, 6] Glimm's classification of U.H.F. algebras is reobtained by means of the first homotopy group $\pi_1(\mathscr{U}(\mathscr{A}))$ of the unitary group $\mathscr{U}(\mathscr{A})$ of a U.H.F. C^*-algebra $\mathscr{A}$ and the canonical homomorphism $\varphi: \pi_1(\mathscr{Z}\mathscr{U}(\mathscr{A})) \to \pi_1(\mathscr{U}(\mathscr{A}))$ where $\mathscr{Z}\mathscr{U}(\mathscr{A})$ denotes the center of $\mathscr{U}(\mathscr{A})$. The present note is motivated by a desire to investigate the analogous situation for a von Neumann algebra factor acting on a seperable Hilbert space.

As a preliminary step we study the projections $\boldsymbol{P}(\mathscr{A})$ of a von Neumann algebra $\mathscr{A}$. Two projections E and F are said to be equivalent [4] (denoted by $E \sim F$) if and only if there exists an operator V in $\mathscr{A}$ such that $V^*V = E$ and $VV^* = F$. (Such an operator V is called a partial isometry, it maps the range of E isometrically onto the range of F.) It is shown that for a factor $\mathscr{A}$ there exists a norm continuous one parameter family $\mathrm{E}(\lambda)$, $0 \leqq \lambda \leqq 1$, of projections with initial point $E = \mathrm{E}(0)$ and terminal point $F = \mathrm{E}(1)$ if and only if $E \sim F$ and $I - E \sim I - F$, where I is the identity operator in $\mathscr{A}$. This enables us to relate the path components of $\boldsymbol{P}(\mathscr{A})$ to analytic properties of projections.

We next begin our study of the first homotopy group, $\pi_1(\mathscr{U}(\mathscr{A}))$, of the unitary group, $\mathscr{U}(\mathscr{A})$, of the von Neumann algebra $\mathscr{A}$. The elements of $\pi_1(\mathscr{U}(\mathscr{A}))$ are certain equivalence classes of one parameter families $U(\lambda)$, $0 \leqq \lambda \leqq 1$, of unitary operators in $\mathscr{A}$, depending continously on λ relative to the operator norm topology of $\mathscr{A}$ and such that $U(0) = I = U(1)$. We call such a family a loop in $\mathscr{U}(\mathscr{A})$. A loop in $\mathscr{U}(\mathscr{A})$ is said to be *simple* if and only if $U(\lambda) = \exp 2\pi i \lambda S$ for a fixed self adjoint operator S in $\mathscr{A}$. We next show that in a factor of infinite type (I_∞, II_∞, III) a simple loop is homotopic to zero. Thus, since we show that the simple loops generate $\pi_1(\mathscr{U}(\mathscr{A}))$ for all $\mathscr{A}$, we conclude that $\pi_1(\mathscr{U}(\mathscr{A})) = 0$ for $\mathscr{A}$ a factor of infinite type. For a factor of finite type a sum of simple loops can be deformed (that is, is homotopic) to a single simple loop $\exp 2\pi i \lambda S$, $0 \leqq \lambda \leqq 1$. A complete homotopy invariant of such a loop is given by $\varphi(S)$ where φ is the trace on $\mathscr{A}$. In particular, $\pi_1(\mathscr{U}(\mathscr{A})) \cong \boldsymbol{R}$, $\varphi(\pi_1(\mathscr{Z}\mathscr{U}(\mathscr{A})) \cong \boldsymbol{Z} \subset \boldsymbol{R}$ for type II_1 factors and $\pi_1(\mathscr{U}(\mathscr{A})) \cong \boldsymbol{Z}$, $\varphi(\pi_1(\mathscr{Z}\mathscr{U}(\mathscr{A})) \cong n\boldsymbol{Z} \subset \boldsymbol{L}$ for type I_n factors, this latter result being well known.

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§ 1. Continuous Deformations of Projections

Let $\mathscr{H}$ be a separable Hilbert space $\mathscr{L}(\mathscr{H})$ the set of all bounded linear operators on $\mathscr{H}$, $\mathscr{A} \subset \mathscr{L}(\mathscr{H})$ a von Neumann algebra, $\boldsymbol{P}(\mathscr{A}) \subset \mathscr{A}$ the set of all (orthogonal) projections in $\mathscr{A}$ and $\mathscr{U}(\mathscr{A})$ the set of all unitary elements in $\mathscr{A}$.

For any $T \in \mathscr{L}(\mathscr{H})$ we define $\ker T = \{x \in \mathscr{H};\ Tx = 0\}$ and $\operatorname{coker} T = \ker T^*$. For a closed subspace $\mathscr{K} \subset \mathscr{H}$, $E_{\mathscr{K}}$ denotes the orthogonal projection onto $\mathscr{K}$. The orthogonal complement of $\mathscr{K}$ in H will be denoted by $\mathscr{K}^\perp$.

We recall the polar decomposition theorem in the following:

Polar Decomposition Theorem. Let $T \in \mathscr{L}(\mathscr{H})$. The polar decomposition of T is $W|T| = T$, where W is a partial isometry such that $W^*W = E_{(\ker T)^\perp}$, $WW^* = E_{(\operatorname{coker} T)^\perp}$ and $|T| = (T^*T)^{1/2}$. If $T \in \mathscr{A}$ then $W, |T| \in \mathscr{A}$ also.

Lemma 1.1. *If E, F are orthogonal projections and $\|E - F\| < 1$ then* $\ker EF = (I - F)\mathscr{H}$ *and* $\operatorname{coker} EF = (I - E)\mathscr{H}$.

Proof. It is clear that

$$\ker EF \supset (I - F)\mathscr{H},$$

$$\operatorname{coker} EF \supset (I - E)\mathscr{H}.$$

Suppose that $EFx = 0$ but $y = Fx \neq 0$. Then $\|(E - F)y\| = \|-y\| = \|y\|$. Hence $\|E - F\| = 1$, contrary to hypothesis. Therefore $EFx = 0 \Rightarrow Fx = 0$ so that $x \in (I - F)\mathscr{H}$. Similarly $FEx = 0$ implies $Ex = 0$, thus $\operatorname{coker} EF \subset (I - E)\mathscr{H}$. Q.E.D.

Lemma 1.2. *Let E and F be projections in $\mathscr{A}$, $\|E - F\| < 1$. Then $E \sim F$ and $I - E \sim I - F$.*

Proof. Applying the polar decempositіon theorem to EF we obtain a partial isometry $W \in \mathscr{A}$ and the operator $|EF| \in \mathscr{A}$ such that $EF = W|EF|$ where in view of (1.1) $W^*W = F$, $WW^* = E$. Therefore $E \sim F$. Since

$$\|(I - E) - (I - F)\| = \|E - F\| < 1$$

the same argument shows that $I - E \sim I - F$. Q.E.D.

Proposition 1.3. *Let $E, F \in \boldsymbol{P}(\mathscr{A})$. Suppose that E and F can be connected by a norm continuous path in $\boldsymbol{P}(\mathscr{A})$. Then $E \sim F$ and $I - E \sim I - F$.*

Proof. Let $\mathrm{P}(t): 0 \leqq t \leqq 1$, be a norm continuous path in $\boldsymbol{P}(\mathscr{A})$ connecting $E = \mathrm{P}(0)$ to $F = \mathrm{P}(1)$. Using the compactness of the unit interval $J = \{0 \leqq t \leqq 1\}$ we may find numbers

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

such that

$$\|\mathrm{P}(t_{i+1}) - \mathrm{P}(t_i)\| < 1: \qquad i = 0, \cdots, n-1.$$

Applying (1. 2) we find

$$E = \mathrm{P}(0) \sim \mathrm{P}(t_1) \sim \cdots \sim \mathrm{P}(t_n) = F$$

and

$$I - E = I - \mathrm{P}(0) \sim \cdots \sim I - \mathrm{P}(t_n) = \mathrm{I} - \mathrm{F}$$

from which the result follows by transitivity of the relation $\sim$. Q.E.D.

Proposition 1.4. *Let E and F be two projections in $\mathscr{A}$ with $E \sim F$ and $I-E \sim I-F$. Then E and F may be connected by a norm continuous path lying in $\boldsymbol{P}(\mathscr{A})$.*

Proof. Let U be a partial isometry from E to F and V a partial isometry from $I-E$ to $I-F$, $U, V \in \mathscr{A}$.

Thus

$$E = U^*U, \qquad F = UU^*,$$

$$I-E = V^*V, \qquad I-F = VV^*.$$

Let $W = U + V$. Then $W \in \mathscr{A}$, and W is actually a unitary operator in $\mathscr{A}$, since W is an isometry from $E\mathscr{H}$ to $F\mathscr{H}$ and $(E\mathscr{H})^{\perp}$ to $(F\mathscr{H})^{\perp}$.
Note that by construction

$$W|_{E\mathscr{H}} = U|_{E\mathscr{H}}, \qquad W|_{(E\mathscr{H})^{\perp}} = V|_{(F\mathscr{H})^{\perp}}.$$

Hence $U = WE$. Now note

$$WEW^* = UW^* = U(U^* + V^*) = UU^* + UV^* = F + UV^*.$$

Next note that $V^*x \in (E\mathscr{H})^{\perp} = (I-E)\mathscr{H}$ for any $x \in \mathscr{H}$. Since $I-E = \ker U$, we have that $UV^* = 0$. Thus

$$WEW^* = F.$$

By the spectral theorem there exists a self adjoint operator $T \in \mathscr{A}$ such that $W = e^{iT}$ with $-\pi I < T \leqq \pi I$. Let

$$\mathrm{P}(t) = e^{itT} E e^{-itT} : 0 \leqq t \leqq 1.$$

Since $T \in \mathscr{A}$, $e^{itT} \in \mathscr{A}$ and $e^{-itT} \in \mathscr{A}$ for all $0 \leqq t \leqq 1$. Therefore $\mathrm{P}(t) \in \mathscr{A}$. In fact $\mathrm{P}(t)$ is a projection for each t and hence $\mathrm{P}(t) \in \mathscr{P}(\mathscr{A})$. Clearly $\mathrm{P}(t)$ is a norm continuous function of t, and since $\mathrm{P}(0) = E$, $\mathrm{P}(1) = F$, constitutes a norm continuous path in $\mathscr{P}(\mathscr{A})$ from E to F. Q.E.D.

We may summarize the results of this section in the following:

Theorem 1.5. *Let E and F be two projections in $\mathscr{A}$. Then E may be connected to F by a norm continuous path of projections in $\mathscr{A}$ if and only if $E \sim F$ and $I-E \sim I-F$.*

§ 2. Reduction of General Loops to Simple Loops

The aim of this section is to provide a proof of the following theorem:
In the unitary group of a von Neumann algebra, any loop is homotopic to a sum of simple loops.

The proof will be accomplished with the aid of a technical lemma whose statement and proof are deferred to the appendix. Reference to this lemma is made at a key point in the argument.

We shall require several preliminary steps. The first Lemma is well-known.

Lemma 2.1. *Let* $f_t(z)$ *be a continuous function of* (t, z), $t \in [0, 1]$, $z \in \boldsymbol{C}$ *and* $\mathcal{N}$ *be the set of all bounded normal linear operators with the norm topology. Then the mapping from* $(t, Q) \in [0, 1] \times \mathcal{N}$ *to* $f_t(Q) \in \mathcal{N}$ *is continuous.*

Proof. Let K be a compact set in $\boldsymbol{C}$ and $\mathcal{N}(K)$ be the set of $Q \in \mathcal{N}$ with its spectrum in K. Let $\varepsilon > 0$ be given. Let $\delta > 0$ be such that

$$|f_{t'}(z) - f_{t''}(z)| < \varepsilon/4$$

for all $z \in K$ and t', $t'' \in [0, 1]$ satisfying $|t' - t''| < \delta$. Let $P_\varepsilon(z, t)$ be a polynomial of t, z and $\bar{z}$ such that

$$|P_\varepsilon(z, t) - f_t(z)| < \varepsilon/4$$

for all $t \in [0, 1]$ and $z \in K$. Such a P_ε exists by the Weierstrass approximation theorem.

Let $\bar{\delta} > 0$ be such that

$$\|Q' - Q''\| < \bar{\delta}, \quad Q', Q'' \in N(K), t \in [0, 1]$$

implies

$$\| P_\varepsilon(Q', t) - P_\varepsilon(Q'', t)\| \leqq \varepsilon/4 \, .$$

Such a $\bar{\delta}$ is seen to exist from the following type of estimates:

$$\begin{aligned} \|Q'^n - Q''^n\| &= \left\| \sum_{k=1}^{n} Q'^{n-k}(Q' - Q'') \, Q''^{k-1} \right\| \\ &\leqq \sum_{k=1}^{n} \|Q'\|^{n-k} \, \|Q' - Q''\| \, \|Q''\|^{k-1} \\ &\leqq n L^{n-1} \|Q' - Q''\| \end{aligned}$$

where L is a bound for $|z|$, $z \in K$.

We now have

$$\begin{aligned} &\|f_{t'}(Q') - f_{t''}(Q'')\| \\ &\quad \leqq \|f_{t'}(Q') - P_\varepsilon(Q', t')\| + \|P_\varepsilon(Q', t') - P_\varepsilon(Q'', t')\| \\ &\quad + \|P_\varepsilon(Q'', t') - f_{t'}(Q'')\| + \|f_{t'}(Q'') - f_{t''}(Q'')\| \\ &\quad < \varepsilon \end{aligned}$$

whenever t', $t'' \in [0, 1]$, $|t' - t''| < \delta$, $Q' Q'' \in \mathcal{N}(K)$ and $\|Q' - Q''\| < \bar{\delta}$.

Q.E.D.

Lemma 2.2. *A loop* $U(\lambda)$: $0 \leqq \lambda \leqq 1$ *in the unitary group* $\mathcal{U}(\mathcal{A})$ *of a von Neumann algebra* $\mathcal{A}$ *is null homotopic in* $\mathcal{U}(\mathcal{A})$ *if* $\|U(\lambda) - I\| < 2$ *for all* λ.

Proof. Since $\mathrm{U}(\lambda)$ is norm continuous,

$$\sup_{\lambda\in[0,1]} \|\mathrm{U}(\lambda)-1\|<2.$$

Hence there exists a, $0<a<\pi$, such that the spectrum of $\mathrm{U}(\lambda)$ lies in the set $\{\exp i\theta:\ -a\leqq\theta\leqq a\}$ for $0\leqq\lambda\leqq 1$.

Let $f_t(z)$ be a continuous function of (t,z), $0\leqq t\leqq 1$, $z\in \boldsymbol{C}$ such that $f_t(\exp i\theta)=\exp it\theta$ for $-a\leqq\theta\leqq a$, $0\leqq t\leqq 1$. Then $f_t(\mathrm{U}(\lambda))$ is unitary in $\mathscr{A}$, norm continuous in (t,λ) with $f_1(\mathrm{U}(\lambda))=\mathrm{U}(\lambda)$ and $f_0(\mathrm{U}(\lambda))=I$, where the continuity is due to Lemma 2.1. Q.E.D.

Given unitary operators U_1, U_2, satisfying $\|U_1-U_2\|<2$ we reserve the notation $\mathrm{L}(U_1,U_2)$ for the path connecting U_1 and U_2 in the explicit manner now to be explained. Since $\|U_1-U_2\|<2$ we have a unique self adjoint operator Q in $\{U_1^* U_2\}''$ satisfying the following conditions:

$$\|Q\|<\pi,$$
$$U_1^* U_2=\exp iQ.$$

The path $\mathrm{L}(U_1,U_2)$ is defined by

$$\mathrm{L}(U_1,U_2)(\lambda)=U_1\exp i\lambda Q:0\leqq\lambda\leqq 1.$$

Note that the distance between any two points on $\mathrm{L}(U_1,U_2)$ is bounded by $\|U_1-U_2\|$. For

$$\begin{aligned}&\|\mathrm{L}(U_1,U_2)(\lambda')-\mathrm{L}(U_1,U_2)(\lambda'')\|\\ &\quad=\|I-\exp i(\lambda'-\lambda'')Q\|\leqq|1-\exp i\|Q\||=\|U_1-U_2\|.\end{aligned}$$

Notations and Conventions. We fix throughout the remainder of this section a von Neumann algebra $\mathscr{A}$ acting on a Hilbert space $\mathscr{H}$. All loops and paths that we consider lie in the unitary group $\mathscr{U}(\mathscr{A})$ of $\mathscr{A}$. All operators lie in $\mathscr{A}$. If a lemma asserts the existence of a loop path or operator it is understood that the operator lies in $\mathscr{A}$ and the loop or path in $\mathscr{U}(\mathscr{A})$. If this is not explicitly proved then it is an easy verification left to the reader.

Lemma 2.3. *Any loop is homotopic to a sum of triangular loops* Δ_j *with three sides consisting of*:

$$L_j=\{\exp i\lambda Q_j;\ 0\leqq\lambda\leqq 1\},$$
$$L_{j,j+1}=L(\exp iQ_j,\ \exp iQ_{j+1}),$$
$$\tilde{L}_{j+1}=\{\exp i(1-\lambda)Q_{j+1};\ 0\leqq\lambda\leqq 1\}$$

where $Q_0,\ldots,Q_n$ *are self-adjoint operators satisfying*

$$\|Q_j\|\leqq\pi,\qquad i=0,\ldots,n,$$
$$\|\exp iQ_j-\exp iQ_{j+1}\|<\delta,\qquad i=0,\ldots,n-1$$

and δ *is a fixed number* $0<\delta<2$.

Proof. Any loop can be divided into several arcs $\{U(\lambda_j), U(\lambda_{j+1})\}$ $0=\lambda_0<\lambda_1\cdots<\lambda_n=1$ such that

$$\|U(\lambda_j)-U(\lambda)\|<\delta:\lambda_j\leqq\lambda\leqq\lambda_{j+1},$$

$j=0,\ldots,n-1$. Let Q_j be defined by

$$U(\lambda_j)=\exp iQ_j$$

with the spectrum of Q_j contained in $[-\pi,\pi]$. Note $Q_j\in\{U(\lambda_j)\}''$ and $\|Q_j\|\leqq\pi$.

Using (2.1) we see that the loop consisting of the two sides $\{U(\lambda_j)^*U(\lambda);\ \lambda_j\leqq\lambda\leqq\lambda_{j+1}\}$ and $U(\lambda_j)^*L_{j,j+1}$ is homotopic to 0. Therefore the path consisting of $\{U(\lambda);\ \lambda_j\leqq\lambda\leqq\lambda_{j+1}\}$ is homotopic, with end points fixed, to the path $L_{j,j+1}$ (Fig. 1).

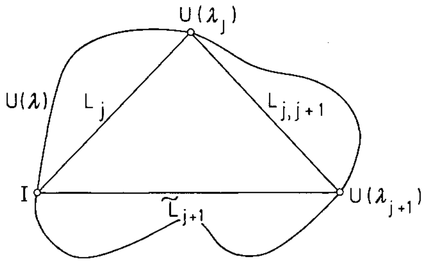

Fig. 1

Next note that the closed path consisting of the two arcs L_k, $\tilde{L}_k$ is null homotopic. Thus we see that the loop $\{U(\lambda):0\leqq\lambda\leqq N\}$ is homotopic to the sum of triangular loops $\Delta_j, j=0,\ldots,n-1$. Q.E.D.

Lemma 2.4. *Suppose that Q_1 and Q_2 are self adjoint operators in $\mathscr{A}$ such that $\|Q_1\|$, $\|Q_2\|\leqq\pi$ and $\|Q_1-Q_2\|<2e^{-\pi}$. Then the triangular loop with three sides*

$$L_1=\{\exp i\lambda Q_1;\ 0\leqq\lambda\leqq1\},$$
$$L_{1,2}=L(\exp iQ_1,\ \exp iQ_2),$$
$$L_2=\{\exp i(1-\lambda)Q_2;\ 0\leqq\lambda\leqq1\}$$

is homotopic to 0.

Proof. Let $Q(\mu)=\mu Q_1+(1-\mu)Q_2:0\leqq\mu\leqq1$. We have $\|Q(\mu')-Q(\mu'')\|=|\mu'-\mu''|\,\|Q_1-Q_2\|<2e^{-\pi}$ for $\mu',\mu''\in[0,1]$. Hence

$$\begin{aligned}&\|\exp iQ(\mu')-\exp iQ(\mu'')\|\\&\leqq\|Q(\mu')-Q(\mu'')\|\exp\max\{\|Q(\mu')\|,\ \|Q(\mu'')\|\}\\&<2.\end{aligned}$$

Thus by (2.2) the loop consisting of the two sides $(\exp iQ_1)^* L_{1,2}$ and $\{(\exp iQ_1)^* \exp iQ(\mu) \mid 0 \leqq \mu \leqq 1\}$ is null homotopic. Let $\Delta(\mu)$ be the triangular loop with sides

$$\{\exp i\lambda Q(\mu);\ 0 \leqq \lambda \leqq 1\},$$
$$\{\exp iQ(\mu');\ \mu \leqq \mu' \leqq 1\},$$
$$\{\exp i(1-\lambda) Q_1;\ 0 \leqq \lambda \leqq 1\}.$$

Then the preceeding discussion shows that the triangular loop $\{L_1, L_{1,2}, \tilde{L}_2\}$ is homotopic to $\Delta(1)$ (Fig. 2).

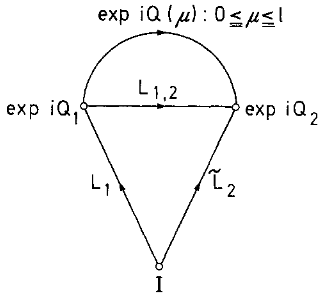

Fig. 2

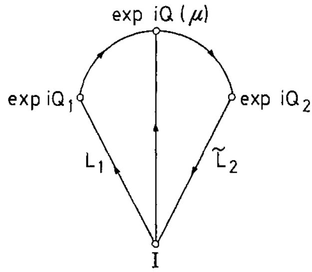

Fig. 3

The triangular loops $\Delta(\mu)$, $0 \leqq \mu \leqq 1$, provide a continuous deformation of $\Delta(1)$ to $\Delta(0)$ (Fig. 3). Since $\Delta(0)$ is clearly null homotopic it follows that the triangular loop $\{L_1, L_{1,2}, \tilde{L}_2\}$ is also null homotopic. Q.E.D.

Lemma 2.5. *Let U_1, U_2 be unitary operators in $\mathscr{A}$. Let Δ_1, Δ_2 be compact connected arcs on the unit circle with mutual distance $r > 0$. Let the length of the arc Δ_1 be $a > 0$, and let ε be a given positive number. Then there exists $\delta(\varepsilon, r, a)$, depending only on $\varepsilon > 0$, $r > 0$, $a > 0$, such that whenever E_1 and E_2 are spectral projections of U_1 and U_2 for Δ_1 and Δ_2 respectively, $\|E_1 \cdot E_2\| < \varepsilon$ whenever $\|U_1 - U_2\| < \delta(\varepsilon, r, a)$.*

Proof. Let $f(z)$, $z \in \boldsymbol{C}$, be a continuous function which is equal to 1 on a fixed Δ_1^0 of length a and 0 at any point on the unit circle S^1 with distance from Δ_1^0 larger than r. Since $f(U)$ is norm continuous in $U \in \mathscr{U}(\mathscr{A})$ by (2.1) (set $f_t(z) = f(z)$ in (2.1)), there exists $\delta(\varepsilon, r, \Delta_1^0) > 0$ such that

$$\|f(U') - f(U'')\| < \varepsilon$$

whenever

$$\|U' - U''\| < \delta(\varepsilon, r, \Delta_1^0),\ U',\ U'' \in \mathscr{U}(\mathscr{A}).$$

Since

$$f(U_1) E_1 = E_1, \qquad f(U_2) E_2 = 0$$

we have

$$\|E_1 E_2\| = \|E_1(\mathrm{f}(U_1) - \mathrm{f}(U_2))E_2\| < \varepsilon$$

for $\|U_1 - U_2\| < \delta(\varepsilon, r, \Delta_1^0)$. For any other arc Δ_1 of length a there exists a real number θ such that $\Delta_1 = e^{i\theta}\Delta_1^0$ and if we use the function $\mathrm{f}_\theta(z) = \mathrm{f}(e^{-i\theta}z)$ instead of $\mathrm{f}(z)$ the preceeding computations are still valid.

Q.E.D.

Lemma 2.6. *Let Q_1 and Q_2 be self adjoint operators in $\mathscr{A}$. Suppose that*

$$Q_1 = \sum_{n=-N}^{n=N} n(\pi/N)E_n\,,$$

$$Q_2 = \sum_{n=-N}^{n=N} (n+\tfrac{1}{2})(\pi/N)F_n$$

where E_n and F_n are spectral projections of Q_1 and Q_2 respectively, and N is a natural number. If

$$\|F_n(I - E_n - E_{n+1})\| < \varepsilon = (2N)^{-2}$$

and

$$F_N = 0$$

then

$$\|Q_1 - Q_2\| < 2\pi/N\,.$$

Proof. We have

$$\begin{aligned} Q_1 - Q_2 &= \sum_{n,m} F_n(Q_1 - Q_2)E_m \\ &= \sum_{n,m} (F_n E_m Q_1 - Q_2 F_n E_m)\,. \end{aligned}$$

For $m = n$ or $n+1$ we see that

$$\begin{aligned} F_n(Q_1 - Q_2)E_m &= F_n E_m(\pi/N)(m - n - 1/2) \\ &= \pm(2N)^{-1}\pi F_n E_m\,. \end{aligned}$$

For the rest, from the hypotheses we have

$$\Bigg\|F_n \sum_{\substack{m\neq n\\ m\neq n+1}} E_m\Bigg\| < \varepsilon\,.$$

Hence

$$\|Q_1 - Q_2\| < \varepsilon(\|Q_1\| + \|Q_2\|)\sum_n 1 + (2N)^{-1}\pi\left(\left\|\sum_n F_n E_n\right\| + \left\|\sum_n F_n E_{n+1}\right\|\right).$$

Since

$$\left\|\sum_n F_n E_n \psi\right\|^2 = \sum_n \|F_n E_n \psi\|^2 \leqq \sum_n \|E_n \psi\|^2 = \|\psi\|^2$$

for $\psi \in \mathscr{H}$, we see that

$$\left\|\sum_n F_n E_n\right\| \leqq 1 .$$

Similarly

$$\|\Sigma F_n E_{n+1}\| \leqq 1 .$$

Hence

$$\|Q_1 - Q_2\| < 4N\pi\varepsilon + (\pi/N) = (2\pi/N)$$

where we have used the estimates $\|Q_1\| \leqq \pi$, $\|Q_2\| \leqq \pi$. Q.E.D.

Lemma 2.7. *There exists $\delta > 0$ with the following property: Whenever Q_1 and Q_2 are self-adjoint elements in $\mathscr{A}$ satisfying*

$$-\pi I < Q_j \leqq \pi I , \quad j \leqq 1, 2 ,$$

$$\|\exp iQ_1 - \exp iQ_2\| < \delta ,$$

then the triangular loop with sides

$$L_1 = \{\exp i\lambda Q_1 \colon\ 0 \leqq \lambda \leqq 1\} ,$$

$$L_{1,2} = \mathrm{L}(\exp iQ_1,\ \exp iQ_2) ,$$

$$\tilde{L}_2 = \{\exp i(1-\lambda) Q_2 \colon\ 0 \leqq \lambda \leqq 1\}$$

is homotopic to a sum of simple loops.

Proof. Let $U_j = \exp iQ_j, j = 1, 2$. Let E_n be the spectral projection for Q_1 on the half open interval $((n-(1/2))\pi/N,\ (n+(1/2))\pi/N]$, $n = -N, -N+1, \ldots, N$. Similarly, let F_n be the spectral projection of Q_2 for the half open interval $(n\pi/N, (n+1)\pi/N]$, $n = -N, -N+1, \ldots, N-1$, where N is an integer chosen so that $N > \pi e^{\pi}$.

By (2.5) there exists $\delta(\varepsilon, \pi/(2N), \pi/N)$ such that if $\|Q_1 - Q_2\| < \delta(\varepsilon, \pi/(2N), \pi/N)$ then

$$\|F_n(I - E_n - E_{n+1})\| < \varepsilon \quad \text{for} \quad n = -N+1, \ldots, N-2 ,$$

$$\|F_{-N}(I - E_{-N} - E_{-N+1} - E_N)\| < \varepsilon ,$$

$$\|F_{N-1}(I - E_{N-1} - E_N - E_{-N})\| < \varepsilon ,$$

and

$$\|(E_N + E_{-N})(I - F_{-N} - F_{N-1})\| < \varepsilon .$$

Since $\|F_\alpha E_\beta\| < \varepsilon$ implies $\|F_\alpha E'\| = \|F_\alpha E_\beta E'\| < \varepsilon$ for any subprojection E' of E_β, the assumptions of the appendix are satisfied with $E_A = F_{-N}$, $E_B = F_{N-1}$, $E_C = I - F_{-N} - F_{N-1}$, $E_0 = E_N + E_{-N}$, $E_\alpha = E_{-N+1}$, $E_\beta = E_{N-1}$, $E_\gamma = I - E_0 - E_\alpha - E_\beta$. Therefore there exists projections E_{01}, E_{02} with

$E_{01} \perp E_{02}$ and

$$E_{01} + E_{02} = E_0 = E_N + E_{-N},$$
$$\|E_A E_{02}\| = \|F_{-N} E_{02}\| < \varepsilon'(\varepsilon),$$
$$\|E_B E_{01}\| = \|F_{N-1} E_{01}\| < \varepsilon''(\varepsilon),$$

where

$$\lim_{\varepsilon \to 0} \varepsilon'(\varepsilon) = 0 = \lim_{\varepsilon \to 0} \varepsilon''(\varepsilon).$$

We define

$$Q_1' = \sum_{n=-N}^{N} (n\pi/N) E_n,$$
$$Q_2' = \sum_{n=-N}^{N-1} (n + (1/2))(\pi/N) F_n,$$

and

$$Q_1'' = Q_1' - 2\pi E_N + 2\pi E_{02}$$
$$= \sum_{n=-N+1}^{N-1} (n\pi/N) E_n - \pi E_{01} + \pi E_{02}.$$

Obviously

$$\|Q_1' - Q_1\| \leqq \pi/(2N)$$

and

$$\|Q_2' - Q_2\| \leqq \pi/(2N).$$

Also

$$\|F_{-N}(I - E_{01} - E_{-N+1})\|$$
$$\leqq \|F_{-N}(I - E_0 - E_{-N+1})\| + \|F_{-N} E_{02}\| < \varepsilon + \varepsilon'(\varepsilon)$$

and

$$\|F_{N-1}(I - E_{N-1} - E_{02})\|$$
$$\leqq \|F_{N-1}(I - E_{N-1} - E_0)\| + \|F_{N-1} E_{01}\| < \varepsilon + \varepsilon''(\varepsilon).$$

Replacing E_N by E_{02}, E_{-N} by E_{01} we see that Q_1'' replaces the role of Q_1. Letting

$$E_n'' = \begin{cases} E_n & \text{if} \quad n \neq N, \ -N, \\ E_{01} & \text{if} \quad n = -N, \\ E_{02} & \text{if} \quad n = N, \end{cases}$$

we may write

$$Q_1'' = \sum_{n=-N}^{N} n(\pi/N) E_n''$$

We then see that the hypotheses of (2.6) are satisfied for Q_1'' and Q_2' with $\varepsilon + \varepsilon'(\varepsilon) < (2N)^{-2}$. Hence for such ε, $\|Q_1'' - Q_2'\| < 2\pi/N$. By our choice of N

$$\left.\begin{array}{l} \|Q_1' - Q_1\| \\ \|Q_2' - Q_2\| \\ \|Q_1'' - Q_2'\| \end{array}\right\} < 2e^{-\pi}.$$

By applying (2.4) we may therefore conclude that the following three triangular loops are homotopic to 0:

The triangular loop with sides $\begin{cases} L_1 = \{\exp i\lambda Q_1 \colon 0 \leqq \lambda \leqq 1\} \\ L_{1,1'} = \mathrm{L}(\exp iQ_1,\ \exp iQ_1') \\ \tilde{L}_{1'} = \{\exp i(1-\lambda)Q_1' \colon 0 \leqq \lambda \leqq 1\}. \end{cases}$

The triangular loop with sides $\begin{cases} L_2 = \{\exp i\lambda Q_2 \colon 0 \leqq \lambda \leqq 1\} \\ L_{2,2'} = \mathrm{L}(\exp iQ_2,\ \exp iQ_2') \\ \tilde{L}_{2'} = \{\exp i(1-\lambda)Q_2' \colon 0 \leqq \lambda \leqq 1\}. \end{cases}$

The triangular loop with sides $\begin{cases} L_{2'} = \{\exp i\lambda Q_2' \colon 0 \leqq \lambda \leqq 1\} \\ L_{2',1''} = \mathrm{L}(\exp iQ_2',\ \exp iQ_1'') \\ \tilde{L}_{1''} = \{\exp i(1-\lambda)Q_1'' \colon 0 \leqq \lambda \leqq 1\}. \end{cases}$

Note that $\exp iQ_1' = \exp iQ_1''$ because $[Q_1', E_N] = [E_{02}, Q_1''] = 0$ and thus $\exp i(Q_1' - 2\pi E_N + 2\pi E_{02}) = \exp iQ_1'$. Note also that the distance from any point on $L_{1,1'}$, $L_{2,2'}$ or $L_{2',1''}$ to $U_1 = \exp iQ_1$ is smaller than

$$\|Q_1 - Q_1'\| + \|Q_1'' - Q_2'\| + \|Q_2' - Q_2\|$$
$$< \frac{\pi}{2N} + \frac{\pi}{2N} + \frac{2\pi}{N} = 3e^{-\pi} < 2\,.$$

Therefore the four paths $U_1^* L_{1,1'}$, $U_1^* L_{1'',2'}$, $U_1^* L_{2',2}$ and $U_1^* L_{2,1}$ form a loop which by (2.2) is null homotopic.

Combining all the preceeding observations, we see that the original loop is homotopic to the loop consisting of L_1, and $\tilde{L}_{1''}$. But since

$$\exp i\lambda Q_1' = \exp[2\pi i\lambda E_N]\exp[-2\pi i\lambda E_{02}]\exp[i\lambda Q_1'']$$

due to $[Q_1', E_N] = [E_{02}, Q_1''] = 0$, the loop consisting of $L_{1'}$ and $L_{1''}$ is homotopic to the sum of the two simple loops $\{\exp 2\pi i\lambda E_N \colon 0 \leqq \lambda \leqq 1\}$ and $\{\exp(-2\pi i\lambda E_{02}) \colon 0 \leqq \lambda \leqq 1\}$ completing the proof. Q.E.D.

Summing up (2.3) and (2.7) we have the following:

Theorem 2.8. *Let $\mathscr{A}$ be a von Neumann algebra with unitary group $\mathscr{U}(\mathscr{A})$. Then $\pi_1(\mathscr{U}(\mathscr{A}))$ is generated by the homotopy classes of the simple loops.*

Proof. Note that taken together (2.3) and (2.7) say that every loop in $\mathscr{U}(\mathscr{A})$ is homotopic to a sum of simple loops. Q.E.D.

§ 3. The First Homotopy Group of the Unitary Group of a Factor

In this section we will apply the theory developed so far to the special case of a von Neumann algebra factor.

Theorem 3.1. *If $\mathscr{A}$ is a factor of infinite type (that is $\mathscr{A}$ is of type I_∞, II_∞ or III), then $\pi_1(\mathscr{U}(\mathscr{A}))=0$.*

Proof. By (2.8) we have only to show that a simple loop $\{\exp 2\pi i\lambda Q: 0\leqq\lambda\leqq 1\}$ is homotopic to 0. Since $\exp 2\pi i Q=1$ we see that $Q=\sum_n nE_n^Q$ for $n=0,\pm 1,\pm 2,\ldots$ and E_n^Q are mutually orthogonal projections. Hence we need only consider the case of a simple loop $\{\exp 2\pi i\lambda E|\ 0\leqq\lambda\leqq 1\}$ where E is a projection.

First we consider the case where E is a projection of infinite relative dimension in $\mathscr{A}$. There exist in $\mathscr{A}$ mutually orthogonal projections E_1, E_2 with infinite relative dimension such that $E=E_1+E_2$. Also there exist mutually orthogonal projections F_1, F_2, F_3, F_4 of infinite relative dimension with $F_1+F_2+F_3+F_4=I$. By (1.1) there exist norm continuous paths of projections

$$\mathrm{F}_i(\mu): 0\leqq\mu\leqq 1\,,\qquad i=1,2,3$$

such that

$$\mathrm{F}_i(0)=F_i\,,\qquad i=1,2,3\,,$$
$$\mathrm{F}_1(1)=E_1,\ \mathrm{F}_2(1)=E_2,\ \mathrm{F}_3(1)=I-F_4\,.$$

Let

$$\mathrm{U}(\lambda,\mu)=[\exp 2\pi i\lambda\mathrm{F}_1(\mu)]\,[\exp 2\pi i\lambda\mathrm{F}_2(\mu)]\cdot[\exp 2\pi i\lambda\mathrm{F}_3(\mu)]\,[\exp -2\pi i\lambda(I-F_4)]\,.$$

Clearly

$$\mathrm{U}(\lambda,0)=I\,,$$
$$\mathrm{U}(\lambda,1)=\exp 2\pi i\lambda(E_1+E_2)=\exp 2\pi i E\,.$$

Thus the loop $\{\exp 2\pi i\lambda E: 0\leqq\lambda\leqq 1\}$ is null homotopic in $\mathscr{U}(\mathscr{A})$.

Next we consider the case where E is a projection of finite relative dimension. Then $I-E$ has infinite relative dimension and $E=I-(I-E)$. Since I and $I-E$ commute with each other $\{\exp 2\pi i\lambda E: 0\leqq\lambda\leqq 1\}$ is homotopic to the difference of the two simple loops $\{\exp 2\pi i\lambda(I-E): 0\leqq\lambda\leqq 1\}$ and $\{\exp 2\pi i\lambda I: 0\leqq\lambda\leqq 1\}$. Since $\mathscr{A}$ is of infinite type both I and $I-E$ have infinite relative dimension and thus the loops $\{\exp\, 2\pi i\lambda(I-E): 0\leqq\lambda\leqq 1\}$, $\{\exp 2\pi i\lambda I: 0\leqq\lambda\leqq 1\}$ are null homotopic in $\mathscr{U}(\mathscr{A})$ by the earlier part of the argument and the result follows. Q.E.D.

Remark. Kuiper [3] has shown that $\mathscr{U}(\mathscr{A})$ is actually contractable for a von Neumann algebra factor of type I_∞. Breuer [1] has obtained a similar result for certain von Neumann algebra factors of type $II_\otimes$. We conjecture that $\mathscr{U}(\mathscr{A})$ is always contractable for a factor of infinite type.

We wish now to deal with the case where $\mathscr{A}$ is a factor of finite type. First we introduce a homotopy invariant for simple loops in such a factor.

Notations and Conventions. Henceforth $\mathscr{A}$ will denote a von Neumann algebra factor of finite type. We denote by φ the normalized trace function on $\mathscr{A}$.

Definition. Let a loop $L=\{\mathrm{U}(\lambda):0\leqq\lambda\leqq 1\}$ in $\mathscr{U}(\mathscr{A})$ be divided into several arcs at

$$0=\lambda_0<\lambda_1<\cdots<\lambda_n=1\,,$$

such that for a fixed positive number δ, $0<\delta<1$

$$\|\mathrm{U}(\lambda')-\mathrm{U}(\lambda'')\|<\delta$$

whenever

$$\lambda_i\leqq\lambda'<\lambda''\leqq\lambda_{i+1}: i=0,\ldots,n-1\,.$$

That is the distance between any two points on the same arc is bounded by δ. Then using

$$\log Q=\sum_{m=1}^{\infty}(-1)^{m-1}\frac{1}{m}(Q-I)^m$$

we define

$$\mathrm{I}_\varphi(L)=\sum_{j=1}^{n}\varphi(\log U_{j-1}^*U_j)$$

where $U_i=\mathrm{U}(\lambda_i)$, $i=0,\ldots,n$.

Theorem 3.2. *With the notations preceeding, if δ is chosen sufficiently small, $\mathrm{I}_\varphi(L)$ is well defined, independent of the points of division, and an invariant of the homotopy class of the loop L in $\mathscr{U}(\mathscr{A})$.*

Proof. There exists $\delta_0>0$ such that for any Q_1, Q_2 with $\|Q_1\|<\delta_0$, $\|Q_2\|<\delta_0$, $\log e^{Q_1}e^{Q_2}-Q_1-Q_2$ can be written as a norm convergent infinite sum of multiple commutators of Q_1 and Q_2 by the Baker-Hausdorff formula. Since φ vanishes on commutators we have

$$\begin{aligned}\varphi(\log e^{Q_1}e^{Q_2})&=\varphi(Q_1)+\varphi(Q_2)\\&=\varphi(\log e^{Q_1})+\varphi(\log e^{Q_2})\end{aligned}$$

whenever $\|e^{Q_1}-I\|<\delta$ and $\|e^{Q_2}-I\|<\delta$ for some small δ.

Therefore whenever the mutual distance of the U_j's is small we have, from $\prod_{j=m+1}^{m'}(U_{j-1}^*U_j)=U_m^*U_{m'}$,

$$\sum_{j=m+1}^{m'}\varphi(\log U_{j-1}^*U_j)=\varphi(\log U_m^*U_{m'})\,.$$

Let $0=\lambda_0<\lambda_1<\cdots<\lambda_n=1$ and $0=\mu_0<\mu_1<\cdots<\mu_m=1$ be two given divisions of $[0,1]$. Consider the union of the two divisions, that is the subdivision using all the λ's and all the μ's. Provided that δ is chosen in the foregoing manner, $I_\varphi(L)$ for the λ division and $I_\varphi(L)$ for the μ division are equal to $I_\varphi(L)$ for the joint division because of the additivity computed in the previous paragraph. Hence $I_\varphi(L)$ is well defined.

Any continuous deformation of loops in $\mathscr{U}(\mathscr{A})$ can be divided into small triangular deformations. Using again the above additivity, $I_\varphi(L)$ is invariant under each triangular deformation and hence $I_\varphi(L)$ is a homotopy invariant. Q.E.D.

Theorem 3.3. *If $\mathscr{A}$ is a factor of type II_1 then $\pi_1(\mathscr{U}(\mathscr{A}))$ is isomorphic to the additive group of reals, in which $\pi_1(\mathscr{Z}\mathscr{U}(\mathscr{A}))$ is the integers.*

Proof. By (2.8) a general loop in $\mathscr{U}(\mathscr{A})$ is homotopic to a sum of simple loops of the form $\{\exp 2\pi i n_j \lambda E_j : 0 \leqq \lambda \leqq j\}$, $j = 1, \ldots, N$, where the n_j are integers and the E_j are projections.

Since $\mathscr{A}$ is of type II_1, each E_j can be divided into m_j mutually orthogonal subprojections with equal relative dimension in $\mathscr{A}: E_j = \sum_{k=1}^{m_j} E_{jk}$. Thus each $\{\exp 2\pi i n_j \lambda E_j : 0 \leqq \lambda \leqq 1\}$ is homotopic to a sum of m_j loops $\{\exp 2\pi i n_j \lambda E_{jk} : 0 \leqq \lambda \leqq 1\}$, $k = 1, \ldots, m_j$. Since $\mathscr{A}$ is a finite factor $\dim(I - E) = 1 - \dim E$ for any projection E in $\mathscr{A}$. Thus in particular

$$\dim(1 - E_{jk}) = 1 - \dim E_{jk} : k = 1, 2, \ldots, m_j$$

and since

$$\dim E_{jk} = \dim E_{j1} : k = 1, \ldots, m_j$$

we see that each projection E_{jk} can be deformed through projections in $\mathscr{A}$ to E_{j1} by (1.5). This gives a deformation of the corresponding loops to $\{\exp 2\pi i n_j \lambda E_{j1} : 0 \leqq \lambda \leqq 1\}$. Thus each $\{\exp 2\pi i n_j \lambda E_j : 0 \leqq \lambda \leqq 1\}$ is homotopic to $\{exp\, 2\pi i n_j m_j E_{j1} : 0 \leqq \lambda \leqq 1\}$.

In this manner we can make all $n_j m_j$ equal to some fixed integer n and $\dim E_{j1}, j = 1, \ldots, N$ smaller than $1/N$. Note that n will be a common multiple of $n_1, \ldots, n_N$ big enough so that $\dim E_j < 1/N$, $j = 1, \ldots, N$.

There exist mutually orthogonal projections $E_j', j = 1, 2, \ldots, N$, with $\dim E_j' = \dim E_{j1}$, $j = 1, \ldots, N$. Hence, since $\mathscr{A}$ is a finite factor $\dim(I - E_j') = \dim(I - E_{j1})$, for $j = 1, \ldots, N$. We may thus apply (1.5) to continuously deform E_j' to $E_{j1}, j = 1, \ldots, N$, through projections in $\mathscr{A}$. Thus we see that the original loop L is homotopic to $\{\exp 2\pi i n \lambda E : 0 \leqq \lambda \leqq 1\}$ where $E = \sum_{j=1}^{N} E_j'$ is a projection.

Suppose next that we are given two loops of the final form, namely

$$L_a = \{\exp 2\pi i n_a \lambda E_a : 0 \leqq \lambda \leqq 1\},$$

$$L_b = \{\exp 2\pi i n_b \lambda E_b : 0 \leqq \lambda \leqq 1\},$$

where E_a, E_b are projections and n_a, n_b are integers. By the same argument as before we can deform each of the above through loops in $\mathscr{U}(\mathscr{A})$ to

$$L'_a = \{\exp 2\pi i n \lambda E'_a : 0 \leqq \lambda \leqq 1\},$$
$$L'_b = \{\exp 2\pi i n \lambda E'_b : 0 \leqq \lambda \leqq 1\}$$

respectively where $n = n_a n_b$, and E'_a, E'_b are projections. The invariant of (3.2) can be calculated immediately for the loop $L = \{\exp 2\pi i m \lambda E : 0 \leqq \lambda \leqq 1\}$ and is given by

$$\mathrm{I}_\varphi(L) = 2\pi i m \dim E,$$

and is an invariant of the homotopy class of the loop. Thus if L_a and L_b are homotopic $\dim E'_a = \dim E'_b$. On the other hand if $\dim E'_a = \dim E'_b$ we may, since $\mathscr{A}$ is a finite factor, apply (1.5) to conclude E'_a may be deformed through projections in $\mathscr{A}$ to E'_b. Thus L'_a is homotopic to L'_b through loops lying in $\mathscr{U}(\mathscr{A})$ and hence the same is true for L_a and L_b.

Therefore $\mathrm{I}_\varphi(\ )$ completely determines the homotopy class of a loop in $\mathscr{U}(\mathscr{A})$. The range of $\mathrm{I}_\varphi(\ )$ is the set of complex numbers $2\pi i n \dim E$ where n is an integer and E a projection. Define

$$\mathrm{I}'_\varphi : \pi_1(\mathscr{U}(\mathscr{A})) \to \boldsymbol{R}$$

by

$$\mathrm{I}'_\varphi(L) = (2\pi i)^{-1} \mathrm{I}_\varphi(L).$$

Since $\mathrm{I}_\varphi(\ )$ is additive, so is $\mathrm{I}'_\varphi(\ )$. Since $\mathscr{A}$ is of type II_1 the range of $\dim E$ is all of $[0, 1]$ and hence I'_φ is surjective. Since $\mathrm{I}'_\varphi(\ )$ is a complete homotopy invariant for loops in $\mathscr{U}(\mathscr{A})$ it is also injective, and hence is an isomorphism of $\pi_1(\mathscr{U}(\mathscr{A}))$ onto the additive group of reals in which $\pi_1(\mathscr{Z}\mathscr{U}(\mathscr{A}))$ is mapped onto the subgroup $\boldsymbol{Z}$ of integers. Q.E.D.

Remark. If $\mathscr{A}$ is a factor of type I_n then substantially the same argument with the invariant $\mathrm{I}_\varphi(\ )$ shows that $\pi_1(\mathscr{U}(\mathscr{A})) \cong \boldsymbol{Z}$ by an isomorphism taking $\pi_1(\mathscr{Z}\mathscr{U}(\mathscr{A}))$ to $n\boldsymbol{Z}$. This result is classical and the details are left to the reader.

Remark. In a von Neumann algebra of finite type, but not necessarily a factor, it should be possible to use the center valued trace and substantially the same argument to compute $\pi_1(\mathscr{U}(\mathscr{A}))$.

Appendix (by L. Pitt): A Technical Point

Theorem. Let $\mathscr{H}$ be a Hilbert space, and

$$\mathscr{H} = \mathscr{H}_0 + \mathscr{H}_\alpha + \mathscr{H}_\beta + \mathscr{H}_\gamma,$$
$$\mathscr{H} = \mathscr{H}_A + \mathscr{H}_B + \mathscr{H}_C$$

be two orthogonal splittings of $\mathscr{H}_0$. Let E_j be the orthogonal projection onto $\mathscr{H}_j$, $j = 0, \alpha, \beta, \gamma, A, B, C$.

If $\|E_0 E_C\|$, $\|E_\alpha E_B\|$, $\|E_\beta E_A\|$, *and* $\|E_\gamma E_B\| \leqq \varepsilon$, *then*

(1) $\|E_A E_0 E_B\| \leqq 3\varepsilon$.

(2) *There exist projections* E_{0A}, E_{0B} *onto* $\mathscr{H}_{0A}, \mathscr{H}_{0B}$ *with* $\mathscr{H}_0 = \mathscr{H}_{0A} \oplus \mathscr{H}_{0B}$ *such that*

$$\|E_B E_{0A}\| \leqq 12\varepsilon$$

and

$$\|E_A E_{0B}\| \leqq 32\varepsilon .$$

Proof. First we show (1). Since

$$\begin{aligned} E_A E_0 E_B &= E_A(I - E_\alpha - E_\beta - E_\gamma)E_B \\ &= 0 - E_A E_\alpha E_B - E_A E_\beta E_B - E_A E_\gamma E_B , \end{aligned}$$

we have

$$\begin{aligned} \|E_A E_0 E_B\| &\leqq \|E_A E_\alpha E_B\| + \|E_A E_\beta E_B\| + \|E_A E_\gamma E_B\| \\ &\leqq \|E_\alpha E_B\| + \|E_A E_\beta\| + \|E_\gamma E_B\| \leqq 3\varepsilon . \end{aligned}$$

To prove (2), let $F = E_0 E_A E_0$ and $F = \int_0^1 \lambda \, dF_\lambda$ be the spectral representation of F. Let $E_{0A} = F([a, 1])$, where $a > 0$ is to be determined later.

If $Q_1^* Q_1 - Q_2^* Q_2 \geqq 0$, then $E Q_1^* Q_1 E - E Q_2^* Q_2 E \geqq 0$ for any projection E and hence $\|Q_1 E x\|^2 \geqq \|Q_2 E x\|^2$ for all x, namely $\|Q_2 E\| \leqq \|Q_1 E\|$. Applying this to $Q_1 = a^{-1} F$, $Q_2 = E_{0A}$ and $E = E_B$, we obtain

$$\|E_{0A} E_B\| \leqq a^{-1} \|F E_B\| \leqq a^{-1} \|E_A E_0 E_B\| \leqq 3\varepsilon a^{-1} .$$

Hence

$$\|E_B E_{0A}\| = \|(E_B E_{0A})^*\| = \|E_{0A} E_B\| \leqq 3\varepsilon a^{-1} .$$

Next let $E_{0B} = E_0 - E_{0A}$. Then $(E_A E_{0B})^* (E_A E_{0B}) = F E_{0B}$ and hence $\|E_A E_{0B}\| = \|F E_{0B}\|^{1/2} \leqq a^{1/2}$, where $\|Q^* Q\| = \|Q\|^2$ is used. Substituting $\|E_A E_{0B} E_A\| = \|E_A E_{0B}\|^2$, $\|E_A E_{0B} E_B\| \leqq \|E_A E_0 E_B\| + \|E_A(E_{0A} E_B)\| \leqq 3\varepsilon(1 + a^{-1})$ and $\|E_A E_{0B} E_C\| \leqq \|E_A E_{0B}\| \, \|E_0 E_C\| \leqq \varepsilon$, into

$$\|E_A E_{0B}\| \leqq \|E_A E_{0B} E_A\| + \|E_A E_{0B} E_B\| + \|E_A E_{0B} E_C\| ,$$

we obtain

$$\|E_A E_{0B}\| \, (1 - \|E_A E_{0B}\|) \leqq \varepsilon(4 + 3a^{-1}) .$$

By using $\|E_A E_{0B}\| \leqq a^{1/2}$, we have

$$\|E_A E_{0B}\| \leqq (1 - a^{1/2})^{-1} (4 + 3a^{-1}) \varepsilon .$$

By choosing $a = 1/4$, we obtain (2). Q.E.D.

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