ON THE HOMOTOPY CLASSIFICATION OF THE EXTENSIONS OF A FIXED MAP

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1. **Introduction.** In considering the homotopy classification of the maps of a CW complex into any topological space X, we are led to the problem of enumerating the homotopy classes of extensions of a given map $u: K \rightarrow X$ over a larger complex $L \supset K$. We examine this for the case in which $L-K$ consists only of disjoint cells, for maps and homotopies relative to a base point $k_0 \in K$.

For a given map $u: (K, k_0) \rightarrow (X, x_0)$, we define in §2 for each $\alpha \in \pi_q(K, k_0)$ a homomorphism

$$\alpha_u: \pi_1(\mathfrak{F}, u) \rightarrow \pi_{q+1}(X, x_0)$$

where $\mathfrak{F}$ is the function space of maps $(K, k_0) \rightarrow (X, x_0)$. If $L = K \cup e^{q+1}$ is formed by attaching the cell e^{q+1} by a map in the class α, and if u extends over L, then we prove that the homotopy classes (rel k_0) of extensions are in 1-1 correspondence with the cokernel of α_u. This may easily be generalized to a complex $L = K \cup \{e^{q_i+1}\}$ such that the e^{q_i+1} are disjoint.

The difficulty lies in computing α_u, even when the group $\pi_1(\mathfrak{F}, u)$ is known. We show how α_u can be computed when K is a cluster of spheres: the result is given in terms of α, its Hopf invariants (including the higher Hopf invariants in the sense of Hilton [3]), the homotopy groups of X, and the operations of composition, suspension, and formation of Whitehead products. This covers, for example, the case when L is a sphere bundle over a sphere with a cross-section, such as the product of two spheres.

In §7 we give applications of the theory to two other problems; the more important of these is a formula for expanding a Whitehead product of the form $[\alpha \circ \gamma, \beta]$. It should be noted that the Whitehead product we use (§4) differs from that defined by J. H. C. Whitehead by a sign.

2. **Homotopy groups of function spaces.** Let K be a CW complex. The function space X^K of maps (=continuous functions) is given the compact-open topology. Then the natural function $\theta: X^{(K \times T)} \rightarrow (X^K)^T$, given by

$$(\theta f)(t)(k) = f(k, t), \qquad k \in K, t \in T,$$

is a homeomorphism if T is a CW complex such that $K \times T$, given the product topology, is also a CW complex (the proof is elementary; cf. [2] and [9] for other cases in which θ is a homeomorphism). Notice that if I is the unit

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interval, then $K\times I$ is always a CW complex. It is convenient to identify $X^{K\times I}$ and $(X^K)^I$ by means of θ.

NOTATION. A fixed base point will always be chosen in each space, and denoted by a subscript 0: thus, $k_0\in K$, $x_0\in X$. The only exception is that $0=(0, \cdots, 0)$ will be the base point in I^n; the base point in $K\times I^n$ will be $(k_0, 0)$. The function space, with the compact-open topology, of maps $(K, k_0)\to(X, x_0)$ will in the future be denoted by X^K; no ambiguity will arise, since no further reference will be made to the space of maps $K\to X$. The domain space K will always be assumed to be a CW complex, k_0 a vertex.

Let u: $(K, k_0)\to(X, x_0)$ be a map; it follows from the first paragraph of this section that we may equally well represent elements of $\pi_1(X^K, u)$ as homotopy classes of maps

$$\hat{F}: I \to X^K \text{ such that } \hat{F}(0) = u = \hat{F}(1),$$

or

$$F: (K\times I, k_0\times I) \to (X, x_0) \text{ such that } F(k, 0) = u(k) = F(k, 1), \qquad k \in K.$$

Therefore a map g: $(Q, q_0)\to(K, k_0)$ induces a homomorphism

$$g^*: \pi_1(X^K, u) \to \pi_1(X^Q, ug)$$

by $g^*\{F\}=\{F(g\times 1)\}$, where 1 is the identity map of I and

$$g \times 1: (Q \times I, q_0 \times I) \to (K \times I, k_0 \times I)$$

is the product map.

Now a path $\hat{L}$ in X^K from u_0 to u_1 is equivalent to a homotopy L: $(K\times I, k_0\times I)\to(X, x_0)$ from u_0 to u_1; the path $\hat{L}$ defines an isomorphism in the usual way from the homotopy groups based at u_1 to those based at u_0: we write for this

$$L_\#: \pi_1(X^K, u_1) \to \pi_1(X^K, u_0). \tag{2.1}$$

LEMMA (2.2). $g^*L_\#=(L(g\times 1))_\# g^*: \pi_1(X^K, u_1)\to\pi_1(X^Q, u_0g)$.

Let g_0, g_1: $(Q, q_0)\to(K, k_0)$, and let G: $(Q\times I, q_0\times I)\to(K, k_0)$ be a homotopy from g_0 to g_1. Then

LEMMA (2.3). $g_0^*=(uG)_\# g_1^*: \pi_1(X^K, u)\to\pi_1(X^Q, ug_0)$.

The proofs of these elementary lemmas are omitted; it is easy to deduce from them

COROLLARY (2.4). *If g is a homotopy equivalence, then g^* is an isomorphism.*

Suppose now that $(Q, q_0)=(S^q, s_0)$, where we consider $S^q=s_0\cup e^q$ as a CW complex with a characteristic map

$$i^q: (I^q, \dot{I}^q) \to (S^q, s_0)$$

which is a homeomorphism of $I^q - \dot{I}^q$ onto e^q of degree $+1$. Let $v: (S^q, s_0) \rightarrow (X, x_0)$, and define $v^\flat: (S^q \times I, s_0 \times I) \rightarrow (X, x_0)$ by $v^\flat(s, t) = v(s)$, for $s \in S^q$, $t \in I$. We then define

$$v_\natural : \pi_1(X^{S^q}, v) \rightarrow \pi_{q+1}(X, x_0) \tag{2.5}$$

as follows: for $\{F\} \in \pi_1(X^{S^q}, v)$, $v_\natural\{F\}$ is the value of the separation element[1] $d(F, v^\flat)$ on the cell $(e^q \times e^1, s_0 \times 0)$ with the product orientation, where $e^1 = I - \dot{I}$. It is readily verified that

LEMMA (2.6). *$v_\natural$ is an isomorphism; and if M is a homotopy from v to v', then $v'_\natural = v_\natural M_\#$.*

Now let $g_0, g_1: (S^q, s_0) \rightarrow (K, k_0)$, let G be a homotopy from g_0 to g_1, and let $u \in X^K$. Then

$$\begin{aligned} (ug_0)_\natural \overset{*}{g_0} &= (ug_0)_\natural (uG)_\# \overset{*}{g_1} && \text{by (2.3)}, \\ &= (ug_1)_\natural \overset{*}{g_1} && \text{by (2.6)}. \end{aligned}$$

Hence the homomorphism $(ug)_\natural g^*$ depends only on the homotopy class $\alpha \in \pi_q(K, k_0)$ of g, and we may define

$$\alpha_u = (ug)_\natural g^* : \pi_1(X^K, u) \rightarrow \pi_{q+1}(X, x_0). \tag{2.7}$$

LEMMA (2.8). *If L is a homotopy from u_0 to u_1, then $\alpha_{u_1} = \alpha_{u_0} L_\#$.*

The lemma follows from (2.2) and (2.6).

3. **The classification theorems.** We now explain the use of α_u in homotopy classification. First, let $L = K \cup e^{q+1}$, where e^{q+1} has an attaching map $g: (S^q, s_0) \rightarrow (K, k_0)$ in a homotopy class $\alpha \in \pi_q(K, k_0)$. Then a map $u: (K, k_0) \rightarrow (X, x_0)$ has an extension to (L, k_0) if and only if

$$u_* \alpha = 0. \tag{3.1}$$

If this is satisfied, let f_0, f_1 be two extensions of u such that there is a homotopy $\overline{H}: (L \times I, k_0 \times I) \rightarrow (X, x_0)$ from f_0 to f_1. Then $H = \overline{H} | (K \times I, k_0 \times I)$ determines an element $\{H\} \in \pi_1(X^K, u)$: we shall prove

LEMMA (3.2). *The value of the separation element $d(f_1, f_0)$ on the cell (e^{q+1}, k_0) is $\alpha_u \{H\} \in \pi_{q+1}(X, x_0)$.*

From the lemma we deduce

THEOREM (3.3). *Let $u: (K, k_0) \rightarrow (X, x_0)$ extend to (L, k_0). Then the homotopy classes* rel k_0 *of extensions are in* 1-1 *correspondence with the elements of the cokernel of α_u, i.e. of $\pi_{q+1}(X, x_0)/\alpha_u \pi_1(X^K, u)$.*

The lemma leads in fact to a more general result: let $L = K \cup \{e^{q_i+1}\}$, where the cells e^{q_i+1} are disjoint, and each possesses an attaching map

[1] Cf. Appendix.

$g_i\colon (S^{q_i}, s_0)\to(K, k_0)$ in a class $\alpha_i\in\pi_{q_i}(K, k_0)$. Set $C(L, K)=\sum\pi_{q_i+1}(X, x_0)$, the strong sum, where the homotopy groups are indexed by the cells of $L-K$; a map $u\colon (K, k_0)\to(X, x_0)$ extends to (L, k_0) if and only if $u_*\alpha_i=0$ for all i. Then the homomorphisms $(\alpha_i)_u$ together define

$$\alpha_u\colon \pi_1(X^K, u)\to C(L, K)$$

such that the coordinate of $\alpha_u(\xi)$ in $\pi_{q_i+1}(X, x_0)$ is $(\alpha_i)_u(\xi)$.

THEOREM (3.4). *Let $u\colon (K, k_0)\to(X, x_0)$ extend to (L, k_0). Then the homotopy classes* rel k_0 *of extensions are in* 1-1 *correspondence with the elements of the cokernel of α_u, i.e. with the cosets $C(L, K)/\alpha_u\pi_1(X^K, u)$.*

We now prove (3.2)–(3.4); we first need an elementary lemma which will be used again later.

Let P be a finite CW complex on I^n such that $0=(0,\cdots,0)$ is a vertex. Let $\{\sigma^n\}$ be the set of n-cells of P, and let the orientation of each, given by the chosen characteristic map $c_\sigma\colon (I^n, \dot I^n, 0)\to(\bar\sigma^n, \dot\sigma^n, p_\sigma)$, agree with the orientation induced by inclusion in I^n. For each σ let $T_\sigma\colon (I, 0, 1)\to(P, 0, p_\sigma)$ be a path in P. Suppose that $h', h\colon (P, 0)\to(X, x_0)$ agree on P^{n-1}. Then the separation element $d(h', h)$ on (σ, p_σ) has a value $\delta_\sigma\in\pi_n(X, hp_\sigma)$. Treating I^n as a CW complex with just one n-cell in the usual way, we also have a separation element $d(h', h)$ on $(I^n, 0)$ with a value $\delta\in\pi_n(X, x_0)$.

LEMMA (3.5). *$\delta=\sum(hT_\sigma)_\#\delta_\sigma$, where $\#$ denotes the operation of the path on the homotopy group, and the summation is over all $\sigma\in\{\sigma^n\}$.*

Since all paths T_σ for a given p_σ are homotopic in I^n, $(hT_\sigma)_\#$ does not depend on the choice of T_σ. Notice that an equivalent result holds with I^n replaced by a sphere S^n, taking i^n as the characteristic map of the cell $e^n=S^n-s_0$.

The proof of this lemma is omitted.

Proof of (3.2). We identify (S^q, s_0) with $(\dot I^{q+1}, 0)$, and write $j=j^q\colon (S^q, s_0)\to(\dot I^{q+1}, 0)$ for the identity map. We first show that the triple $(L, K, k_0)=(I^{q+1}, S^q, 0)$ is a universal example. Let the cell e^{q+1} in $L=K\cup e^{q+1}$ have characteristic map $\bar g\colon (I^{q+1}, S^q, 0)\to(L, K, k_0)$, and attaching map $g=\bar g|(S^q, 0)$. Let $e_0^{q+1}=I^{q+1}-S^q$ have characteristic and attaching maps $\bar j, j$, the identity maps; and using the notation of (3.2), set $f_1'=f_1\bar g$, $f_0'=f_0\bar g$, $\overline{H}'=\overline{H}(g\times 1)$, $H'=H(g\times 1)$, $u'=ug$. Suppose that (3.2) holds for the universal example, so that $d(f_1', f_0')=\iota_{u'}\{H'\}$, where ι is the class of j, and the separation element is evaluated on $(e_0^{q+1}, 0)$. Then if $d(f_1, f_0)$ is evaluated on (e^{q+1}, k_0) we have

$$\begin{aligned} d(f_1, f_0) &= d(f_1', f_0') = \iota u'\{H'\} = u_\natural' j^*\{H'\} \\ &= u_\natural'\{H'\} = (ug)_\natural\{H(g\times 1)\} \\ &= (ug)_\natural g^*\{H\} = \alpha_u\{H\}, \end{aligned}$$

from the definitions.

We now prove (3.2) for the universal example by means of an explicit construction. The separation element $d(f_1, f_0)$ on $(e_0^{q+1}, 0)$ is represented by the map E: $((I^{q+1}\times I)^{\cdot}, 0)\to(X, x_0)$ given by

$$E(p, t) = \begin{cases} f_1(p), & t = 1 \\ f_1(p) = f_0(p), & 0 < t < 1 \\ f_0(p), & t = 0. \end{cases} \left.\begin{matrix} p \in I^{q+1}, t \in I, \\ (p, t) \in (I^{q+1} \times I)^{\cdot} \\ \end{matrix}\right\}$$

Take a cellular decomposition of $\dot{I}^{q+2}=(I^{q+1}\times I)^{\cdot}$ such that $\dot{I}^{q+1}=0\cup e^q$; $I^{q+1}=\dot{I}^{q+1}\cup e^{q+1}$; $I=0\cup 1\cup e^1$. Thus

$$\dot{I}^{q+2} = (\dot{I}^{q+1} \times \dot{I} \cup 0 \times I) \cup (e^{q+1} \times 0) \cup (e^{q+1} \times 1) \cup (e^q \times e^1).$$

Now $\overline{H}$ agrees with E on the q-section of $\dot{I}^{q+2}$, and also on the cells $e^{q+1}\times 0$, $e^{q+1}\times 1$. Hence, by using Lemma (3.5) for a sphere, and noting that the orientation of $e^q\times e^1$ is *opposite* to that induced by inclusion in $\dot{I}^{q+2}$, the separation element $d(\overline{H}|(I^{q+1}\times I)^{\cdot}, E)$ on $(\dot{I}^{q+2}, 0)$ is equal to minus the element $d(\overline{H}|\dot{I}^{q+1}\times I, E|\dot{I}^{q+1}\times I)$ on $(e^q\times e^1, 0)$. But maps of $(\dot{I}^{q+2}, 0)$ into (X, x_0) determine elements of $\pi_{q+1}(X, x_0)$, so that the former separation element is

$$\{\overline{H}\,|\,(I^{q+1} \times I)^{\cdot}\} - \{E\} = 0 - \{E\} = -\{E\};$$

and since $\overline{H}|\dot{I}^{q+1}\times I=H$, $E|\dot{I}^{q+1}\times I=u^\flat$, the latter separation element is

$$\begin{aligned} d(H, u^\flat)(e^q \times e^1, 0) &= u_\natural\{H\} \text{ by definition,} \\ &= u_\natural j^*\{H\} = \iota_u\{H\}. \end{aligned}$$

Hence $d(f_1, f_0)(I^{q+1}, 0)=\iota_u\{H\}$, which proves (3.2) for the universal example.

Proof of (3.3). The homotopy classes rel K of extensions of u are in 1-1 correspondence with the elements of $\pi_{q+1}(X, x_0)$; they may be distinguished by the separation elements of representative maps. Let f_0, f_1 be two extensions of u for which there is a homotopy $\overline{H}$ rel k_0 from f_0 to f_1. Then by (3.2), the separation element on (e^{q+1}, k_0) is contained in $\alpha_u\pi_1(X^K, u)$. Conversely, if f_0, f_1 are two extensions of u such that $d(f_1, f_0)=\alpha_u\{H\}$, with H: $(K\times I, k_0\times I)\to(X, x_0)$, let $\overline{H}$ be an extension of H to $L\times I$ such that $\overline{H}(p, 0)=f_0(p)$, $p\in L$, and define f_1': $(L, k_0)\to(X, x_0)$ by $f_1'(p)=\overline{H}(p, 1)$. Then by (3.2), $d(f_1', f_0)=\alpha_u\{H\}=d(f_1, f_0)$. Hence $d(f_1, f_1')=0$, and $f_1\simeq f_1'$ rel K. Since $f_1'\simeq f_0$ rel k_0, $f_1\simeq f_0$ rel k_0.

Proof of (3.4). If $L=K\cup\{e^{q_i+1}\}$ is formed by attaching a set of cells to K, we may alter K within homotopy type so that the base point k_0 lies on the boundary of each cell; this does not change the group $\pi_1(X^K, u)$ by more than an isomorphism. Then, if u extends to two maps f_0, f_1: $(L, k_0)\to(X, x_0)$, the maps determine an element $d(f_1, f_0)\in C(L, K)$ such that the coordinate of $d(f_1, f_0)$ in $\pi_{q_i+1}(X, x_0)$ is $d(f_1, f_0)(e^{q_i+1}, k_0)$ (which may be defined in the subcomplex $K\cup e^{q_i+1}$). Then it is easy to show from Lemma (3.2) by the method used in the proof of (3.3) that f_0 and f_1 are homotopic if and only if there exists $\{H\}\in\pi_1(X^K, u)$ such that $d(f_1, f_0)(e^{q_i+1}, k_0)=(\alpha_i)_u\{H\}$ for all i. The theorem then follows at once.

An alternative proof of the above two theorems can be obtained by considering homotopy sequences of the fibering $X^L\to X^K$ induced by the inclusion $K\subset L$.

4. **The addition, product, and composition theorems.** In this section we give three theorems which are useful in the computation of the homomorphism α_u.

Let $\alpha, \beta\in\pi_q(K, k_0)$, $u\in X^K$, $\xi\in\pi_1(X^K, u)$; and let $\cdot$ denote the operation of π_1 on π_r.

THEOREM (4.1) (ADDITION THEOREM). *If* $q>1$,

$$(\alpha+\beta)_u(\xi) = \alpha_u(\xi)+\beta_u(\xi);$$

if $q=1$ *(so that* $u_*\alpha\in\pi_1(X, x_0)$*), then*

$$(\alpha+\beta)_u(\xi) = \alpha_u(\xi)+(u_*\alpha)\cdot\beta_u(\xi).$$

Thus if $q>1$, the transformation $(\alpha, \xi)\to\alpha_u(\xi)$ is a pairing of $\pi_q(K, k_0)$ and $\pi_1(X^K, u)$ to $\pi_{q+1}(X, x_0)$; if $q=1$, the transformation might be called a crossed pairing. We shall prove the theorem later, by means of an explicit construction.

Now let $\gamma\in\pi_m(K, k_0)$, $\delta\in\pi_n(K, k_0)$ be represented by maps $f\colon (I^m, \dot I^m)\to(K, k_0)$ and $g\colon (I^n, \dot I^n)\to(K, k_0)$ respectively. Then the *Whitehead product* $[\gamma, \delta]$ is defined to be the class of the map $p\colon (\dot I^{m+n}, 0)=(I^m\times\dot I^n\cup\dot I^m\times I^n, 0)\to(K, k_0)$ given by

$$\begin{aligned} p(s, t) &= f(s), \ s\in I^m, t\in \dot I^n \\ &\ g(t), \ s\in \dot I^m, t\in I^n. \end{aligned}$$

Notice that because of our orientation conventions (cf. Appendix), $[\gamma, \delta]$ is not the same as that defined by J. H. C. Whitehead in [8]; we write the latter, defined by using homology orientations, as $[\gamma, \delta]'$. The relation is easily seen to be $[\gamma, \delta]=(-1)^{m+n-1}[\gamma, \delta]'$.

Let $u\in X^K$, $\xi\in\pi_1(X^K, u)$.

THEOREM (4.2) (PRODUCT THEOREM). $[\gamma, \delta]_u(\xi)$ *is given by*

(i) $-[u_*\gamma, \delta_u(\xi)]+(-1)^{n+1}[\gamma_u(\xi), u_*\delta]$ if $m, n>1$;

(ii) $-[u_*\gamma, \delta_u(\xi)]+(-1)^{n+1}[\gamma_u(\xi), u_*\gamma\cdot u_*\delta]$ if $m=1, n>1$;

(iii) $-[u_*\delta\cdot u_*\gamma, \delta_u(\xi)]+(-1)^{n+1}[\gamma_u(\xi), u_*\delta]$ if $m>1, n=1$;

(iv) $-[u_*\delta\cdot u_*\gamma, \delta_u(\xi)]-(-1)^{n+1}u_*\delta\cdot[\gamma_u(\xi), -(u_*\gamma\cdot u_*\delta)]$ if $m=n=1$.

If we agree to use π_r for $r>1$ as a trivial group of operators, then (iv) is seen to include the other formulae. The proof will be given in §8.

Two simple consequences of (4.2) are the following:

COROLLARY (4.3). *If* $\gamma \in \pi_1(K, k_0)$, $\delta \in \pi_n(K, k_0)$, $n>1$, *then*

$$(\gamma \cdot \delta)_u(\xi) = u_*\gamma \cdot \delta_u(\xi) - [\gamma_u(\xi), u_*\gamma \cdot u_*\delta].$$

This follows from (4.2) (ii) and (4.1), since $\gamma \cdot \delta = [\gamma, \delta]' + \delta = (-1)^n[\gamma, \delta] + \delta$.

Now let $\alpha = P(\delta_1, \cdots, \delta_s)$ be a multiple Whitehead product formed from the ordered set $\delta_1, \cdots, \delta_s$ ($\delta_i \in \pi_{n_i}(K, k_0)$, $n_i > 1$) by the insertion of $s-1$ brackets [ ]. Let P_i denote the product $P(u_*\delta_1, \cdots, (\delta_i)_u(\xi), \cdots, u_*\delta_s)$ formed in the same way, but with δ_j replaced by $u_*\delta_j$ if $j \neq i$, and δ_i by $(\delta_i)_u(\xi)$.

COROLLARY (4.4). $\alpha_u(\xi) = \sum_i \pm P_i$, *where the signs are determined by* P *and the integers* n_i.

The proof is by repeated application of (4.2)(i). For example,

$$\begin{aligned}[\delta_1, [\delta_2, \delta_3]]_u(\xi) = {} & [u_*\delta_1, [u_*\delta_2, (\delta_3)_u\xi]] \\ & + (-1)^{n_3}[u_*\delta_1, [(\delta_2)_u\xi, u_*\delta_3]] \\ & + (-1)^{n_2+n_3-1}[(\delta_1)_u\xi, [u_*\delta_2, u_*\delta_3]].\end{aligned}$$

We now use (4.1) and (4.4) to simplify α_u when $\alpha = \beta \circ \phi$, ($\beta \in \pi_n(K, k_0)$, $\phi \in \pi_q(S^n, s_0)$). To express the result we need certain of the higher Hopf invariants of ϕ (cf. [3]); the definition of these depends on a choice of *basic products* $\omega_i \in \pi_{r_i}(S^n \vee S^n_0, s_0)$, $n \geqq 2$, as defined and ordered in [3], with $\omega_{-2} = \iota^n$, $\omega_{-1} = \iota^n_0$, respectively the generators of $\pi_n(S^n \vee S^n_0, s_0)$ represented by maps of degree $+1$ of S^n onto S^n and S^n_0. Then it is shown in [3] that

$$(4.5) \qquad (\iota^n + \iota^n_0) \circ \phi = \iota^n \circ \phi + \iota^n_0 \circ \phi + \sum_0^\infty \omega_i \circ H_i(\phi),$$

where $H_i(\phi) \in \pi_q(S^{r_i})$ is termed a *higher Hopf invariant* of ϕ.

For elements γ, δ in the homotopy groups of any space Y, define inductively $\sigma_0(\gamma, \delta) = [\gamma, \delta], \cdots, \sigma_p(\gamma, \delta) = [\gamma, \sigma_{p-1}(\gamma, \delta)]$. Then it follows from the ordering $\iota^n < \iota^n_0$ chosen above that $\sigma_p(\iota^n, \iota^n_0)$ is a basic product of weight $p+2$ for $p \geqq 0$. If $\sigma_p(\iota^n, \iota^n_0) = \omega_{i_p}$, write $B_p(\phi) = H_{i_p}(\phi)$, the corresponding higher Hopf invariant. Let S_* be the suspension homomorphism.

THEOREM (4.6) (SPHERE THEOREM). *Let* $\phi \in \pi_q(S^n)$, $n \geqq 2$, $v \in X^{S^n}$, *and let* $\zeta \in \pi_{n+1}(X, x_0)$. *Then*

$$\phi_v v_\natural^{-1}(\zeta) = \zeta \circ S_*\phi + \sum_0^\infty (-1)^{p+1}\sigma_p(v_*\iota^n, \zeta) \circ S_*B_p(\phi).$$

In particular, the sphere theorem allows us to compute any homomorphism of the fundamental groups of the loop spaces $\pi_1(\Omega^n X, v) \to \pi_1(\Omega^q X, vf)$ induced by a map $f: S^q \to S^n$.

Let β, ϕ, u, ξ be as above, and let $b\colon (S^n, s_0)\to(K, k_0)$ be a representative map for β. Then it follows from the definitions that

$$(\beta\circ\phi)_u(\xi) = \phi_{ub}(ub)_\natural^{-1}\beta_u(\xi). \tag{4.7}$$

Theorem (4.6), together with (4.7), yields

COROLLARY (4.8) (COMPOSITION THEOREM).

$$(\beta\circ\phi)_u(\xi) = \beta_u(\xi)\circ S_*\phi + \sum_0^\infty (-1)^{p+1}\sigma_p(u_*\beta, \beta_u(\xi))\circ S_*B_p(\phi).$$

In particular, if $q<3n-2$, then $B_p(\phi)=0$ for all $p>0$, and $B_0(\phi)=H(\phi)$, the generalized Hopf invariant. The formula then reduces to

$$(\beta\circ\phi)_u(\xi) = \beta_u(\xi)\circ S_*\phi - [u_*\beta, \beta_u(\xi)]\circ S_*H(\phi). \tag{4.9}$$

Proof of (4.1). Let a, $b\colon (S^q, s_0)\to(K, k_0)$ represent α, β respectively. Denoting by $i=i^q\colon (I^q, \dot I^q)\to(S^q, s_0)$ a characteristic map for the cell $e^q=S^q-s_0$ as before, we can represent $\alpha+\beta$ by $c\colon (S^q, s_0)\to(K, k_0)$, defined by

$$\begin{aligned} ci(t_1, \cdots, t_q) &= ai(2t_1, t_2, \cdots, t_q) && \text{if } t_1\leqq 1/2\\ &= bi(2t_1-1, t_2, \cdots, t_q) && \text{if } t_1\geqq 1/2.\end{aligned}$$

Let $F\colon (K\times I, k_0\times I)\to(X, x_0)$ represent $\xi\in\pi_1(X^K, u)$; then

$$(\alpha+\beta)_u(\xi) = d(F(c\times 1), u^\flat(c\times 1))(e^q\times e^1, s_0\times 0), \tag{4.10}$$

where $e^1=I-\dot I$.

Let the subsets I_1^q, $I_2^q\subset I^q$ be determined by $t_1\leqq 1/2$, $t_1\geqq 1/2$, respectively, and define cells σ_1, $\sigma_2\subset I^q\times I$ as the interiors of $I_1^q\times I$, $I_2^q\times I$, with base points $p_1=0=(0, \cdots, 0)$, $p_2=(1/2, 0, \cdots, 0)$ respectively. Let T be a path from 0 to p_2 given by $T(t)=(t/2, 0, \cdots, 0)$. Applying (3.5) to the separation element in (4.10), we obtain

$$\begin{aligned}(\alpha+\beta)_u(\xi) &= d(F(ci\times 1), u^\flat(ci\times 1))(\sigma_1, p_1)\\ &\quad + (u^\flat(ci\times 1)T)_\# d(F(ci\times 1), u^\flat(ci\times 1))(\sigma_2, p_2)\\ &= \alpha_u(\xi) + (uci\, T)_\#\beta_u(\xi).\end{aligned}$$

If $q>1$, $uciT$ is the constant path; if $q=1$, it represents $u_*\alpha$. This proves (4.1).

In order to prove (4.6) we need the following lemma:

LEMMA (4.11). *Let* $\phi\in\pi_q(S^n, s_0)$, $\zeta\in\pi_{n+1}(X, x_0)$. *Then*

$$\phi_{x_0}(x_0)_\natural^{-1}(\zeta) = \zeta\circ S_*\phi.$$

Proof. Let $F\colon (S^n\times I, s_0\times I)\to(X, x_0)$ represent $(x_0)_\natural^{-1}\zeta$ (so that $F(S^n\times\dot I)=x_0$), and let $r\colon S^k\times I\to S^{k+1}$ be the identification map, of degree $+1$, which pinches $S^k\times\dot I\cup s_0\times I$ to a point. Then the following diagram commutes, where f represents ϕ, and $F'=Fr^{-1}$:

$$\begin{tikzcd}
(S^q \times I, S^q \times \dot{I} \cup s_0 \times I) \arrow[r, "f \times 1"] \arrow[d, "r"] & (S^n \times I, S^n \times \dot{I} \cup s_0 \times I) \arrow[r, "F"] \arrow[d, "r"] & (X, x_0) \\
(S^{q+1}, s_0) \arrow[r, "Sf"] & (S^{n+1}, s_0) \arrow[ur, "F'"'] &
\end{tikzcd}$$

Clearly

$$\zeta = (x_0)_\natural\{F\} = d(F, \overset{\flat}{x_0})(e^n \times e^1, s_0 \times 0) = \{F'\}.$$

And similarly

$$\begin{aligned}\phi_{x_0}\{F\} &= d(F(f \times 1), \overset{\flat}{x_0}(f \times 1))(e^q \times e^1, s_0 \times 0)\\ &= \{F'(Sf)\} = \{F'\} \circ S_*\phi = \zeta \circ S_*\phi.\end{aligned}$$

Proof of (4.6). Let $g: S^n \to S^n \vee S_0^n$ represent $\iota^n + \iota_0^n$, and let $u = v \vee x_0$: $S^n \vee S_0^n \to X$. We identify

$$\pi_1(X^{S^n \vee S_0^n}, u) = \pi_1(X^{S^n}, v) + \pi_1(X^{S_0^n}, x_0)$$

in the natural way, so that elements of the group may be written $(v_\natural^{-1}\eta,$ $(x_0)_\natural^{-1}\zeta)$, for $\eta, \zeta \in \pi_{n+1}(X, x_0)$; and we further abbreviate this notation to (η, ζ). It is easily verified that

$$\text{(4.12)} \qquad \overset{n}{\iota}_u(\eta, \zeta) = \eta, \qquad (\overset{n}{\iota}_0)_u(\eta, \zeta) = \zeta.$$

Then

$$\text{(4.13)} \qquad \begin{aligned}((\overset{n}{\iota} + \overset{n}{\iota}_0) \circ \phi)_u(0, \zeta) &= \phi_{ug}(ug)_\natural^{-1}(\overset{n}{\iota} + \overset{n}{\iota}_0)_u(0, \zeta)\\ &= \phi_{ug}(ug)_\natural^{-1}\zeta \text{ by (4.1), (4.12),}\\ &= \phi_v v_\natural^{-1}\zeta \text{ by (2.6), (2.8) since } v \simeq ug.\end{aligned}$$

On the other hand, we have the expansion of (4.5)

$$\text{(4.14)} \qquad (\overset{n}{\iota} + \overset{n}{\iota}_0) \circ \phi = \overset{n}{\iota} \circ \phi + \overset{n}{\iota}_0 \circ \phi + \sum_0^\infty \omega_i \circ H_i(\phi),$$

and we may apply the addition theorem to the left-hand side of (4.13) in this expanded form. Since $u_*\iota_0^n = 0$, it follows from (4.4) that the expression

$$(\omega_i \circ H_i(\phi))_u(0, \zeta) = (H_i(\phi))_{x_0}(x_0)_\natural^{-1}(\omega_i)_u(0, \zeta)$$

is 0 if ω_i involves ι_0^n more than once. By definition $\{\sigma_p(\iota^n, \iota_0^n)\}$, $p = -1, 0,$ $1, \cdots$ consists of those basic products which involve ι_0^n only once. If $\omega_{i_p} = \sigma_p$, then writing $B_p(\phi) = H_{i_p}(\phi)$, we have by induction

$$\begin{aligned}\sigma_p(\overset{n}{\iota}, \overset{n}{\iota}_0)_u(0, \zeta) &= -[u_*\overset{n}{\iota}, \sigma_{p-1}(\overset{n}{\iota}, \overset{n}{\iota}_0)_u(0, \zeta)]\\ &= (-1)^{p+1}\sigma_p(u_*\overset{n}{\iota}, (\overset{n}{\iota}_0)_u(0, \zeta))\\ &= (-1)^{p+1}\sigma_p(v_*\overset{n}{\iota}, \zeta),\end{aligned}$$

using (4.12) and the fact that $u_*\iota^n = v_*\iota^n$. Hence

$$(4.15)\qquad \begin{aligned}(\sigma_p(\overset{n}{\iota}, \overset{n}{\iota_0}) \circ B_p(\phi))_u(0, \zeta) &= (B_p(\phi))_{x_0}(x_0)_\natural^{-1}((-1)^{p+1}\sigma_p(v_*\overset{n}{\iota}, \zeta)) \\ &= (-1)^{p+1}\sigma_p(v_*\overset{n}{\iota}, \zeta) \circ S_*B_p(\phi)\end{aligned}$$

by (4.11).

Applying the addition theorem to the left-hand side of (4.13), expanded as in (4.14), and using (4.12) and (4.15) to calculate the terms, we obtain the expression in Theorem (4.6).

5. **Examples.** Using the notation of (3.3), let $L = K \cup e^{q+1}$, where the class of the attaching map is $\alpha \in \pi_q(K, k_0)$, and let $u: (K, k_0) \to (X, x_0)$ have an extension over L. Then to classify the extensions of u, we must compute α_u; and the theorems of the preceding section allow this to be done in certain cases. In particular, if we know the homomorphisms $(\delta_t)_u$ for certain elements $\delta_t \in \pi_{n_t}(K, k_0)$, then we may compute α_u for any α formed from the δ_t by the operations of addition, formation of Whitehead products, and composition with elements of homotopy groups of spheres. In the special case $K = S^{n_1} \vee \cdots \vee S^{n_r}$, Hilton has shown that all elements of the homotopy groups of K can be so formed from the generators $\iota^{n_1}, \cdots, \iota^{n_r}$.

As an example, let $K = S^m \vee S^n$, with $m \leqq n$, and suppose for simplicity that $q < 3m-2$. Let ν, ω denote the classes of $v = u \mid S^m$, $w = u \mid S^n$ respectively. We identify

$$\pi_1(X^K, u) = \pi_1(X^{S^m}, v) + \pi_1(X^{S^n}, w)$$

in the natural way. Abbreviating $(\eta, \zeta) = (v_\natural^{-1}\eta, w_\natural^{-1}\zeta)$, $\eta \in \pi_{m+1}(X, x_0)$, $\zeta \in \pi_{n+1}(X, x_0)$, we compute $\alpha_u(\eta, \zeta) \in \pi_{q+1}(X, x_0)$. Leaving aside the cases $m = 1$ or $n = 1$,

$$\pi_q(K) = \pi_q(S^m) + \pi_q(S^n) + [\iota^m, \iota^n] \circ \pi_q(S^{m+n-1});$$

let $\alpha = \alpha_1 + \alpha_2 + [\iota^m, \iota^n] \circ \beta$, where $\alpha_1 \in \pi_q(S^m)$, $\alpha_2 \in \pi_q(S^n)$, $\beta \in \pi_q(S^{m+n-1})$. Then

$$\alpha_u(\eta, \zeta) = (\alpha_1)_u(\eta, \zeta) + (\alpha_2)_u(\eta, \zeta) + ([\iota^m, \iota^n] \circ \beta)_u(\eta, \zeta).$$

Now from (4.6)

$$\begin{aligned}(\alpha_1)_u(\eta, \zeta) &= \eta \circ S_*\alpha_1 - [\nu, \eta] \circ S_*H(\alpha_1), \\ (\alpha_2)_u(\eta, \zeta) &= \zeta \circ S_*\alpha_2 - [\omega, \zeta] \circ S_*H(\alpha_2)\end{aligned}$$

and from (4.2) and (4.8), since β is a suspension,

$$([\iota^m, \iota^n] \circ \beta)_u(\eta, \zeta) = (-[\nu, \zeta] + (-1)^{n+1}[\eta, \omega]) \circ S_*\beta.$$

This determines $\alpha_u(\eta, \zeta)$ as a sum of these expressions. If $m = n = 1$, then α_u can be found by the addition theorem. If $m = 1 < n$, then α is a sum $\sum \xi_i \cdot \alpha_i$, $\xi_i \in \pi_1(S^1)$, $\alpha_i \in \pi_q(S^n)$. α_u is then given by the addition theorem and (4.3).

As a special case of the example, we consider maps $S_1^n \times S_2^n \to S^n$, $n \geqq 2$; here $\alpha = [\iota_1^n, \iota_2^n]$. If v, w have degrees p, q respectively, $p, q \neq 0$, we say that an extension of u is of type (p, q). The obstruction to such an extension is $u_*[\iota_1^n, \iota_2^n] = pq[\iota^n, \iota^n]$. Suppose that u has an extension: then the homotopy classes of extensions are in 1-1 correspondence with $\pi_{2n}(S^n)/\alpha_u(v_\natural^{-1}\pi_{n+1}(S^n), w_\natural^{-1}\pi_{n+1}(S^n))$. The subgroup contains only the elements 0, $q[\iota^n, \eta]$, $p[\iota^n, \eta]$, if $n \geqq 3$, where η is the generator of $\pi_{n+1}(S^n)$. Now Hilton and Whitehead have shown [4] that $[\iota^n, \eta] \neq 0$ if and only if $n \equiv 1 \bmod 4$. Hence, using known results on Whitehead products,

EXAMPLE (5.1). There exist maps $S_1^n \times S_2^n \to S^n$, $n \geqq 2$, of type (p, q), $p, q \neq 0$, if and only if n is odd, and either pq is even or $\pi_{2n+1}(S^{n+1})$ has an element of Hopf invariant 1. Suppose that p, q, and n are such that maps do exist. Then the homotopy classes of such maps are in 1-1 correspondence with the elements of $\pi_{2n}(S^n)$ if p and q are both even, or if $n \equiv -1 \bmod 4$; otherwise they are in 1-1 correspondence with the elements of

$$\pi_{2n}(S^n)/[\iota^n, \pi_{n+1}(S^n)] = \pi_{2n}(S^n)/Z_2.$$

Other examples are easily given; for instance

EXAMPLE (5.2). The identity map $S^n \to S^n$ always extends to maps $S^n \times S^{n-1} \to S^n$; the homotopy classes of extensions are in 1-1 correspondence with the elements of $\pi_{2n-1}(S^n)/[\iota^n, \pi_n(S^n)] \approx S_*\pi_{2n-1}(S^n)$.

EXAMPLE (5.3). Let u be a map of $S^1 \vee S^1$ into the real projective plane which is nontrivial on both circles. Then there are two homotopy classes rel s_0 of extensions of u to $S^1 \times S^1$.

6. **An application: the group of homotopy equivalences.** We shall outline an application of the above methods to the group of homotopy classes of homotopy equivalences of a space with itself, denoted Eq.

Let K be a 1-connected CW complex, and let $K \cup e^{q+1}$ be formed by attaching a cell e^{q+1}, $q > \dim K$, with $\alpha \in \pi_q(K)$ the class of the attaching map and $\bar{\alpha} \in \pi_{q+1}(K \cup e^{q+1}, K)$ the class of the characteristic map. Let

$$i\colon K \subset K \cup e^{q+1}$$

be the inclusion, and define a homomorphism

$$d^*\colon i_*\pi_{q+1}(K) \to Eq(K \cup e^{q+1})$$

as follows: $d^*(\beta)$ is the homotopy class of an extension g of i such that $d(g, 1)(e^{q+1}) = \beta$, where 1 denotes the identity map of $K \cup e^{q+1}$. Since $q > \dim K$ if f is a homotopy equivalence of $K \cup e^{q+1}$, then $f_*\bar{\alpha} = \epsilon(f)\bar{\alpha}$, where $\epsilon(f) = \pm 1$. We also define homomorphisms

$$j^*\colon Eq(K \cup e^{q+1}) \to Eq(K), \qquad j_0^*\colon Eq(K \cup e^{q+1}) \to Eq(S^{q+1}),$$

by $j^*\{f\} = \{f | K\}$, $j_0^*\{f\} = \epsilon(f)\iota^{q+1}$.

THEOREM (6.1). *The following sequences are exact*:

$$i_*\pi_{q+1}(K) \xrightarrow{d^*} Eq(K \cup e^{q+1}) \xrightarrow{j^*} Eq(K), \qquad \text{if } 2\alpha \neq 0;$$

$$i_*\pi_{q+1}(K) \xrightarrow{d^*} Eq(K \cup e^{q+1}) \xrightarrow{j^* + j_0^*} Eq(K) + Eq(S^{q+1}), \quad \text{if } 2\alpha = 0.$$

From Lemma 7 of [6] it follows that the image of j^* is the set of classes $\{h\}$ such that $h_*\alpha = \pm\alpha$; denote this subgroup by $Eq_e(K)$. The image of $j^*+j_0^*$ is then $Eq_e(K)+Eq(S^{q+1})$, if $2\alpha=0$. The kernel of d^* is

$$i_*\pi_{q+1}(K) \cap \alpha_i\pi_1((K \cup e^{q+1})^K, i)$$

where the base point $k_0 \in K$ is any point of $\dot{e}^{q+1}$. Methods were given in the previous sections for calculating α_i if K is a bunch of spheres, so that in this case we can find $Eq(K \cup e^{q+1})$ up to extension.

The operations of $Eq_e(K)$, or $Eq_e(K)+Eq(S^{q+1})$, on $i_*\pi_{q+1}(K)/i_*\pi_{q+1}(K) \cap \alpha_i\pi_1$ are given as follows: Let $\gamma \in \pi_{q+1}(K)$, $\psi = \{h\} \in Eq(K)$, $\epsilon\iota^{q+1} \in Eq(S^{q+1})$. Then

(i) If $2\alpha \neq 0$, then $\psi \cdot (i_*\gamma) = i_*h_*\gamma$;

(ii) If $2\alpha = 0$, then $(\psi, \epsilon\iota^{q+1}) \cdot (i_*\gamma) = \epsilon i_*h_*\gamma$.

The extension is not known to us, in general.

7. **Further applications.** In this section we shall show how the theory of §§2–4 can be applied to obtain information about Whitehead products.

THEOREM (7.1). *If $\gamma \in \pi_q(S^m)$, then in $\pi_{q+n-1}(S^m \vee S^n)$ we have $[\iota^m{}_0\gamma, \iota^n] = [\iota^m, \iota^n] \circ S_*^{n-1}\gamma + \sum_0^\infty (-1)^{(p+1)(n+1)}\sigma_{p+1}(\iota^m, \iota^n) \circ S_*^{n-1}B_p(\gamma)$, for $m, n > 1$, where $\sigma_{p+1}(\iota^m, \iota^n)$ and $B_p(\gamma)$ are defined as in* (4.6).

Proof. Using the elementary relation

$$[\eta, \iota^1] = \iota^1 \cdot \eta - \eta, \qquad \text{for } \eta \in \pi_q(S^m), \tag{7.2}$$

to expand both sides of the identity $(\iota^1 \cdot \iota^m) \circ \gamma = \iota^1 \cdot (\iota^m \circ \gamma)$, we obtain

$$([\iota^m, \iota^1] + \iota^m) \circ \gamma = [\iota^m \circ \gamma, \iota^1] + \iota^m \circ \gamma. \tag{7.3}$$

Now as shown in the addition theorem, if $u \in X^K$, $\xi \in \pi_1(X^K, u)$, then the transformation $(u, \xi): \pi_q(K, k_0) \to \pi_{q+1}(X, x_0)$ given by $(u, \xi)\alpha = \alpha_u(\xi)$ is a homomorphism for $q > 1$. Taking $K = S^m \vee S^1$, $X = S^m \vee S^2$, u such that $u_*\iota^m = \iota^m$, $u_*\iota^1 = 0$, and ξ such that $\iota_u^m(\xi) = 0$, $\iota_u^1(\xi) = \iota^2$, and applying (u, ξ) to both sides of (7.3), we obtain by use of the composition theorem

$$(-[\iota^m, \iota^2] + 0) \circ S_*\gamma + \sum_0^\infty (-1)^{p+1}\sigma_p(0 + \iota^m, -[\iota^m, \iota^2] + 0) \circ S_*B_p(\gamma)$$
$$= - [\iota^m \circ \gamma, \iota^2] + 0;$$

using the definition of $\sigma_{p+1}(\iota^m, \iota^n)$, this yields the equation in (7.1) for the case $n = 2$.

We can now prove (7.1) by induction on n. Suppose that (7.1) holds for

n, and apply (u, ξ) to both sides of the equation, with $K=S^m\vee S^n$, $X=S^m\vee S^{n+1}$, u such that $u_*\iota^m=\iota^m$, $u_*\iota^n=0$, and ξ such that $\iota_u^m(\xi)=0$, $\iota_u^n(\xi)=\iota^{n+1}$. We obtain

$$-[\iota^m\circ\gamma, \iota^{n+1}] = -[\iota^m, \iota^{n+1}]\circ S_*^n\gamma + \sum_0^\infty(-1)^{(p+1)(n+1)}(-1)^{p+2}\sigma_{p+1}(\iota^m, \iota^{n+1})\circ S_*^nB_p(\gamma)$$

which yields the required equation for $n+1$. This proves (7.1).

Theorem (7.1) may be used as a universal example to derive

COROLLARY (7.4). *If* $\gamma\in\pi_q(S^m)$, $\alpha\in\pi_m(X)$, $\beta\in\pi_n(X)$, m, $n>1$, *then* $[\alpha\circ\gamma, \beta]=[\alpha, \beta]\circ S_*^{n-1}\gamma+\sum_0^\infty(-1)^{(p+1)(n+1)}\sigma_{p+1}(\alpha, \beta)\circ S_*^{n-1}B_p(\gamma)$.

The corollary generalizes a formula of G. W. Whitehead [5, (3.59)] for the case in which γ is a suspension.

As a further application, we give a simple inductive proof of the Jacobi identity for Whitehead products in $\pi_{p+q+r-2}(S^p\vee S^q\vee S^r)$ (cf. [3] et al.). With our conventions for the Whitehead product, the identity is given by

THEOREM (7.5).

$$(-1)^{(p+1)r}[[\iota^p, \iota^q], \iota^r] + (-1)^{(r+1)q}[[\iota^r, \iota^p], \iota^q] + (-1)^{(q+1)p}[[\iota^q, \iota^r], \iota^p] = 0, \qquad p, q, r \geqq 2.$$

Proof. It is elementary that the following relation holds in $\pi_2(S^2\vee S_0^2\vee S^1)$: $\iota^1\cdot[\iota^2, \iota_0^2]=[\iota^1\cdot\iota^2, \iota^1\cdot\iota_0^2]$. Expanding both sides by (7.2),

$$[[\iota^2, \iota_0^2], \iota^1] + [\iota^2, \iota_0^2] = [[\iota^2, \iota^1] + \iota^2, [\iota_0^2, \iota^1] + \iota_0^2]. \tag{7.6}$$

Choosing $K=S^2\vee S_0^2\vee S^1$, $X=S^2\vee S_0^2\vee S_1^2$ $u\in X^K$ such that $u_*\iota^2=\iota^2$, $u_*\iota_0^2=\iota_0^2$, $u_*\iota^1=0$, $\xi\in\pi_1(X^K, u)$ such that $\iota_u^2(\xi)=(\iota_0^2)_u(\xi)=0$, $\iota_u^1(\xi)=\iota_1^2$, and applying (u, ξ) to both sides of (7.6) we obtain $-[[\iota^2, \iota_0^2], \iota_1^2]=[\iota_0^2, [\iota_0^2, \iota_1^2]]+[[\iota^2, \iota_1^2], \iota_0^2]$ which yields (7.5) for $p=q=r=2$.

Suppose inductively that the identity of (7.5) holds for p, q, r. Taking $K=S^p\vee S^q\vee S^r$, $X=S^p\vee S^q\vee S^{r+1}$, $u\in X^K$ such that $u_*\iota^p=\iota^p$, $u_*\iota^q=\iota^q$, $u_*\iota^r=0$, and $\xi\in\pi_1(X^K, u)$ such that $\iota_u^p(\xi)=\iota_u^q(\xi)=0$, $\iota_u^r(\xi)=\iota^{r+1}$, by applying (u, ξ) to both sides of the equality in (7.5) we obtain the same equality with r replaced by $r+1$. This proves (7.5).

The proof could equally well start with the relation $\iota^1\cdot[\iota_0^1, \iota_1^1]=[\iota^1\cdot\iota_0^1, \iota^1\cdot\iota_1^1]$ which can be verified purely formally.

Notice that if we apply homomorphisms (u, ξ) to both sides of (7.6) with u and ξ appropriately chosen to raise the dimensions of ι^2 and ι_0^2, but leave ι^1 fixed, then we obtain a generalization of the Jacobi identity for the case in which one factor is of dimension 1; this can be written

$$(7.7)\quad \begin{aligned}&(-1)^{p+1}[[\iota^p, \iota^q], \iota^1] + [[\iota^1, \iota^p], \iota^q]\\ &\qquad + (-1)^{(q+1)p}[[\iota^q, \iota^1], \iota^p] + [[\iota^1, \iota^p], [\iota^q, \iota^1]] = 0.\end{aligned}$$

Equation (7.7) also follows directly from the properties of the operation of π_1, in the manner of (7.6)

Theorem (7.5) is a universal example for the Jacobi identity in the homotopy groups of any space.

8. **Proof of the product theorem.** We shall now prove Theorem (4.2). As universal examples for K, γ, and δ we take $S^m \vee S^n$, ι^m, and ι^n respectively. Then if K, γ, and δ are arbitrary, there is a map $h: S^m \vee S^n \rightarrow K$ such that $h_* \iota^m = \gamma$ and $h_* \iota^n = \delta$. Since

$$[\gamma, \delta]_u = (h_*[\iota^m, \iota^n])_u = [\iota^m, \iota^n]_{uh} h^*,$$

and $\gamma_u = \iota^m_{uh} h^*$, $\delta_u = \iota^n_{uh} h^*$, one verifies immediately that Theorem (4.2) for the general case follows if we have proved it for the universal example.

Let $w: S^m \rightarrow X$, $v: S^n \rightarrow X$ define $u = w \vee v: S^m \vee S^n \rightarrow X$; we identify $\pi_1(X^{S^m \vee S^n}, u) = \pi_1(X^{S^m}, w) + \pi_1(X^{S^n}, v)$ under the natural isomorphism, so that there is a natural isomorphism $(w_\natural^{-1}, v_\natural^{-1}): \pi_{m+1} + \pi_{n+1} \rightarrow \pi_1(X^{S^m \vee S^n}, u)$, where $\pi_k = \pi_k(X, x_0)$. Let

$$\kappa = [\iota^m, \iota^n]_u (w_\natural^{-1}, v_\natural^{-1}): \pi_{m+1} + \pi_{n+1} \rightarrow \pi_{m+n};$$

let ω, ν denote $w_* \iota^m$, $v_* \iota^n$, and let $\lambda \in \pi_{m+1}$, $\rho \in \pi_{n+1}$. Then Theorem (4.2) for the universal example can be written

$$(8.1)\quad \begin{array}{llll} \text{(i)} & \kappa(\lambda, \rho) = -[\omega, \rho] + (-1)^{n+1}[\lambda, \nu] & & \text{if } m, n > 1,\\ \text{(ii)} & \kappa(\lambda, \rho) = -[\omega, \rho] + (-1)^{n+1}[\lambda, \omega \cdot \nu] & & \text{if } m = 1, n > 1,\\ \text{(iii)} & \kappa(\lambda, \rho) = -[\nu \cdot \omega, \rho] + (-1)^{n+1}[\lambda, \nu] & & \text{if } m > 1, n = 1,\\ \text{(iv)} & \kappa(\lambda, \rho) = -[\nu \cdot \omega, \rho] - \nu \cdot [\lambda, -(\omega \cdot \nu)] & & \text{if } m = n = 1. \end{array}$$

We write the fundamental group additively, and shall first deduce (8.1) (iv) from the addition theorem. In this case $S^m \vee S^n = S^1 \vee S^1$, and we set $\iota = \iota^m$, $\iota' = \iota^n$. Then for

$$\xi = (w_\natural^{-1}\lambda, v_\natural^{-1}\rho) \in \pi_1(X^{S^1 \vee S^1}, u),$$

it is clear that $\iota_u(\xi) = \lambda$, $\iota'_u(\xi) = \rho$. Now $[\iota, \iota'] = (\iota' + \iota) - (\iota + \iota')$, and

$$(\iota' + \iota)_u(\xi) = \iota'_u(\xi) + u_* \iota' \cdot \iota_u(\xi) = \rho + \nu \cdot \lambda.$$

Since

$$\begin{aligned}(-(\iota + \iota') + (\iota + \iota'))_u(\xi) &= (-(\iota + \iota'))_u(\xi) + (-(\omega + \nu)) \cdot (\iota + \iota')_u(\xi),\\ (-(\iota + \iota'))_u(\xi) &= -(-\nu - \omega) \cdot (\lambda + \omega \cdot \rho).\end{aligned}$$

Therefore

$$\begin{aligned}\kappa(\lambda, \rho) &= [\iota, \iota']_u(\xi) = ((\iota' + \iota) - (\iota + \iota'))_u(\xi)\\ &= \rho + \nu\cdot\lambda - (\nu + \omega)\cdot((-\nu - \omega)\cdot(\lambda + \omega\cdot\rho))\\ &= \nu\cdot(\lambda - (\omega - \nu - \omega)\cdot\lambda) + \rho - (\nu + \omega - \nu)\cdot\rho\\ &= -\nu\cdot[\lambda, \omega\cdot(-\nu)] - [\rho, \nu\cdot\omega].\end{aligned} \tag{8.2}$$

This proves (8.1)(iv).

We can now suppose that $m+n>2$; if we prove that

$$\begin{aligned}\kappa(0, \rho) &= -[\omega, \rho] && \text{if } n > 1,\\ &= -[\nu\cdot\omega, \rho] && \text{if } n = 1,\end{aligned} \tag{8.3}$$

it will follow that

$$\begin{aligned}\kappa(\lambda, 0) &= [\overset{m}{\iota}, \overset{n}{\iota}]_u(\overset{-1}{w_\natural}\lambda, \overset{-1}{v_\natural}0) = ((-1)^{mn}[\overset{n}{\iota}, \overset{m}{\iota}])_u(\overset{-1}{w_\natural}\lambda, \overset{-1}{v_\natural}0)\\ &= (-1)^{mn+1}[\nu, \lambda] \quad\text{or}\quad (-1)^{n+1}[\omega\cdot\nu, \lambda]\\ &= (-1)^{n+1}[\lambda, \nu] \quad\text{or}\quad (-1)^{n+1}[\lambda, \omega\cdot\nu]\end{aligned}$$

according as $m>1$ or $m=1$. Then (8.3) implies that $\kappa(\lambda, \rho)=\kappa(\lambda, 0)+\kappa(0, \rho)$ is given by (8.1)(i), (ii), or (iii), and we need only prove (8.3).

Consider the case $(X, x_0)=(S^m \vee S^{n+1}, s_0)$, $w=j^m$, the identity map of S^m, $v=s_0$, $\rho=\iota^{n+1}$, where $m, n\geqq 1$ and $m+n>2$; we prove by considering representative maps that

LEMMA (8.4). *In this case* $\kappa(0, \iota^{n+1})=-[\iota^m, \iota^{n+1}]$.

Proof. $[\iota^m, \iota^n]$ is represented by $p\colon \dot{I}^{m+n}=(I^m\times I^n)^\cdot\to S^m\vee S^n$,

$$\begin{aligned}p(y, y') &= i^m(y) && \text{if } y\in I^m,\ y'\in \dot{I}^n,\\ &= i^n(y') && \text{if } y\in \dot{I}^m,\ y'\in I^n.\end{aligned}$$

Define maps E, F: $((S^m\vee S^n)\times I, s_0\times I)\to(S^m\vee S^{n+1}, s_0)$ by

$$\begin{aligned}E(z, t) &= z, && z\in S^m,\ t\in I,\\ &= s_0, && z\in S^n;\\ F(z, t) &= z, && z\in S^m,\\ &= i^{n+1}((i^n)^{-1}(z), t), && z\in S^n.\end{aligned}$$

Then $F(p\times 1)$, $E(p\times 1)\colon(\dot{I}^{m+n}\times I,\ 0\times I)\to(S^m\vee S^{n+1},\ s_0)$ agree on $\dot{I}^{m+n}\times\dot{I}\cup 0\times I$; and since F represents $((j^m)_\natural^{-1}0, s_{0\natural}^{-1}\iota^{n+1})\in\pi_1(X^{S^m\vee S^n}, u)$, we have

$$\kappa(0, \iota^{n+1}) = d(F(p\times 1), (E(p\times 1))((\dot{I}^{m+n}-0)\times I, 0\times 0).$$

Extend $F(p\times 1)$, $E(p\times 1)$ over $\dot{I}^{m+n+1}=\dot{I}^{m+n}\times I\cup I^{m+n}\times\dot{I}$ to $\overline{F}$, $\overline{E}$ respectively, as follows: for $y\in I^m$, $y'\in I^n$, $t\in\dot{I}$ define

$$\begin{aligned}\overline{F}(y, y', t) &= i^m(y) && \text{if } y'\in\dot{I}^n,\\ &= i^{n+1}(y', t) = s_0 && \text{if } y\in\dot{I}^m;\end{aligned}$$

and the same for $\bar{E}$. $\bar{F}$ and $\bar{E}$ are readily seen to be the canonical maps representing $[\iota^m, \iota^{n+1}]$ and $[\iota^m, 0]$ respectively; also, $\bar{F}$ and $\bar{E}$ agree on $I^{m+n}\times \dot{I}\cup 0\times I$. Setting $e^{m+n+1}=\dot{I}^{m+n+1}-0$, and applying Lemma (3.5) as in the proof of (3.2), it follows that $d(F(p\times 1), E(p\times 1))=d(\bar{F}|\dot{I}^{m+n}\times I, \bar{E}|\dot{I}^{m+n}\times I)$ on $((\dot{I}^{m+n}-0)\times I, 0\times 0)$ is equal to $-d(\bar{F}, \bar{E})(e^{m+n+1}, 0)$. Thus

$$\kappa(0, \iota^{n+1}) = -d(\bar{F}, \bar{E})(e^{m+n+1}, 0) = -(\{\bar{F}\}-\{\bar{E}\}) = -\{\bar{F}\} = -[\iota^m, \iota^{n+1}]$$

which proves (8.4).

The space $S^m\vee S^{n+1}$ in (8.4) is a universal example for the case $v=x_0$; for, given any (X, x_0), w, ρ, there exists $g\colon (S^m\vee S^n, s_0)\to(X, x_0)$ such that $g|(S^m, s_0)=w$ and $\{g|S^{n+1}\}=\rho$.

COROLLARY (8.5). *$\kappa(0, \rho)=-[\omega, \rho]$ if $v=x_0$, with X, w, ρ arbitrary.*

Now let all of X, w, v, ρ be arbitrary. Define $h=j^m\vee h'\colon (S^m\vee S^n, s_0)\to(S^m\vee S_1^n\vee S_2^n, s_0)$, where $h'\colon S^n\to S_1^n\vee S_2^n$ is such that $h'_*\iota^n=\iota_1^n+\iota_2^n$. Let $\bar{v}=(x_0\vee v)h'$; then $h^*\colon \pi_1(X^{S^m\vee S_1^n\vee S_2^n}, w\vee x_0\vee v)\to\pi_1(X^{S^m\vee S^n}, w\vee\bar{v})$. We identify the second group with $\pi_1(X^{S^m}, w)+\pi_1(X^{S^n}, \bar{v})$, and treat the first similarly. Then it is clear from the definition of $\natural$ as a separation element, and from the definition of h^*, that

$$\overset{*}{h}(\overset{-1}{w_\natural}\lambda, \overset{-1}{x_{0\natural}}\rho_1, \overset{-1}{v_\natural}\rho_2) = (\overset{-1}{w_\natural}\lambda, \overset{-1}{\bar{v}_\natural}(\rho_1+\rho_2)).$$

Let M be a homotopy rel S^m from $w\vee\bar{v}$ to $u=w\vee v$; under the above identification $M_\#=((M|S^m)_\#, (M|S^n)_\#)$. Since $v_\natural=(M|S^n)_\#\bar{v}_\natural$ by (2.6), and $(M|S^m)_\#$ is the identity,

$$M_\#(\overset{-1}{w_\natural}\lambda, \overset{-1}{\bar{v}_\natural}(\rho_1+\rho_2)) = (\overset{-1}{w_\natural}\lambda, \overset{-1}{v_\natural}(\rho_1+\rho_2)).$$

Setting $r=w\vee x_0\vee v$,

$$[\iota^m\ \iota_1^n+\iota_2^n]_r = (h_*[\iota^m, \iota^n])_r = [\iota^m, \iota^n]_{rh}h^* = [\iota^m, \iota^n]_u M_\# h^*$$

and hence

$$\begin{aligned} \kappa(0, \rho) &= [\iota^m, \iota^n]_u(\overset{-1}{w_\natural}0, \overset{-1}{v_\natural}\rho) \\ &= [\iota^m, \iota^n]_u M_\# h^*(\overset{-1}{w_\natural}0, \overset{-1}{x_{0\natural}}\rho, \overset{-1}{v_\natural}0) \qquad (8.6)\\ &= [\iota^m, \iota_1^n+\iota_2^n]_r(\overset{-1}{w_\natural}0, \overset{-1}{x_{0\natural}}\rho, \overset{-1}{v_\natural}0). \end{aligned}$$

If $m\geqq 1$, $n>1$, then it follows from the addition theorem that $[\iota^m, \iota_1^n+\iota_2^n]_r=[\iota^m, \iota_1^n]_r+[\iota^m, \iota_2^n]_r$. The first term yields $[\iota^m, \iota_1^n]_r(w_\natural^{-1}0, x_{0\natural}^{-1}\rho, v_\natural^{-1}0)=-[\omega, \rho]$ by (8.5), while the second term yields 0. Hence

$$\kappa(0, \rho) = -[\omega, \rho] \qquad \text{for } m\geqq 1,\ n>1. \tag{8.7}$$

If $m>1$, $n=1$, then

$$[\iota^m, \iota_1^n+\iota_2^n] = (\iota_1^n+\iota_2^n)\cdot\iota^m - \iota^m = [\iota_2^n\cdot\iota^m, \iota_1^n] + [\iota^m, \iota_2^n].$$

Applying the addition theorem, and noting that the second term again gives 0,

$$(8.8)\qquad [\iota^m, \iota_1^n + \iota_2^n]_r(w_\natural^{-1}0,\ x_{0\natural}^{-1}\rho,\ v_\natural^{-1}0) = [\iota_2^n\cdot\iota^m, \iota_1^n]_r(w_\natural^{-1}0,\ x_{0\natural}^{-1}\rho,\ v_\natural^{-1}0).$$

Let $k=l\vee j_1^n\colon (S^m\vee S^n, s_0)\to(S^m\vee S_1^n\vee S_2^n, s_0)$, where l represents $\iota_2^n\cdot\iota^m$, so that rk represents $\nu\cdot\omega$ on S^m and 0 on S^n. Then $k^*\colon \pi_1(X^{S^m\vee S_1^n\vee S_2^n}, r)\to\pi_1(X^{S^m\vee S^n}, rk)$ is clearly such that

$$k^*(w_\natural^{-1}0,\ x_{0\natural}^{-1}\rho,\ v_\natural^{-1}0) = (l_\natural^{-1}0,\ x_{0\natural}^{-1}\rho).$$

Since $k_*[\iota^m, \iota^n]=[\iota_2^n\cdot\iota^m, \iota_1^n]$,

$$(8.9)\qquad \begin{aligned}[\iota_2^n\cdot\iota^m, \iota_1^n]_r(w_\natural^{-1}0,\ x_{0\natural}^{-1}\rho,\ v_\natural^{-1}0) &= [\iota^m, \iota^n]_{rk}k^*(w_\natural^{-1}0,\ x_{0\natural}^{-1}\rho,\ v_\natural^{-1}0)\\ &= [\iota^m, \iota^n]_{rk}(l_\natural^{-1}0,\ x_{0\natural}^{-1}\rho) = -[\nu\cdot\omega, \rho] \text{ by (8.5).}\end{aligned}$$

Equations (8.6), (8.8), and (8.9) yield

$$(8.10)\qquad \kappa(0, \rho) = -[\nu\cdot\omega, \rho] \qquad \text{if } m>1,\ n=1.$$

Equations (8.7) and (8.10) together prove (8.3), and hence Theorem (4.2).

Appendix. Separation elements

Let I^n be the subset of Euclidean n-space consisting of n-tuples of real numbers $(y_1, \cdots, y_n)$, $0\leqq y_i\leqq 1$, oriented by the generator of $H_n(I^n, \dot I^n)$ represented by the identity map of I^n in the cubical singular theory. Let J^{n-1} be the closure of the subset of $\dot I^n$ for which $y_n<1$, and let I_1^{n-1} be the subset of $\dot I^n$ for which $y_n=1$. If $x_0\in A\subset X$, then elements of $\pi_n(X, A, x_0)$ are represented by maps $f\colon (I^n, \dot I^n, J^{n-1})\to(X, A, x_0)$, and the boundary operator

$$\partial\colon \pi_n(X, A, x_0)\to\pi_{n-1}(A, x_0, x_0) = \pi_{n-1}(A, x_0)$$

is defined by $\partial\{f\}=\{f| I_1^{n-1}\}$. If we identify $(S^{n-1}, s_0)=(\dot I^n, 0)$, where $0=(0, \cdots, 0)$, then the specification of the boundary operator determines an orientation of S^{n-1} (cf. [7, §4]). It is to be noted that this is not the orientation given by the homology boundary.

Let $h_t\colon (I^n, \dot I^n, J^{n-1}, 0)\to(I^n, \dot I^n, J^{n-1}, 0)$ be a homotopy such that h_0 = identity, $h_1(J^{n-1})=0$. h_t determines a 1-1 correspondence between the sets of homotopy classes of maps $g\colon (I^n, \dot I^n, 0)\to(X, A, x_0)$ and the elements of $\pi_n(X, A, x_0)$ by $\{g\}\to\{gh_1\}$, and similarly between the homotopy classes of maps $g'\colon(\dot I^n, 0)\to(A, x_0)$ and the elements of $\pi_{n-1}(A, x_0)$. Using this correspondence, we may represent elements of $\pi_n(X, A, x_0)$ and $\pi_{n-1}(A, x_0)$ by maps of $(I^n, \dot I^n, 0)$ and $(\dot I^n, 0)$ respectively.

We define separation elements as follows (cf. [1] for the original definition). Let K be a CW complex(2) and let $\sigma\in K$ be an n-cell with characteristic map $c_\sigma\colon (I^n, \dot I^n, 0)\to(\bar\sigma, \dot\sigma, p_\sigma)$, where $p_\sigma\in\dot\sigma$ is a point. If $f, g\colon(\bar\sigma, p_\sigma)\to(X, x_0)$

(2) A fixed choice of characteristic map for each cell is implied in the definition of a CW complex.

agree on $\dot{\sigma}$, they determine a *separation element* $d(f, g)(\sigma, p_\sigma) \in \pi_n(X, x_0)$, represented by $F: (\dot{I}^{n+1}, 0) \to (X, x_0)$,

$$F(y_1, \cdots, y_{n+1}) = \begin{cases} fc_\sigma(y_1, \cdots, y_n) & \text{if } y_{n+1} = 1, \\ fc_\sigma(y_1, \cdots, y_n) = gc_\sigma(y_1, \cdots, y_n) & 0 < y_{n+1} < 1, \\ gc_\sigma(y_1, \cdots, y_n) & \text{if } y_{n+1} = 0. \end{cases}$$

Thus $d(f, g)(\sigma, p_\sigma) = d(fc_\sigma, gc_\sigma)(I^n, 0)$ (we shall not bother to distinguish between the open and the closed cell, provided this causes no confusion).

It follows from the orientation convention that if $f(\dot{\sigma}) = x_0$, $g(\bar{\sigma}) = x_0$, then $d(f, g)(\sigma, p_\sigma) = \{fc_\sigma\}$.

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