On the Hydrostatic Theory of the Figure of the Earth

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Summary

A table is provided of the external and dynamical ellipticities of the Earth for different values of the ratio I/Mb^2, I being the mean moment of inertia and b the mean radius, on the first order Radau theory. A simplified model is used to find the corrections for the slight inaccuracy of the Radau approximation and the terms of the second order.

The dynamical ellipticity and J_2 as given by artificial satellites are used to estimate I/Mb^2, and it is shown that on the hydrostatic theory $10^6 J_2$ would be $1072 \cdot 1 \pm 0 \cdot 4$, in contradiction to the direct determination $1082 \cdot 78 \pm 0 \cdot 05$. The departures from the hydrostatic state indicated by J_2 and J_3 imply stress differences of 4×10^7 to 8×10^7 dyn/cm^2 in the interior.

1. The first-order theory of the figure of the Earth, on the assumption of hydrostatic pressure in the interior, goes back to Clairaut, and an important numerical simplification was given by Radau. Theories taking account of second-order terms were given by Callandreau, Darwin and de Sitter. Since the work of Radau the results have been used chiefly to estimate the ellipticity from the precessional constant (dynamical ellipticity); it is shown that over a wide range of possible structures the relation between these quantities is nearly independent of structure.

Results from artificial satellites have however shown a definite discrepancy, and an entirely new approach has been made by S. W. Henriksen (1960). The perturbations of the satellites give a very accurate determination of J, the coefficient of the second harmonic in the Earth's field, and the ratio of J to the dynamical ellipticity H gives C/Ma^2, where C is the moment of inertia about the polar axis, M the mass, and a the equatorial radius. Now the mass, the radius of the sphere of equal volume b, and the mean moment of inertia I are presumably nearly unalterable, and I/Mb^2 can be found from C/Ma^2 with a second-order error. With this value the theory can be used to estimate all of e, J and H and compared with observation. However Henriksen appears to have used de Sitter's formulae, in which there are some numerical mistakes (Jeffreys 1953).

The Radau first-order theory puts a certain function $\psi(\eta)$ equal to 1; this approximation is very close through the whole range of η. I shall speak of this as the simplified theory. The corrections needed are for the slight difference between ψ and 1 and for the second-order terms. It seems desirable that they should be presented in three parts: (1) an interpolable table of e and H in terms of I/Mb^2 based on the simplified theory, (2) an estimate of the correction for $\psi - 1$, (3) an estimate of the effects of second-order terms. Since (2) and (3) are small it will be enough to evaluate them for any approximate model. With this method there will be no need to carry out the full calculation for every new model that may be suggested.

2. On the simplified theory the parameter η is defined by

$$\eta = \frac{r}{e}\frac{de}{dr} \tag{1}$$

and its value at the surface is

$$\eta(1) = \frac{5}{2}\frac{m}{e(1)} - 2 \tag{2}$$

where

$$m = 0{\cdot}0034498 \tag{3}$$

to five significant figures. We have also

$$J = e(1) - \tfrac{1}{2}m = \frac{3}{2}\frac{C-A}{Ma^2}; \quad H = \frac{C-A}{C} = \frac{J}{3C/2Ma^2} \tag{4}$$

$$\frac{3}{2}\frac{I(r)}{M(r)r^2} = 1 - \tfrac{2}{5}\sqrt{(1-\eta)}. \tag{5}$$

I find the following values; suffix 0 indicates the simplified theory.

Table 1

$\frac{3}{2}\frac{I}{Mb^2}$	$\eta_0(1)$	$100e_0(1)$	$100(e_0(1)-\frac{1}{2}m)$	$100H_0$
0·48	0·6900	0·32061	0·14812	0·30858
0·49	0·6256	0·32848	0·15599	0·31835
0·50	0·5626	0·33657	0·16408	0·32816
0·51	0·5006	0·34490	0·17241	0·33806
0·52	0·4400	0·35346	0·18097	0·34802

These values of e, J and H are really the terms of order m in the complete expansions in powers of m.

3. When η is found, for any distribution of density, e follows by integration except for a constant factor. Bullard (1948) used a distribution based on one of Bullen's (the same, I think, as the one I interpolated to decimals of the radius in "The Earth") but this is rather complicated, and in particular the jump in density at the inner core may be too great. Even one discontinuity was found to lead to considerable trouble in the numerical integration. Since details are needed only for the estimation of corrections, I thought it best to use a simplified model, representing the differences of density between the top of the shell* and the upper side of the core boundary, and between the top of the core and just outside the inner core; constants were added to give mean density about 5·517 and $\frac{3}{2}\ I/Mb^2$

* Wiechert's word was Mantel, which I have always translated by *shell*. I think the current use of *mantle* is unfortunate, since in English it suggests something soft and floppy. *Mantel*, according to my dictionary, has a secondary meaning *casing* (of a cylinder) and Wiechert may have had this in mind.

about 0·5. Then the adopted form is:

$$\begin{aligned} 0 \leqslant r < 0{\cdot}55, \quad \rho &= 12{\cdot}010 - 7{\cdot}339r^2 \\ 0{\cdot}55 < r \leqslant 1, \quad \rho &= 6{\cdot}799 - 3{\cdot}384r^2. \end{aligned} \tag{6}$$

The radius is taken as 1. These make

$$\bar{\rho} = 5{\cdot}515, \frac{3}{2}\frac{I}{Mb^2} = 0{\cdot}5006. \tag{7}$$

It is to be emphasized that this smoothed distribution is not intended as an improvement on the various distributions that have been worked out, especially by Bullen, Bullard and Birch. It was chosen simply to make numerical integration easier. The densities agree with the model used in "The Earth" (§ 4·06) within 0·2 except in the crustal layers and the inner core. The simplified theory must give the same result at the outer surface for all models with the same value of I/Mb^2; but the density distribution does affect the distribution of e in the interior. I think that it would be premature to attempt a solution in more detail at present, since it is sure to be revised on account of changes found in the bulk modulus at small depths and in the interpretation of the inner core and the curious region just outside it. But it is to be expected that any conclusions drawn from the present model about the small corrections that are the main object of the present paper will be right within about 3 per cent, and this is ample for present application.

Intervals of 0·05 of r were used in most of the range, but η varies rapidly just outside the core, and from 0·55 to 0·75 intervals of 0·02 were used. Specimen results are in Table 2.

Table 2

r	ρ	η	$100e$	r	ρ	η	$100e$
1·00	3·415	0·5587	0·33707	0·55	9·790	0·0708	0·26178
0·90	4·058	0·5232	0·31824	0·50	10·175	0·0584	0·26016
0·80	4·633	0·4962	0·29991	0·40	10·836	0·0373	0·25748
0·75	4·895	0·4757	0·29063	0·30	11·349	0·0205	0·25538
0·65	5·369	0·3739	0·27265	0·20	11·716	0·0090	0·25394
0·63	5·456	0·3350	0·26964	0·10	11·992	0·0020	0·25308
0·61	5·540	0·2870	0·26693	0·00	12·010	0·0000	0·25285
0·59	5·621	0·2294	0·26463				
0·57	5·700	0·1603	0·26286				
0·55	5·775	0·0708	0·26178				

The surface value of e^{-1} is 296·67.

4. **Effect of variation of ψ.**

$$\psi(\eta) = \frac{1 + \frac{1}{2}\eta - \frac{1}{10}\eta^2}{\sqrt{(1+\eta)}}, \tag{8}$$

in all models that have been used, is close to 1; I find the following values of $10^4(\psi - 1)$. Only one figure is needed.

	0	0·05	0·10	0·15	0·20	0·25	0·30	0·35	0·40	0·45	0·50	0·55	0·60
$10^4(\psi-1)$	0	0	2	3	5	7	7	8	7	5	2	−2	−7

We have, in the notation used in "The Earth", especially § 4·03 (38)

$$\rho_0 r^5 \sqrt{(1+\eta)} = \int_0^r 5\rho_0 r^4 \psi(\eta) d\eta \tag{9}$$

and $\rho_0 r^3$ is a simple multiple of $M(r)$. To find the correction to the surface value of $\sqrt{(1+\eta)}$ we therefore need only compare the integrals of $M(r)r$ and $M(r)r(\psi-1)$. This is found to make the mean value of $\psi-1$ equal to $+1{\cdot}3\times 10^{-4}$. This small value is due, as Bullard pointed out, to the negative values where $\eta > 0{\cdot}52$ or so being multiplied by the largest values of r^4.

Then this change multiplies $1+\eta$ by $1{\cdot}00026$, and the revised value of e given by (2) is $0{\cdot}0033701$. The simplified theory with $^{3/2}I/Mb^2 = 0{\cdot}5006$ gives $0{\cdot}0033707$; hence the correction to e for $\psi-1$ is -6×10^{-7}. The same correction applies to J, and that to H will be -18×10^{-7}.

5. The second order terms.

The integral equation for e', in the notation of Jeffreys (1953) is

$$(e'+\tfrac{2}{7}e^2)\rho_0 - \tfrac{3}{5}(S+T) - \tfrac{1}{2}\bar{\rho}m = \tfrac{4}{21}e(\bar{\rho}m-3T), \tag{10}$$

and if e' is e_1, the solution so far obtained, and the terms in e^2 and em are neglected, the left side will vanish. Put then

$$e' = e_1 + \delta e \tag{11}$$

and rearrange; we have

$$\rho_0\delta e - \frac{3}{5}\left\{\frac{1}{r^5}\int_0^r \rho d(a'^5\delta e) + \int_r^b \rho d\delta e\right\}$$

$$= -\frac{2}{7}\rho_0 e^2 + \frac{5}{35}\frac{1}{r^5}\int_0^r \rho d(a^{15}e^2) + \frac{16}{35}\int_r^b \rho de^2$$

$$+\frac{4}{21}e\left(\bar{\rho}m - 3\int_r^b \rho de\right) = f(r). \tag{12}$$

The procedure hitherto has been to convert (10) into a second-order differential equation with two boundary conditions, expressing that the outer surface is hydrostatic and that e is finite at the centre; then (2), corrected for second-order terms, is a consequence. Darwin and de Sitter allowed for the second-order terms in the differential equation by a modification of ψ. I suggested that it is probably easier and less productive of mistakes in arithmetic to solve (12) directly as an integral equation by successive approximation. The first approximation given by the simplified theory is already very close, and the function on the right can be worked out directly. It ranges from $+28\times 10^{-7}$ at the surface to $+49\times 10^{-7}$ at $r = 0{\cdot}57$, does a curious but apparently genuine dip to 32×10^{-7} at $r = 0{\cdot}55$, and rises again to $+64\times 10^{-7}$ at the centre. I showed that a convergent solution can be found by simply dropping the two integrals on the left and proceeding by successive substitution; but convergence was likely to be slow. I therefore took an approximation in the form

$$\delta e = A - 10^4 B e_0^2$$

with A, B constant, and adjusted A, B so that the residuals at $r = 1$, $0{\cdot}55$, and $0{\cdot}00$ would have equal modulii, about $\pm\, 8 \times 10^{-7}$. The required integrals had already been worked out. The results were

$$A = 13{\cdot}3 \times 10^{-7}; \quad B = 49 \times 10^{-7}$$

and the calculated values of δe were $+7{\cdot}7 \times 10^{-7}$, $+9{\cdot}9 \times 10^{-7}$, and $+10{\cdot}2 \times 10^{-7}$. A rough further approximation gave

r	1·00	0·55	0·00
$10^7\,\delta e$	9·2	9·1	11·0

The total correction to e' at the surface is therefore $+3 \times 10^{-7}$, which is much smaller than e^2.

Finally we have to allow for the difference between e' and the ellipticity e:

$$e = e' + \frac{5}{42} e^2$$

and the second term at the surface is $+13{\cdot}6 \times 10^{-7}$. The total correction at the surface is about 16×10^{-7}.

J is defined as in "The Earth", p. 136(13):

$$J = \frac{3}{2}\frac{C-A}{Ma^2} = e - \tfrac{1}{2}m + e(-\tfrac{1}{2}e + \tfrac{1}{7}m) \tag{13}$$

and the correction is δe + the second order term $= -24 \times 10^{-7}$.

Note that this J is not the same as J of my 1953 paper. J_2 of artificial satellite theory is $\frac{2}{3}J$.

The dynamical ellipticity is $(C-A)/C$, but in a second-order theory we must distinguish C in the denominator from I, the mean moment of inertia:

$$C = I(1 + \tfrac{2}{3} H). \tag{14}$$

Then the correction to H (including that mentioned at the end of § 2) is

$$\frac{\delta J}{3I/2Ma^2} - \frac{2}{3}\left(\frac{J}{3I/2Ma^2}\right)^2 = -120 \times 10^{-7}. \tag{15}$$

The integral equation for the fourth harmonic is

$$8\kappa\rho_0 - \frac{8}{3r^7}\int_0^r \rho\, d(a'^7\kappa) - \frac{8}{3}\int_r^b \rho\, d\left(\frac{\kappa}{a'^2}\right)$$

$$= 3\rho_0 e^2 - \frac{6e}{r^5}\int_0^r \rho\, d(a'^5 e) + \frac{3}{r^7}\int_r^b \rho\, d(a'^7 e^2) = g(r) \tag{16}$$

The three terms in $g(r)$ nearly cancel for $r \leqslant 0{\cdot}7$; $10^4 g(r)$ ranges from $0{\cdot}077$ to $0{\cdot}234$ from $r = 0{\cdot}95$ to $1{\cdot}00$. Then the first approximation to $\kappa(1)$ is $+5{\cdot}3 \times 10^{-7}$, but the second term on the left will be about $\frac{1}{3}\,\rho(1)/\bar{\rho} = 0{\cdot}20$ of the first and the second approximation will give $\kappa(1) = 6{\cdot}4 \times 10^{-7}$.

For given I/Mb^2, then, e and H in Table 1 must be increased by $+16$ and -120 in the last place given. With $H = 0{\cdot}0032726 \pm 0{\cdot}0000007$ this would make

$$I/Mb^2 = 0{\cdot}5003 \pm 0{\cdot}0001,\ e = 0{\cdot}0033698 \pm 0{\cdot}0000006,\ e^{-1} = 296{\cdot}75 \pm 0{\cdot}05 \tag{17}$$

The result for e^{-1} appears to differ appreciably from Bullard's; he gets $e^{-1} = 297{\cdot}338 \pm 0{\cdot}050$. His $\kappa(1) = 68 \times 10^{-8}$ is in reasonable agreement with mine, especially if it is noted that I have corrected an error in one of de Sitter's coefficients. He uses $H = 0{\cdot}003327237 \pm 0{\cdot}000\,00059$ and de Sitter's definition of e, which depends somewhat on κ, but these do not appear to account for the differences.

Bullard's model and mine give for $10^6 e$ in the first approximation:

r	E.C.B.	H.J.
1·00	3364	3371
0·55	2567	2618
0·00	2132	2528

The difference at the surface is small, corresponding to slightly different values of I/Mb^2, but the other differences are substantial, especially at the centre. They give a warning that, though results for the surface depend little on the model, provided I/Mb^2 is kept the same, this does not apply to the interior, and further revision would have little value for reasons already stated.

6. **Use of artificial satellites**

In the formula taken from de Sitter (Jeffreys 1948p. 240(6)) connecting the lunar inequality with the mass of the Moon, I now take my value for the lunar inequality

$$L' = 6''{\cdot}4378 \pm 0''{\cdot}0017 \tag{18}$$

and Rabe's solar parallax

$$\pi_\odot = 8''{\cdot}79835 \pm 0''{\cdot}00039. \tag{19}$$

I use the Earth's equatorial radius and mean gravity as in the same paper (later corrections are well within the uncertainty arising from L) and find

$$\mu^{-1} = 81{\cdot}299 \pm 0{\cdot}022;\quad z = -0{\cdot}0029 \pm 0{\cdot}00027 \tag{20}$$
$$w = -0{\cdot}00196 \pm 0{\cdot}00018$$

and hence

$$H = 0{\cdot}0032730\,(1 \pm 0{\cdot}00018) \tag{21}$$

The best value of J is probably that of King–Hele, Cook and Rees (1963), based on

$$J_2 = \tfrac{2}{3} J = 10^{-6}\,(1082{\cdot}78 \pm 0{\cdot}05) \tag{22}$$

If the volume of the comparison sphere is to be the same we want

$$\frac{3}{2}(C-A)/Mb^2 = \frac{3}{2} J_2 \left(1 + \frac{2}{3} e\right). \tag{23}$$

Then

$$\frac{3}{2}\frac{I}{Mb^2} = \frac{3J_2}{2H}\left(1 + \frac{2}{3}e - \frac{2}{3}H\right) = 0{\cdot}496262\,(1 + 0{\cdot}00018). \tag{24}$$

Interpolating from Table 1 and applying the corrections we find that on the hydrostatic theory we should have

$$\begin{aligned} 100e &= 0{\cdot}33370 \pm 0{\cdot}00006, \; e^{-1} = 299{\cdot}67 \pm 0{\cdot}05; \\ 100H &= 0{\cdot}32379 \pm 0{\cdot}00006 \end{aligned} \tag{25}$$

and above all

$$J_0 = 0{\cdot}0016105 \pm 0{\cdot}0000006, \quad J = 0{\cdot}0016081 \pm 0{\cdot}0000006,$$

$$J_2 = 0{\cdot}0010721 \pm 0{\cdot}0000004$$

The contradiction between the theoretical and actual values of J_2 found by Henriksen (stated by him in terms of e^{-1}) is fully confirmed.

7. Strength needed for support of the P_2 and P_3 inequalities

On the suppositions that inequalities are supported by strength (1) down to the core (2) to a depth of 0·1 of the radius, I showed (1943) that the strengths S needed in dynes/cm^2 are about as follows, where g_n is the coefficient of the term in gravity stated in gal.

	Case (1)	Case (2)
P_2	$4{\cdot}3 \times 10^9 g_n = 4{\cdot}3 \times 10^7$	$7{\cdot}9 \times 10^9 g_n = 7{\cdot}9 \times 10^7$
P_3	$2{\cdot}7 \times 10^9 g_n = 1{\cdot}4 \times 10^7$	$5{\cdot}3 \times 10^9 g_n = 2{\cdot}6 \times 10^7$

$g_n = (n-1)g$ times the discrepancy in J_n: $g_2 = 0{\cdot}01$ gal, $g_3 = 0{\cdot}0049$ gal.

The strengths indicated are of the same order as those estimated from the tesseral harmonics, and are specially interesting because of discordances found between various estimates of the latter.

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