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ON THE HYPOELLIPTICITY AND THE GLOBAL ANALYTIC-HYPOELLIPTICITY OF PSEUDO- DIFFERENTIAL OPERATORS

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Introduction

In the recent paper [13] Kumano-go and Taniguchi have studied by using oscillatory integrals when pseudo-differential operators in R^n are Fredholm type and examined whether or not the operators $L_k(x, D_x, D_y) = D_x + ix^k D_y$ in Mizohata [15] and $L_{\pm}(x, D_x, D_y) = D_x \pm ix D_y^2$ in Kannai [6] are hypoelliptic by a unified method. In the present paper we shall give the detailed description for results obtained in [13] and study the hypoellipticity for the operator of the form $L = \sum_{|\alpha: m| + |\alpha': m'| \leq 1} a_{\alpha\alpha'} x^\alpha \bar{y}^{\alpha'} D_x^\alpha D_y^{\alpha'}$ with semi-homogeneity in (x, \bar{y}, D_x, D_y) by deriving the similar inequality to that of Grushin [4] for the elliptic case. Then we can treat the semi-elliptic case as well as the elliptic case. We shall also give a theorem on the global analytic-hypoellipticity of a non-elliptic operator, and applying it give a necessary and sufficient condition for the operator $L(x, D_x, D_y)$ to be hypoelliptic, when the coefficients of L are independent of \bar{y} (see Theorem 3.1).

In Section 1 we shall describe pseudo-differential operators of class $S_{\lambda, \rho, \delta}^m$ which is defined by using a basic weight function $\lambda = \lambda(x, \xi)$ varying in x and ξ (cf. [13] and also [1]). In Section 2 we shall study the global analytic-hypoellipticity of a non-elliptic pseudo-differential operator and give an example which indicates that the condition (2.3) is necessary in general. In Section 3 we shall consider the local hypoellipticity for the operator L and give some examples.

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1. Algebras and L^2 -boundedness

DEFINITION 1.1. For $-\infty < m < \infty$, $0 \leq \delta < 1$ and a sequence $\bar{\tau}$; $0 \leq \tau_0 \leq \tau_1 \leq \dots$ we define a Fréchet space $\mathcal{A}_{\delta, \bar{\tau}}^m$ by the set of C^∞ -functions $p(\xi, x)$ in $R_{\xi, x}^{2n}$ for which each semi-norm

$$|p|_{\alpha, \beta}^{(m)} = \sup_{x, \xi} \{ |p_{(\beta)}^{(\alpha)}(\xi, x) \langle x \rangle^{-\tau_1 |\beta|} \langle \xi \rangle^{-m - \delta_1 |\beta|} \}$$

is finite, where $p_{(\beta)}^{(\alpha)} = \partial_{\xi}^{\alpha} D_x^{\beta} p$, $D_{x_j} = -i\partial/\partial x_j$, $\partial_{\xi_j} = \partial/\partial \xi_j$, $j=1, \dots, n$,

$$\langle x \rangle = \sqrt{1 + |x|^2}, \quad \langle \xi \rangle = \sqrt{1 + |\xi|^2}.$$

We define the oscillatory integral $O_s[p]$ for $p(\xi, x) \in \mathcal{A}_{\delta, \tau}^m$ by

$$\begin{aligned} O_s[p] &\equiv O_s - \iint e^{-ix \cdot \xi} p(\xi, x) dx d\xi \\ &= \lim_{\varepsilon \rightarrow 0} \iint e^{-ix \cdot \xi} \chi_{\varepsilon}(\xi, x) p(\xi, x) dx d\xi, \end{aligned}$$

where $d\xi = (2\pi)^{-n} d\xi$, $x \cdot \xi = x_1 \xi_1 + \dots + x_n \xi_n$ and $\chi_{\varepsilon}(\xi, x) = \chi(\varepsilon \xi, \varepsilon x)$ ($0 < \varepsilon \leq 1$) for a $\chi(\xi, x) \in \mathcal{S}$ (the class of rapidly decreasing functions of Schwartz) in $R_{\xi, x}^{2n}$ such that $\chi(0, 0) = 1$ (cf. ([11], [13])).

REMARK. We can easily obtain the following statements (cf. [11]).

1°) For $p \in \mathcal{A}_{\delta, \tau}^m$ we have

$$O_s[p] = \iint e^{-ix \cdot \xi} \langle x \rangle^{-2l'} \langle D_{\xi} \rangle^{2l'} \{ \langle \xi \rangle^{-2l} \langle D_x \rangle^{2l} p(\xi, x) \} dx d\xi$$

by taking integers l, l' such that $-2l(1-\delta) + m < -n$ and $-2l' + \tau_{2l} < -n$.

2°) Let $\{p_{\varepsilon}\}_{0 < \varepsilon < 1}$ be a bounded set in $\mathcal{A}_{\delta, \tau}^m$ and converges to a $p_0(\xi, x) \in \mathcal{A}_{\delta, \tau}^m$ as $\varepsilon \rightarrow 0$ uniformly on any compact set of $R_{\xi, x}^{2n}$. Then we have

$$\lim_{\varepsilon \rightarrow 0} O_s[p_{\varepsilon}] = O_s[p_0].$$

3°) For $p \in \mathcal{A}_{\delta, \tau}^m$ we have

$$O_s[x^{\alpha} p] = O_s[D_{\xi}^{\alpha} p] \quad \text{and} \quad O_s[\xi^{\alpha} p] = O_s[D_x^{\alpha} p].$$

DEFINITION 1.2. We say that a C^{∞} -function $\lambda(x, \xi)$ in $R_{x, \xi}^{2n}$ is a basic weight function when $\lambda(x, \xi)$ satisfies conditions:

$$(1.1) \quad A_0^{-1} \langle \xi \rangle^a \leq \lambda(x, \xi) \leq A_0 (1 + |x|^{\tau_0} + |\xi|) \quad (\tau_0 \geq 0, a > 0),$$

$$(1.2) \quad |\lambda_{(\beta)}^{(\alpha)}(x, \xi)| \leq A_{\alpha\beta} \lambda(x, \xi)^{1 - |\alpha| + \delta |\beta|} \quad (0 \leq \delta < 1),$$

$$(1.3) \quad \lambda(x+y, \xi) \leq A_1 \langle y \rangle^{\tau_1} \lambda(x, \xi) \quad (\tau_1 \geq 0)$$

for positive constants $A_0, A_{\alpha\beta}, A_1$.¹⁾

DEFINITION 1.3. We say that a C^{∞} -function $p(x, \xi)$ in $R_{x, \xi}^{2n}$ belongs to $S_{\lambda, \rho, \delta}^m$, $-\infty < m < \infty$, $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$, when for any multi-index α, β

1) For a basic weight function $\lambda(x, \xi)$ satisfying (1.1)–(1.3) we can always find an equivalent basic weight function $\lambda'(x, \xi)$ with $\delta=0$ in (1.2) to $\lambda(x, \xi)$, i.e., $C^{-1}\lambda(x, \xi) \leq \lambda'(x, \xi) \leq C\lambda(x, \xi)$.

$$(1.4) \quad |p^{(\alpha)}(x, \xi)| \leq C_{\alpha\beta} \lambda(x, \xi)^{m-\rho|\alpha|+\delta|\beta|}.$$

For $p(x, \xi) \in S_{\lambda, \rho, \delta}^m$ we define pseudo-differential operator $P = p(X, D_x)$ with the symbol $\sigma(P)(x, \xi) = p(x, \xi)$ by

$$(1.5) \quad Pu(x) = \int e^{ix \cdot \xi} p(x, \xi) \hat{u}(\xi) d\xi \quad \text{for } u \in \mathcal{S},$$

where $\hat{u}(\xi) = \int e^{-ix \cdot \xi} u(x) dx$ is the Fourier transform of $u \in \mathcal{S}$.

For a $p \in S_{\lambda, \rho, \delta}^m$ we define semi-norms $|p|_{l_1, l_2}^{(m)}$, $l_1, l_2 = 0, 1, \dots$ by

$$|p|_{l_1, l_2}^{(m)} = \text{Max}_{|\alpha| \leq l_1, |\beta| \leq l_2} \left\{ \sup_{x, \xi} |p^{(\alpha)}(x, \xi)| \lambda(x, \xi)^{-m+\rho|\alpha|-\delta|\beta|} \right\}.$$

Then $S_{\lambda, \rho, \delta}^m$ makes a Fréchet space.

In what follows we shall only treat the case: $\delta = \rho = 0$ or $0 = \delta < \rho = 1$ since it simplifies the statements below and is sufficient for our aim.

Theorem 1.4. *Let $P_j = p_j(X, D_x) \in S_{\lambda, \rho, 0}^{m_j}$, $j = 1, 2$. Then $P = P_1 P_2$ belongs to $S_{\lambda, \rho, 0}^{m_1+m_2}$ and we have for any integer $N > 0$*

$$(1.6) \quad \begin{aligned} \sigma(P)(x, \xi) & \quad (\text{denoted also by } p_1 \circ p_2(x, \xi)) \\ & = \sum_{|\alpha| < N} \frac{1}{\alpha!} p_\alpha(x, \xi) + N \sum_{|\gamma| = N} \int_0^1 \frac{(1-\theta)^{N-1}}{\gamma!} r_{\gamma, \theta}(x, \xi) d\theta \end{aligned}$$

where

$$\begin{cases} p_\alpha(x, \xi) = p_1^{(\alpha)}(x, \xi) p_{2(\alpha)}(x, \xi) & (\in S_{\lambda, \rho, 0}^{m_1+m_2-\rho|\alpha|}), \\ r_{\gamma, \theta}(x, \xi) = O_s - \iint e^{-iy \cdot \eta} p_1^{(\gamma)}(x, \xi + \theta\eta) p_{2(\gamma)}(x+y, \xi) dy d\eta. \end{cases}$$

The set $\{r_{\gamma, \theta}(x, \xi)\}_{|\alpha| \leq 1}$ is bounded in $S_{\lambda, \rho, 0}^{m_1+m_2-\rho|\gamma|}$.

Proof. By the same method of the Theorem 2.5 and 2.6 in [11] we can prove the formula (1.6) if we have only to prove $\{r_{\gamma, \theta}\}$ is a bounded set in $S_{\lambda, \rho, 0}^{m_1+m_2-\rho|\gamma|}$. Since $\partial_\xi^\alpha D_x^\beta r_{\gamma, \theta}$ is represented as the linear combination of

$$(1.7) \quad \iint e^{-iy \cdot \eta} p_{1(\beta_1)}^{(\alpha_1+\gamma)}(x, \xi + \theta\eta) p_{2(\beta_2+\gamma)}^{(\alpha_2)}(x+y, \xi) dy d\eta, \\ (\alpha = \alpha_1 + \alpha_2, \beta = \beta_1 + \beta_2)$$

we have only to prove that each term of the form (1.7) is estimated by $C\lambda(x, \xi)^{m_1+m_2-\rho|\gamma|-\rho|\alpha|}$. Here and in what follows we omit the notation O_s^- . We have

$$\begin{aligned} & \left| \iint e^{-iy \cdot \eta} p_{1(\beta_1)}^{(\alpha_1+\gamma)}(x, \xi + \theta\eta) p_{2(\beta_2+\gamma)}^{(\alpha_2)}(x+y, \xi) dy d\eta \right| \\ & = \left| \iint e^{-iy \cdot \eta} \langle y \rangle^{-2l_1} \langle D_\eta \rangle^{2l_1} p_{1(\beta_1)}^{(\alpha_1+\gamma)}(x, \xi + \theta\eta) p_{2(\beta_2+\gamma)}^{(\alpha_2)}(x+y, \xi) dy d\eta \right| \end{aligned}$$

$$\begin{aligned}
 &\leq \left| \int_{|\eta| \leq C_0 \lambda} \langle \eta \rangle^{-n_0} d\eta \int e^{-iy \cdot \eta} \langle D_y \rangle^{n_0} \{ \langle y \rangle^{-2l_1} \langle D_\eta \rangle^{2l_1} p_{1(\beta_1)}^{(\alpha_1 + \gamma)}(x, \xi + \theta\eta) \right. \\
 &\qquad \qquad \qquad \left. \cdot p_{2(\beta_2 + \gamma)}^{(\alpha_2)}(x + y, \xi) \} dy \right| \\
 &+ \left| \int_{|\eta| \geq C_0 \lambda} |\eta|^{-2l_2} d\eta \int e^{-iy \cdot \eta} (-\Delta_y)^{l_2} \{ \langle y \rangle^{-2l_1} \langle D_\eta \rangle^{2l_1} p_{1(\beta_1)}^{(\alpha_1 + \gamma)}(x, \xi + \theta\eta) \right. \\
 &\qquad \qquad \qquad \left. \cdot p_{2(\beta_2 + \gamma)}^{(\alpha_2)}(x + y, \xi) \} dy \right| \\
 &\leq C \left\{ \int_{|\eta| \leq C_0 \lambda} \langle \eta \rangle^{-n_0} d\eta \int \langle y \rangle^{-2l_1} \lambda(x, \xi + \theta\eta)^{m_1 - \rho|\gamma| - \rho|\alpha_1|} \lambda(x + y, \xi)^{m_2 - \rho|\alpha_2|} dy \right. \\
 &\quad \left. + \int_{|\eta| \geq C_0 \lambda} |\eta|^{-2l_2} d\eta \int \langle y \rangle^{-2l_1} \lambda(x, \xi + \theta\eta)^{m_1 - \rho|\gamma| - \rho|\alpha_1|} \lambda(x + y, \xi)^{m_2 - \rho|\alpha_2|} dy \right\} \\
 &\leq C \left\{ \lambda(x, \xi)^{m_1 + m_2 - \rho|\gamma| - \rho|\alpha_1|} \int \langle \eta \rangle^{-n_0} d\eta \int \langle y \rangle^{-2l_1 + \tau_1} \lambda(x + y, \xi)^{m_2 - \rho|\alpha_2|} dy \right. \\
 &\quad \left. + \lambda(x, \xi)^{m_2 - \rho|\alpha_2|} \int_{|\eta| \geq C_0 \lambda} |\eta|^{-2l_2 + m_1} d\eta \int \langle y \rangle^{-2l_1 + \tau_1} \lambda(x + y, \xi)^{m_2 - \rho|\alpha_2|} dy \right\} \\
 &\leq C \lambda(x, \xi)^{m_1 + m_2 - \rho|\gamma| - \rho|\alpha_1|},
 \end{aligned}$$

where $n_0 = 2([n/2] + 1)$, $m_{1+} = \text{Max}(m_1, 0)$, l_1, l_2 are integers such that

$$-2l_1 + \tau_1 |m_2 - \rho|\alpha_2|| < -n, \quad -2l_2 + m_{1+} + n + 1 \leq \text{Min}(0, m_1 - \rho|\gamma| - \rho|\alpha_1|),$$

and C_0 is a constant such that

$$(1.8) \quad \frac{1}{2} \lambda(x, \xi) \leq \lambda(x, \xi + \eta) \leq \frac{3}{2} \lambda(x, \xi) \quad \text{if } |\eta| \leq C_0 \lambda(x, \xi).$$

We can prove the following two theorems by the same method.

Theorem 1.5. *Let $S_{\lambda, \rho, 0}^{m, m'}$ denote a set of double symbols $p(\xi, x', \xi')$, which satisfy*

$$|p_{(\beta)}^{(\alpha, \alpha')}(\xi, x', \xi')| \leq C_{\alpha\alpha\beta} \lambda(x', \xi)^{m - \rho|\alpha|} \lambda(x', \xi')^{m' - \rho|\alpha'|},$$

and define operators $P = p(D_x, X', D_{x'})$ by

$$\widehat{Pu}(\xi) = O_s - \iint e^{-ix' \cdot (\xi - \xi')} p(\xi, x', \xi') \hat{u}(\xi') d\xi' dx' \quad \text{for } u \in S.$$

Then P belongs to $S_{\lambda, \rho, 0}^{m+m', 0}$ and we can write $\sigma(P)(x, \xi)$ in the form (1.6) for any $N > 0$, where

$$\begin{cases} p_\alpha(x, \xi) = p_{(\alpha)}^{(\alpha, 0)}(\xi, x, \xi) & (\in S_{\lambda, \rho, 0}^{m+m' - \rho|\alpha|}) \\ r_{\gamma, \theta}(x, \xi) = O_s - \iint e^{-iy \cdot \eta} p_{(\gamma)}^{(\gamma, 0)}(\xi + \theta\eta, x + y, \xi) dy d\eta. \end{cases}$$

The set $\{r_{\gamma, \theta}(x, \xi)\}_{|\theta| \leq 1}$ is bounded in $S_{\lambda, \rho, 0}^{m+m' - \rho|\gamma|}$.

Theorem 1.6. For $P=p(X, D_x) \in S_{\lambda, \rho, 0}^m$, the operator $P^{(*)}$ defined by

$$(Pu, v) = (u, P^{(*)}v) \quad \text{for } u, v \in \mathcal{S}$$

belongs to $S_{\lambda, \rho, 0}^m$ and we have for any $N > 0$

$$\sigma(P^{(*)})(x, \xi) = \sum_{|\alpha| < N} \frac{1}{\alpha!} p_{\alpha}^{(*)}(x, \xi) + N \sum_{|\gamma| = N} \int_0^1 \frac{(1-\theta)^{N-1}}{\gamma!} r_{\gamma, \theta}^{(*)}(x, \xi) d\theta,$$

where

$$\begin{cases} p_{\alpha}^{(*)}(x, \xi) = (-1)^{|\alpha|} \overline{p_{(\alpha)}^{(\alpha)}(x, \xi)} & (\in S_{\lambda, \rho, 0}^{m-\rho|\alpha|}) \\ r_{\gamma, \theta}^{(*)}(x, \xi) = O_s - \iint e^{-iy \cdot \eta} (-1)^{|\gamma|} \overline{p_{(\gamma)}^{(\gamma)}(x+y, \xi+\theta\eta)} dy d\eta. \end{cases}$$

The set $\{r_{\gamma, \theta}^{(*)}(x, \xi)\}_{|\theta| \leq 1}$ is bounded in $S_{\lambda, \rho, 0}^{m-\rho|\gamma|}$.

REMARK. The maps

$$S_{\lambda, \rho, 0}^{m_1} \times S_{\lambda, \rho, 0}^{m_2} \ni (p_1, p_2) \rightarrow p_1 \circ p_2 \in S_{\lambda, \rho, 0}^{m_1+m_2}$$

and

$$S_{\lambda, \rho, 0}^m \ni p \rightarrow p^{(*)} \in S_{\lambda, \rho, 0}^m$$

are continuous.

Let $q(\sigma)$ be a C^∞ - and even-function such that $q(\sigma) \geq 0$, $\int q(\sigma)^2 d\sigma = 1$ and $\text{supp } q \subset \{\sigma \in R^n; |\sigma| \leq 1\}$, and set

$$F(x, \xi; \zeta) = \lambda(x, \xi)^{-n/4} q((\zeta - \xi)/\lambda(x, \xi)^{1/2}).$$

Theorem 1.7. For $P=p(X, D_x) \in S_{\lambda, 1, 0}^m$, we define the Friedrichs part $P_F = \hat{p}_F(D_x, X', D_x)$ by

$$p_F(\xi, x', \xi') = \int F(x', \xi; \zeta) p(x', \zeta) F(x', \xi'; \zeta) d\zeta.$$

Then we have

- (i) $p_F(\xi, x', \xi')$ belongs to $S_{\sqrt{\lambda}, 1, 0}^{2m, 0}$,
- (ii) The operator P_F belongs to $S_{\lambda, 1, 0}^m$ and $P - P_F \in S_{\lambda, 1, 0}^{m-1}$, and $\sigma(P_F)$ has the form

$$\sigma(P_F)(x, \xi) \sim p(x, \xi) + \sum_{|\alpha| + |\beta| + |\gamma| \geq 2} \psi_{\alpha\beta\gamma}(x, \xi) p_{(\beta)}^{(\alpha)}(x, \xi)$$

where $\psi_{\alpha\beta\gamma} \in S_{\lambda, 1, 0}^{(|\alpha| - |\beta| - |\gamma|)/2}$,

- (iii) If $p(x, \xi)$ is real-valued and non-negative, we have

$$\begin{aligned} (p_F(D_x, X', D_x)u, v) &= (u, p_F(D_x, X', D_x)v) \quad \text{for } u, v \in \mathcal{S}, \\ (p_F(D_x, X', D_x)u, u) &\geq 0 \quad \text{for } u \in \mathcal{S}. \end{aligned}$$

Proof is carried out by the similar way to that in [9].

Theorem 1.8. *We can extend $P=p(X, D_x) \in S_{\lambda,0,0}^0$ to a bounded operator on L^2 and we get*

$$(1.9) \quad \|Pu\|_{L^2} \leq C |p|_{i_0, i_0}^{(0)} \|u\|_{L^2},$$

where C and l_0 are independent of P and u .

Since $S_{\lambda,0,0}^0 \subset S_{\langle \xi \rangle, 0, 0}^0$, this theorem is a corollary of Calderón-Vaillancourt's theorem in [2].

2. Global analytic-hypoellipticity

DEFINITION 2.1. We say that $L \in S_{\lambda,1,0}^m$ is globally analytic-hypoelliptic if the following statement holds for L :

If $u \in L^2(\mathbb{R}^n)$ is a solution of the equation

$$L(X, D_x)u = f \quad \text{for } f \in C^\infty(\mathbb{R}^n)$$

and f satisfies for some $M > 0$

$$(2.1) \quad \|D_x^\alpha f\|_{L^2} \leq M^{1+|\alpha|} \alpha!,$$

then u is analytic and we have

$$(2.2) \quad \|D_x^\alpha u\|_{L^2} \leq M_1^{1+|\alpha|} \alpha!$$

for another constant $M_1 > 0$.

Theorem 2.2. *Let $L \in S_{\lambda,1,0}^m$ ($m > 0$) satisfy the following conditions:*

$$(2.3) \quad |L(x, \xi)| \geq C \lambda(x, \xi)^m \quad \text{for } |\xi| \geq R$$

for some $C > 0$ and $R \geq 0$, and for any multi-index α there exists M_α such that

$$(2.4) \quad |L_{(\beta)}^{(\alpha)}(x, \xi)| \leq M_\alpha^{1+|\beta|} \beta! \lambda(x, \xi)^{m-|\alpha|}.$$

Then the operator $L(X, D_x)$ is globally analytic-hypoelliptic.

EXAMPLE 2.3. Let $L(x_1, x_2, D_{x_1}, D_{x_2}) = D_{x_1}^2 + D_{x_2}^6 + x_1^2 + x_2^6 - 15x_2^4 + 45x_2^2 - 16$. Then we can prove that L satisfies the conditions (2.3) and (2.4) by taking $\lambda(x_1, x_2, \xi_1, \xi_2) = (1 + |L(x_1, x_2, \xi_1, \xi_2)|^2)^{1/2}$ as a basic weight function. The equation $L(X_1, X_2, D_{x_1}, D_{x_2})u = 0$ has a non-trivial solution $e^{-(x_1^2 + x_2^2)/2}$.

As a generalization of the above example we have

EXAMPLE 2.4 (cf. [5]). Let $L(x, D_x) = \sum_{|\alpha| \leq m_1} a_\alpha(x) D_x^\alpha$ be a hypoelliptic differential operator of order m_1 with analytic coefficients. Suppose that L satisfies following conditions for constants $\tau_0 \geq 0$, $0 < \rho \leq 1$, $C_1 > 0$, $C_2 > 0$, $M > 0$,

- (0) $|\partial_x^\beta a_\omega(x)| \leq M^{1+|\beta|} \beta!$ if $|\beta| \geq m_1 \tau_0$ and $|\alpha| \leq m_1$,
- (i) $C_1^{-1} \langle \xi \rangle^{\rho m_1} \leq |L(0, \xi)| \leq C_1 |L(x, \xi)|$ for large $|\xi|$,
- (ii) $|L_{(\beta)}^{(\alpha)}(x, \xi) / L(x, \xi)| \leq M^{1+|\beta|} \beta! (|\xi| + |x|^{\tau_0})^{-\rho|\alpha|}$ for large $|\xi| + |x|^{\tau_0}$,
- (iii) $|L_{(\beta)}(x, \xi)| \leq C_2(1 + |L(0, \xi)|)$ if $|\beta| \geq m_1 \tau_0$.

Then we can see that L satisfies the conditions of Theorem 2.2 by taking $\lambda(x, \xi) = (1 + |L(x, \xi)|^2)^{1/2m}$ for a large m as a basic weight function.

Proof. From (0) we can choose a positive constant m' such that

$$|L(x, \xi)| \leq C(|\xi| + |x|^{\tau_0})^{m'} \quad \text{for } |\xi| + |x|^{\tau_0} \geq 1.$$

We put $m = m' / \rho$ and $\lambda(x, \xi) = (1 + |L(x, \xi)|^2)^{1/2m}$. Then we have (2.4) from (0) and (ii). By usual calculus we have (1.2) for $\delta = 0$. From (i) we have (1.1) for $a = \rho m_1 / m$ and (2.3). Finally we can get (1.3) by (i) and (iii).

EXAMPLE 2.5. Let $L(x_1, x_2, D_{x_1}, D_{x_2}) = iD_{x_1} + D_{x_2}^2 - 2ix_2^3 D_{x_2} + x_1 - x_2^6 - 3x_2^2$. Then L is a semi-elliptic operator and $Lu = 0$ has a non-analytic solution $u = e^{-(x_1^{2/2} + x_2^{4/4})} \sum_{m=0}^{\infty} \frac{f^{(m)}(x_1)}{(2m)!} x_2^{2m} (\in S)$ where $f(x_1) \in C_0^\infty(\mathbb{R}^1)$ and belongs to the Gevrey class $\rho < (3/2)$. This fact means the conditions are necessary in general. In fact let L satisfy (2.3) and (2.4). Then we have the following contrary:

$$1 = |\partial_{x_1} L(-t^2, 0, 0, t)| \leq C \lambda(-t^2, 0, 0, t)^m \leq |L(-t^2, 0, 0, t)| = 0$$

for large t .

Proof of Theorem 2.2. Define $\{E_j(x, \xi)\}_{j=0,1,\dots}$ for $|\xi| \geq R$ inductively by

$$(2.5) \quad \begin{aligned} E_0(x, \xi) &= L(x, \xi)^{-1}, \\ E_j(x, \xi) &= -\sum_{i=0}^{j-1} \sum_{|\gamma|=j-i} \frac{1}{\gamma!} E_i^{(\gamma)}(x, \xi) L_{(\gamma)}(x, \xi) E_0(x, \xi) \quad (j \geq 1), \end{aligned}$$

then we have $|E_{j(\beta)}^{(\alpha)}| \leq C_{j\alpha\beta} \lambda(x, \xi)^{-m-j-|\alpha|}$ if $|\xi| \geq R$. Taking $\varphi_R(\xi) \in C^\infty$ such that $\varphi_R = 1$ if $|\xi| \geq 2R$ and $\varphi_R = 0$ if $|\xi| \leq R$, and an integer N such that $aN \geq 1$, we define

$$(2.6) \quad E(x, \xi) = \varphi_R(\xi) \sum_{j=0}^{N-1} E_j(x, \xi) \in S_{\lambda, 0, 0}^{-m}.$$

Then we have

$$(2.7) \quad EL = I - K, \quad K \in S_{\langle \xi \rangle, 0, 0}^{-1}.$$

In fact by the same method of Theorem 1.4 we have

$$(2.8) \quad \begin{aligned} &\sigma(EL)(x, \xi) - 1 \\ &= \sum_{j=0}^{N-1} \sum_{|\gamma| < N-j} \frac{1}{\gamma!} \varphi_R(\xi) E_j^{(\gamma)}(x, \xi) L_{(\gamma)}(x, \xi) - 1 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{j=0}^{N-1} \sum_{|\gamma_1+\gamma_2| \leq N-j, \gamma_1 \neq 0} \frac{1}{\gamma_1! \gamma_2!} \partial_{\xi}^{\gamma_1} \varphi_R(\xi) E_j^{(\gamma_2)}(x, \xi) L_{\langle \gamma_1+\gamma_2 \rangle}(x, \xi) \\
 & + \sum_{j=0}^{N-1} \sum_{|\gamma_1+\gamma_2|=N-j} (N-j) \int_0^1 \frac{(1-\theta)^{N-j-1}}{\gamma_1! \gamma_2!} r_{j\gamma_1\gamma_2\theta}(x, \xi) d\theta \\
 & \equiv I_1 + I_2 + I_3,
 \end{aligned}$$

where

$$r_{j\gamma_1\gamma_2\theta}(x, \xi) = \iint e^{-iy \cdot \eta} \partial_{\xi}^{\gamma_1} \varphi_R(\xi + \theta\eta) E_j^{(\gamma_2)}(x, \xi + \theta\eta) L_{\langle \gamma_1+\gamma_2 \rangle}(x+y, \xi) dy d\eta.$$

From (2.5) we have

$$(2.9) \quad I_1 = \varphi_R(\xi) - 1 \in S_{\langle \xi \rangle, 0, 0}^{-1}.$$

From the fact that $\partial_{\xi}^{\gamma_1} \varphi_R(\xi)$ has compact support if $\gamma_1 \neq 0$, we get

$$(2.10) \quad I_2 \in S_{\langle \xi \rangle, 0, 0}^{-1}.$$

Next we prove that $\{r_{j\gamma_1\gamma_2\theta}\}_{|\theta| \leq 1}$ is bounded in $S_{\langle \xi \rangle, 0, 0}^{-1}$. Since $\partial_{\xi}^{\alpha} D_{\eta}^{\beta} r_{j\gamma_1\gamma_2\theta}$ is a linear combination of

$$r'_{\theta}(x, \xi) = \iint e^{-iy \cdot \eta} \partial_{\xi}^{\alpha_1+\gamma_1} \varphi_R(\xi + \theta\eta) E_{j(\beta_1)}^{(\alpha_2+\gamma_2)}(x, \xi + \theta\eta) L_{(\beta_2+\gamma_1+\gamma_2)}^{(\alpha_3)}(x+y, \xi) dy d\eta$$

such that $\alpha_1+\alpha_2+\alpha_3=\alpha$, $\beta_1+\beta_2=\beta$. Hence we have only to prove for a constant C

$$|r'_{\theta}| \leq C \langle \xi \rangle^{-1}.$$

We take a constant C_0 such that (1.8) is satisfied and integers l_1, l_2, l_3 such that $-2l_1+m\tau_1 < -n$, $-2l_2+1 < -n$, $-2l_3+n+1 \leq -m-1/a$. Then we have

$$\begin{aligned}
 & |r'_{\theta}(x, \xi)| \\
 & = \left| \iint e^{-iy \cdot \eta} \langle y \rangle^{-2l_1} \langle D_{\eta} \rangle^{2l_1} \{ \partial_{\xi}^{\alpha_1+\gamma_1} \varphi_R(\xi + \theta\eta) E_{j(\beta_1)}^{(\alpha_2+\gamma_2)}(x, \xi + \theta\eta) \right. \\
 & \quad \left. \cdot L_{(\beta_2+\gamma_1+\gamma_2)}^{(\alpha_3)}(x+y, \xi) \} dy d\eta \right| \\
 & \leq \int_{|\eta| \leq C_0\lambda} \langle \eta \rangle^{-2l_2} d\eta \int |\langle D_y \rangle^{2l_2} [\langle y \rangle^{-2l_1} \langle D_{\eta} \rangle^{2l_1} \{ \partial_{\xi}^{\alpha_1+\gamma_1} \varphi_R(\xi + \theta\eta) \\
 & \quad \cdot E_{j(\beta_1)}^{(\alpha_2+\gamma_2)}(x, \xi + \theta\eta) L_{(\beta_2+\gamma_1+\gamma_2)}^{(\alpha_3)}(x+y, \xi) \}] | dy \\
 & + \int_{|\eta| \geq C_0\lambda} |\eta|^{-2l_3} d\eta \int |(-\Delta_y)^{l_3} [\langle y \rangle^{-2l_1} \langle D_{\eta} \rangle^{2l_1} \{ \partial_{\xi}^{\alpha_1+\gamma_1} \varphi_R(\xi + \theta\eta) \\
 & \quad \cdot E_{j(\beta_1)}^{(\alpha_2+\gamma_2)}(x, \xi + \theta\eta) L_{(\beta_2+\gamma_1+\gamma_2)}^{(\alpha_3)}(x+y, \xi) \}] | dy \\
 & \equiv J_1 + J_2.
 \end{aligned}$$

To estimate J_1 we devide into two cases.

(i) When $\alpha_1 + \gamma_1 = 0$ we have, noting that $|\gamma_2| = N - j$

$$J_1 \leq C \int_{|\eta| \leq C_0 \lambda} \langle \eta \rangle^{-2l_2} d\eta \int \langle y \rangle^{-2l_1} \lambda(x, \xi + \theta\eta)^{-m-N} \lambda(x+y, \xi)^m dy$$

$$\leq C \lambda(x, \xi)^{-N} \int \langle \eta \rangle^{-2l_2} d\eta \int \langle y \rangle^{-2l_1+m\tau_1} dy \leq C \langle \xi \rangle^{-1}.$$

(ii) When $\alpha_1 + \gamma_1 \neq 0$ we have, noting that $\partial_{\xi}^{\alpha_1+\gamma_1} \varphi_R$ has compact support

$$J_1 \leq C \int_{|\eta| \leq C_0 \lambda} \langle \eta \rangle^{-2l_2} d\eta \int \langle y \rangle^{-2l_1} \langle \xi + \theta\eta \rangle^{-1} \lambda(x, \xi + \theta\eta)^{-m} \lambda(x+y, \xi)^m dy$$

$$\leq C \langle \xi \rangle^{-1} \int \langle \eta \rangle^{-2l_2+1} d\eta \int \langle y \rangle^{-2l_1+m\tau_1} dy \leq C \langle \xi \rangle^{-1}.$$

Next for J_2 we have

$$J_2 \leq C \int_{|\eta| \geq C_0 \lambda} |\eta|^{-2l_3} d\eta \int \langle y \rangle^{-2l_1} \lambda(x+y, \xi)^m dy$$

$$\leq C \lambda(x, \xi)^{-2l_3+m+n} \int \langle y \rangle^{-2l_1+m\tau_1} dy \leq C \lambda(x, \xi)^{-1/a} \leq C \langle \xi \rangle^{-1}.$$

Hence we get $I_3 \in S_{\langle \xi \rangle, 0, 0}^{-1}$ and combining (2.8)–(2.10) we get (2.7). From (2.4) and (2.6) we see also that there exists M_2 independent of γ such that

$$(2.11) \quad |\sigma(EL_{\langle \gamma \rangle})|_{i_0, i_0}^{(0)} \leq M_2^{1+|\gamma|} \gamma! \quad \text{for } l_0 \text{ in Theorem 1.8.}$$

Moreover from (2.7) there exists constant C_1 such that

$$(2.12) \quad |K(x, \xi) \xi_j|_{i_0, i_0}^{(0)} \leq C_1 \quad \text{for any } j = 1, \dots, n.$$

Suppose that for $u \in L^2$ $Lu = f$ satisfies (2.1). We have $u = ELu + Ku = Ef + Ku$ from (2.7) and so it is clear that u is a C^∞ -function. Therefore we have only to prove that u satisfies (2.2), since (2.2) implies the analyticity of u by Sobolev’s lemma. Take M_1 sufficiently large such that

$$(2.13) \quad 3C_2 C_1 \leq M_1,$$

$$(2.14) \quad 3C_2 M |E|_{i_0, i_0}^{(0)} \leq M_1, \quad M \leq M_1,$$

$$(2.15) \quad 3 \cdot 2^n C_2 M_2^2 \leq M_1, \quad 2M_2 \leq M_1,$$

$$(2.16) \quad \|u\|_{L^2} \leq M_1,$$

where C_2 is a constant satisfying (1.9).

From (2.16), (2.2) is trivial when $\alpha = 0$, so we show (2.2) by induction on $|\alpha|$. From (2.7), $D_x^\alpha u = ELD_x^\alpha u + KD_x^\alpha u$ ($\alpha \neq 0$). Then we have

$$(2.17) \quad \|D_x^\alpha u\| \leq \|ELD_x^\alpha u\| + \|KD_x^\alpha u\|.$$

Since $\alpha \neq 0$ there exists multi-index α_2 such that $|\alpha_2|=1, \alpha=\alpha_1+\alpha_2$. By (2.12), (2.13) and Theorem 1.8 we get

$$(2.18) \quad \|KD_x^\alpha u\| = \|(KD_x^{\alpha_2})D_x^{\alpha_1}u\| \leq C_2 C_1 \|D_x^{\alpha_1}u\| \leq C_2 C_1 M_1^{1+|\alpha_1|} \alpha_1! \leq M_1^{1+|\alpha|} \alpha! / 3.$$

By Leibniz' formula, we have

$$LD_x^\alpha = D_x^\alpha L - \sum_{\alpha_1 < \alpha} \frac{\alpha!}{\alpha_1! (\alpha - \alpha_1)!} L_{(\alpha - \alpha_1)} D_x^{\alpha_1}.$$

Then

$$(2.19) \quad \|ELD_x^\alpha u\| \leq \|ED_x^\alpha f\| + \sum_{\alpha_1 < \alpha} \frac{\alpha!}{\alpha_1! (\alpha - \alpha_1)!} \|EL_{(\alpha - \alpha_1)} D_x^{\alpha_1} u\|.$$

From (2.1), (2.6) and (2.14) we have

$$(2.20) \quad \|ED_x^\alpha f\| \leq C_2 |E|_{i_0, i_0}^{(0)} \|D_x^\alpha f\| \leq C_2 |E|_{i_0, i_0}^{(0)} M^{1+|\alpha|} \alpha! \leq M_1^{1+|\alpha|} \alpha! / 3.$$

Finally we have from (2.11), (2.15) and the assumption of induction

$$(2.21) \quad \begin{aligned} & \sum_{\alpha_1 < \alpha} \frac{\alpha!}{\alpha_1! (\alpha - \alpha_1)!} \|EL_{(\alpha - \alpha_1)} D_x^{\alpha_1} u\| \\ & \leq \sum_{\alpha_1 < \alpha} C_2 \frac{\alpha!}{\alpha_1! (\alpha - \alpha_1)!} M_2^{1+|\alpha - \alpha_1|} (\alpha - \alpha_1)! M_1^{1+|\alpha_1|} \alpha_1! \\ & = M_1^{1+|\alpha|} \alpha! (C_2 M_2^2 / M_1) \sum_{\alpha_1 < \alpha} (M_2 / M_1)^{|\alpha - \alpha_1| - 1} \leq M_1^{1+|\alpha|} \alpha! / 3. \end{aligned}$$

Therefore from (2.17)–(2.21) we get (2.2).

Corollary 2.6. *Let L satisfy the same conditions as Theorem 2.2. If a bounded and continuous function u is a solution of $Lu=f$ and $f \in C^\infty(R^n)$ satisfies for some M_3*

$$(2.22) \quad |D_x^\alpha f| \leq M_3^{1+|\alpha|} \alpha!,$$

then we have for another constant M_4

$$(2.23) \quad |D_x^\alpha u| \leq M_4^{1+|\alpha|} \alpha! \langle x \rangle^{n_0} \quad \text{for an even number } n_0 > n.$$

Proof. We write $Lu=f$ in the form

$$\langle X \rangle^{-n_0} L(X, D_x) \langle X \rangle^{n_0} u_1 = f_1,$$

where $u_1(x) = \langle x \rangle^{-n_0} u(x), f_1(x) = \langle x \rangle^{-n_0} f(x)$.

We write simplified symbol of $\langle X \rangle^{-n_0} L(X, D_x) \langle X \rangle^{n_0}$ by $L_1(X, D_x)$. Then the pair (L_1, u_1, f_1) satisfies the conditions of the theorem and we get $\|D_x^\alpha u_1\| \leq M_5^{1+|\alpha|} \alpha!$ for some $M_5 > 0$. Hence from Sobolev's lemma we can get (2.23).

REMARK. In Theorem 2.2 we may assume (2.4) only for $|\alpha| \leq l_0$ with l_0 in Theorem 1.8, and in Corollary 2.6 for $|\alpha| \leq 2l_0$.

3. Local hypoellipticity

In this section we shall study a differential operator $L(x, \mathfrak{Y}, D_x, D_y)$ in $R_x^n \times R_y^k$ with polynomial coefficients of the form

$$(3.1) \quad L(x, \mathfrak{Y}, \xi, \eta) = \sum_{|\alpha: m| + |\alpha': m'| \leq 1} a_{\alpha\alpha'\gamma\gamma'} x^\alpha \mathfrak{Y}^\gamma \xi^\alpha \eta^{\alpha'}$$

where $y=(\mathfrak{Y}, \tilde{\mathfrak{Y}})$, $\mathfrak{Y}=(y_1, \dots, y_s)$, $\tilde{\mathfrak{Y}}=(y_{s+1}, \dots, y_k)$ for $s \leq k$, $\alpha=(\alpha_1, \dots, \alpha_n)$, $\alpha'=(\alpha'_1, \dots, \alpha'_k)$, $\gamma=(\gamma_1, \dots, \gamma_n)$, $\gamma'=(\gamma'_1, \dots, \gamma'_s, 0, \dots, 0)$ and $|\alpha: m| = \alpha_1/m_1 + \dots + \alpha_n/m_n$, $|\alpha': m'| = \alpha'_1/m'_1 + \dots + \alpha'_k/m'_k$ for multi-indices $m=(m_1, \dots, m_n)$, $m'=(m'_1, \dots, m'_k)$ of positive integers m_j and m'_j . We say that L is hypoelliptic if $u \in \mathcal{D}'(R_{x,y}^{n+k})$ belongs to $C^\infty(\Omega)$ when Lu belongs to $C^\infty(\Omega)$ for any open set Ω of $R_{x,y}^{n+k}$. Now setting $m = \text{Max} \{m_j, m'_j\}$, we assume that there exist four real vectors $\rho, \rho', \sigma, \sigma'$ of the form $\rho=(\rho_1, \dots, \rho_n)$, $\rho'=(\rho'_1, \dots, \rho'_k)$, $\sigma=(\sigma_1, \dots, \sigma_n)$, $\sigma'=(\sigma'_1, \dots, \sigma'_s, 0, \dots, 0)$ such that

$$(3.2) \quad \begin{cases} \text{(i)} & \rho_j = \sigma_j = m/m_j & \text{for } j = 1, \dots, n \\ \text{(ii)} & \rho'_j > \sigma'_j \geq 0, \quad m'_j \rho'_j \geq m & \text{for } j = 1, \dots, k \end{cases}$$

and

$$(3.3) \quad L(t^{-\sigma}x, t^{-\sigma'}\mathfrak{Y}, t^\rho\xi, t^{\rho'}\eta) = t^m L(x, \mathfrak{Y}, \xi, \eta) \quad \text{for } t > 0,$$

where $t^{-\sigma}x=(t^{-\sigma_1}x_1, \dots, t^{-\sigma_n}x_n)$, $t^{-\sigma'}\mathfrak{Y}=(t^{-\sigma'_1}y_1, \dots, t^{-\sigma'_s}y_s)$,

$$t^\rho\xi = (t^{\rho_1}\xi_1, \dots, t^{\rho_n}\xi_n), \quad t^{\rho'}\eta = (t^{\rho'_1}\eta_1, \dots, t^{\rho'_k}\eta_k).$$

Condition 1. If we put

$$(3.4) \quad L_0(x, \mathfrak{Y}, \xi, \eta) = \sum_{|\alpha: m| + |\alpha': m'| = 1} a_{\alpha\alpha'\gamma\gamma'} x^\alpha \mathfrak{Y}^\gamma \xi^\alpha \eta^{\alpha'}$$

then we have

$$(3.5) \quad L_0(x, \mathfrak{Y}, \xi, \eta) \neq 0 \quad \text{for } |x| + |\mathfrak{Y}| \neq 0 \text{ and } (\xi, \eta) \neq 0,$$

which means that $L(x, \mathfrak{Y}, \xi, \eta)$ is semi-elliptic for $|x| + |\mathfrak{Y}| \neq 0$.

Condition 2. The equation $L(X, \mathfrak{Y}, D_x, \eta)v(x)=0$ in R_x^n has no non-trivial solution in $\mathcal{S}(R_x^n)$ for $|\eta|=1$.

Theorem 3.1. *We consider the operator $L(x, \mathfrak{Y}, D_x, D_y)$ under Condition 1 and the assumption*

$$\text{Max}_{1 \leq j \leq k} \{\sigma'_j\} < \text{Min}_{1 \leq j, l \leq k} \{m'_j \rho'_j / m'_l\}.$$

Then we have

(S) *If Condition 2 holds, then $L(x, \mathfrak{Y}, D_x, D_y)$ is hypoelliptic.*

(N) *If the coefficients of L are independent of \mathfrak{Y} , i.e., $s=0$, then Condition 2 is necessary for the hypoellipticity of the operator L .*

EXAMPLES 3.2.

i) $L = (-\Delta_x)^l + |x|^{2\nu}(-\Delta_y)^{l'}$ in $R_x^n \times R_y^k$ (cf. [3], [7], [14]).

We set $\rho_1 = \dots = \rho_n = \sigma_1 = \dots = \sigma_n = l_0/l$, $\rho'_1 = \dots = \rho'_k = (v/l+1)l_0/l'$, $\sigma'_1 = \dots = \sigma'_k = 0$, where $l_0 = \text{Max}(l, l')$. Then we can see that L is always hypoelliptic.

ii) $L_{\pm}(x, D_x, D_y) = D_x \pm ix^l D_y^m$ in $R_x^1 \times R_y^1$ (cf. [6], [8], [15]).

We set $\rho_1 = \sigma_1 = m$, $\rho'_1 = l+1$, $\sigma'_1 = 0$. Then we see the following three cases:

- a) If l is even, $L_+(X, D_x, \pm 1)v = 0$ and $L_-(X, D_x, \pm 1)v = 0$ have no non-trivial solution in \mathcal{S} .
- b) If l is odd and m is even, $L_+(X, D_x, \pm 1)v = 0$ has no non-trivial solution in \mathcal{S} and $L_-(X, D_x, \pm 1)v = 0$ has non-trivial solution $e^{-x^{l+1}/(l+1)} \in \mathcal{S}$.
- c) If l and m are odd, $L_+(X, D_x, -1)v = 0$ has non-trivial solution $e^{-x^{l+1}/(l+1)} \in \mathcal{S}$ and $L_-(X, D_x, 1)v = 0$ has non-trivial solution $e^{-x^{l+1}/(l+1)} \in \mathcal{S}$.

Consequently we see from (N) and (S) that L_+ is hypoelliptic if and only if “ l is even”, or “ l is odd and m is even”, and L_- is hypoelliptic if and only if “ l is even”.

iii) $L = D_{x_1}^2 + D_{x_2}^6 + (x_1^2 + x_2^6)D_y^6 - 15x_2^4 D_y^5 + 45x_2^2 D_y^4 - 16D_y^3$ in $R_x^2 \times R_y^1$.

We set $\rho_1 = \sigma_1 = 3$, $\rho_2 = \sigma_2 = 1$, $\rho'_1 = 2$, $\sigma'_1 = 0$. We can see that L does not satisfy Condition 2. In fact for $\eta = 1$ $L(X_1, X_2, D_{x_1}, D_{x_2}, 1)v(x_1, x_2) = 0$ is an equation given in Example 2.3 and has non-trivial solution $v = e^{(-x_1^2 + x_2^2)/2}$. Therefore applying (N) we can see that L is not hypoelliptic.

For the proof of the theorem we need several lemmas. We introduce

notations: $|x, \mathfrak{Y}|_{(\sigma, \sigma')} = \sum_{j=1}^n |x_j|^{1/\sigma_j} + \sum_{j=1}^s |y_j|^{1/\sigma'_j}$,

$$|\eta|_{\rho'} = \sum_{j=1}^k |\eta_j|^{1/\rho'_j}, \quad \mu(x, \mathfrak{Y}, \eta) = \sum_{j=1}^k |x, \mathfrak{Y}|_{(\sigma, \sigma')}^{(m'_j \rho'_j - m)} |\eta_j|^{m'_j}.$$

First we estimate the monomials of the form $x^\alpha \mathfrak{Y}^{\gamma'} \eta^{\alpha'}$.

Lemma 3.3. *Let α, α', γ and γ' be multi-indices of dimension n, k, n, k , respectively, such that $|\alpha: m| + |\alpha': m'| \leq 1$ and $\gamma'_j = 0$ for $j \geq s+1$. We put*

$$(3.6) \quad \theta = (\sigma, \gamma) + (\sigma', \gamma') + m - (\rho, \alpha) - (\rho', \alpha').$$

If we denote $\rho'_0 = \text{Min}_{1 \leq j \leq k} (m'_j \rho'_j / m)$, then we have

(i) *If there exists $\theta' \geq 0$ such that $m(|\alpha: m| + |\alpha': m'|) + (\theta + \theta')/\rho'_0 \leq m$, we have*

$$(3.7) \quad |x, \mathfrak{Y}|_{(\sigma, \sigma')}^{\theta} |x^\alpha \mathfrak{Y}^{\gamma'} \eta^{\alpha'}| |\eta|_{\rho'}^{\theta + \theta'} \leq C(|\eta|_{\rho'}^m + \mu(x, \mathfrak{Y}, \eta))^{1 - |\alpha: m|}.$$

(ii) *If $m(|\alpha: m| + |\alpha': m'|) + \theta/\rho'_0 > m$, we have*

$$(3.8) \quad |x^\alpha \mathfrak{Y}^{\gamma'} \eta^{\alpha'}| |\eta|_{\rho'}^{(1 - |\alpha: m| - |\alpha': m'|)m\rho'_0} \leq C(|\eta|_{\rho'}^m + \mu(x, \mathfrak{Y}, \eta))^{1 - |\alpha: m|}$$

for $|x| \leq \delta$, $|\mathfrak{Y}| \leq \delta$ and $|\eta| \geq 1$, where δ is some positive constant.

We can prove this by the same method as Lemma 3.1 and 3.2 in [4].

Lemma 3.4. *Under condition 1 we have for a constant $C > 0$*

$$(3.9) \quad C^{-1}|L_0(x, \mathfrak{Y}, \xi, \eta)| \leq \left\{ \sum_{j=1}^n |\xi_j|^{m_j} + \mu(x, \mathfrak{Y}, \eta) \right\} \leq C|L_0(x, \mathfrak{Y}, \xi, \eta)|.$$

Proof. In case $|x| + |\mathfrak{Y}| \neq 0$, it is sufficient for the sake of semi-homogeneity to prove when $|x| + |\mathfrak{Y}| = 1$, and this is true because of Condition 1. In case $|x| + |\mathfrak{Y}| = 0$, (3.9) is clear by letting $|x| + |\mathfrak{Y}| \rightarrow 0$.

Define $\lambda_h(x, \xi)$ with parameter $h = (\mathfrak{Y}, \eta)$ ($|\eta| = 1$) by $\lambda_h(x, \xi) = \{1 + |L(x, \mathfrak{Y}, \xi, \eta)|^2\}^{1/2m}$ and set $p_h(x, \xi) = L(x, \mathfrak{Y}, \xi, \eta)$. Then we have

Proposition 3.5.

- (i) $\lambda_h(x, \xi)$ satisfies (1.1)–(1.3).
- (ii) $\{p_h(x, \xi)\}$ is bounded in $\{S_{\lambda_h, 1, 0}^m\}$ in the sense that for any α, β there exists a bounded function $C_{\alpha\beta}(x, \mathfrak{Y})$ which is independent of η ($|\eta| = 1$) and tends to zero as $|x| + |\mathfrak{Y}| \rightarrow \infty$ when $\beta \neq 0$, such that

$$|p_{h(\beta)}^{(\alpha)}(x, \xi)| \leq C_{\alpha\beta}(x, \mathfrak{Y})\lambda_h(x, \xi)^{m-|\alpha|}.$$

- (iii) There exists a constant C independent of h such that

$$(3.10) \quad |p_h(x, \xi)| \geq C\lambda_h(x, \xi)^m \quad \text{for large } |x| + |\mathfrak{Y}| + |\xi|.$$

Proof. Set $\lambda'_h(x, \xi) = \{1 + \sum_{j=1}^n |\xi_j|^{m_j} + \mu(x, \mathfrak{Y}, \eta)\}^{1/m}$. Then from Lemma 3.3 (i) and Lemma 3.4 we can prove

$$(3.11) \quad |L(x, \mathfrak{Y}, \xi, \eta)| \geq C\lambda'_h(x, \xi)^m \quad \text{for large } |x| + |\mathfrak{Y}| + |\xi|,$$

which induces

$$(3.12) \quad C^{-1}\lambda'_h(x, \xi) \leq \lambda_h(x, \xi) \leq C\lambda'_h(x, \xi).$$

For each term $a_{\alpha\alpha', \gamma\gamma'} x^\gamma \mathfrak{Y}^{\gamma'} \xi^\alpha \eta^{\alpha'}$ in L , we have from Lemma 3.3

$$\begin{aligned} & |\partial_\xi^{\beta_1} \partial_x^{\alpha_1} (a_{\alpha\alpha', \gamma\gamma'} x^\gamma \mathfrak{Y}^{\gamma'} \xi^\alpha \eta^{\alpha'})| \\ & \leq C \text{Min}(1, |x, \mathfrak{Y}|_{(\sigma, \sigma')}^{-(\sigma, \beta_1)}) (1 + \mu(x, \mathfrak{Y}, \eta))^{1-|\alpha: m|} (1 + \sum_{j=1}^n |\xi_j|^{m_j})^{|\alpha: m| - |\alpha_1: m|} \\ & \leq C \text{Min}(1, |x, \mathfrak{Y}|_{(\sigma, \sigma')}^{-(\sigma, \beta_1)}) \lambda'_h(x, \xi)^{m-|\alpha_1|} \quad (\alpha_1 \leq \alpha). \end{aligned}$$

Here we use the fact that $|\eta| = 1$. Therefore we have

$$(3.13) \quad |p_{h(\beta)}^{(\alpha)}(x, \xi)| \leq C \text{Min}(1, |x, \mathfrak{Y}|_{(\sigma, \sigma')}^{-(\sigma, \beta)}) \lambda'_h(x, \xi)^{m-|\alpha|}.$$

First we check (i). From (3.12) λ_h satisfies (1.1) for $a = \text{Min}_{1 \leq j \leq n} \{m_j/m\}$. By usual

calculus (1.2) follows by (3.13). Since p_h is a polynomial in x , we have using Taylor series

$$|p_h(x+z, \xi)| \leq \sum_{|\alpha| \leq N} |z^\alpha p_{h(\omega)}(x, \xi)| / \alpha! \leq C \langle z \rangle^{m\tau_1} \lambda'_h(x, \xi)^m \leq C \langle z \rangle^{m\tau_1} \lambda_h(x, \xi)^m$$

for some τ_1 . So (1.3) holds for λ_h . Consequently we get (i). (ii) and (iii) follow at once by (3.11)–(3.13).

Lemma 3.6. *Let a basic weight function $\lambda(x, \xi)$ satisfy*

$$(3.14) \quad A_0^{-1}(1+|x|+|\xi|)^{\alpha'} \leq \lambda(x, \xi) \leq A_0(1+|x|^{\tau_0}+|\xi|) \\ (\alpha' > 0, A_0 > 0, \tau_0 > 0)$$

instead of (1.1). Suppose that $p(x, \xi) \in S_{\lambda, 1, 0}^m$ ($m > 0$) satisfies

$$|p(x, \xi)| \geq C\lambda(x, \xi)^m \quad \text{for large } |x| + |\xi|.$$

Then for any $u \in L^2(\mathbb{R}^n)$, $Pu = p(X, D_x)u(x) = 0$ implies $u \in \mathcal{S}(\mathbb{R}^n)$.

Proof. Let $Q \in S_{\lambda, 1, 0}^{-m}$ be a parametrix such that $QP = I - K$, $K \in S_{\lambda, 1, 0}^{-\infty}$ ($= \bigcap_{-\infty < m < \infty} S_{\lambda, 1, 0}^m$). Then we have $u = Ku$. For any positive number r and t , $\langle X \rangle^r \langle D_x \rangle^t K(X', D_x)$ belongs to $S_{\lambda, 1, 0}^{-\infty}$ and we get $\langle X \rangle^r \langle D_x \rangle^t u \in L^2$. Therefore we get $u \in \mathcal{S}$.

Proposition 3.7. *If Condition 1 and 2 hold, then for any $v \in C_0^\infty(\mathbb{R}^n)$ we have*

$$(3.15) \quad \|v\|_{L^2}^2 \leq C \int |p_h(X, D_x)v(x)|^2 dx,$$

where C is independent of v and h with $|\eta| = 1$.

Proof. From (3.10) there exists a parametrix $\{Q_h\}$ which is bounded in $\{S_{\lambda_h, 1, 0}^{-m}\}$ such that

$$(3.16) \quad Q_h P_h = I - K_h,$$

where $\{K_h\}$ is bounded in $\{S_{\lambda_h, 1, 0}^{-m}\}$, $\lim_{|\mathcal{Y}| \rightarrow \infty} \sup_{\xi \in \mathbb{R}^{2n}, |\eta|=1} |K_h(x, \xi)| = 0$ and for any multi-index α, β

$$(3.17) \quad \sup_{x, \xi} |K_h^{(\alpha)}(x, \xi) - K_{h_0}^{(\alpha)}(x, \xi)| \rightarrow 0 \quad \text{as } h \rightarrow h_0.$$

Therefore we have

$$\|v\| \leq \|Q_h P_h v\| + \|K_h v\| \leq C \|P_h v\| + \|K_h v\|.$$

Since $\{K_h\}$ is bounded in $\{S_{\lambda_h, 1, 0}^{-m}\}$ and $\lim_{|\mathcal{Y}| \rightarrow \infty} \sup_{(x, \xi) \in \mathbb{R}^{2n}, |\eta|=1} |K_h(x, \xi)| = 0$, we have for a constant l_0 in Theorem 1.8

$$|K_h|_{l_0, l_0}^{(0)} \rightarrow 0 \quad \text{as } |\mathcal{Y}| \rightarrow \infty.$$

Then for a sufficiently large constant $M > 0$

$$\|K_h v\| \leq \frac{1}{2} \|v\| \quad \text{for } |\mathfrak{Y}| \geq M,$$

and we get (3.15) for $|\mathfrak{Y}| \geq M$.

Now assume that for $|\mathfrak{Y}| \leq M$ (3.15) does not hold. Then we can choose sequences $\{h_\nu\}$, $\{v_\nu\}$ such that

$$(3.18) \quad \|v_\nu\| = 1,$$

$$(3.19) \quad \|P_{h_\nu} v_\nu\| \rightarrow 0 \quad \text{as } \nu \rightarrow \infty,$$

$$(3.20) \quad h_\nu = (\mathfrak{Y}^\nu, \eta^\nu), \quad \text{where } |\mathfrak{Y}^\nu| \leq M, \quad |\eta^\nu| = 1.$$

From (3.20) we may assume that

$$(3.21) \quad h_\nu \rightarrow h_0$$

for some $h_0 = (\mathfrak{Y}^0, \eta^0)$. Applying v_ν to (3.16) we get

$$(3.22) \quad Q_{h_\nu} P_{h_\nu} v_\nu = v_\nu - K_{h_\nu} v_\nu.$$

From (3.19) and (3.21) we have $Q_{h_\nu} P_{h_\nu} v_\nu \rightarrow 0$ in L^2 as $\nu \rightarrow \infty$, and from the fact that $\{K_h\}$ is bounded in $\{S_{\lambda, h, 1, 0}^{-m}\}$, $\limsup_{|\mathfrak{Y}| \rightarrow \infty} \sup_{\xi} |K_{h_0}(x, \xi)| = 0$ and (3.17) we get K_h is uniformly continuous and K_{h_0} is a compact operator in L^2 (cf. [10], [12]). So writing $K_{h_\nu} v_\nu = (K_{h_\nu} - K_{h_0})v_\nu + K_{h_0} v_\nu$ we can choose a convergent subsequence $\{K_{h_\nu} v_\nu\}$ in account of (3.18). Therefore from (3.22) we can choose an element $v_0 \in L^2$ such that

$$(3.23) \quad v_{\nu'} \rightarrow v_0 \quad \text{in } L^2.$$

Then from (3.19) and (3.21) $P_{h_0} v_0 = 0$. When $\eta_j^0 = 0$ for all j such that $m'_j \rho'_j \neq m$, we have $v_0 = 0$ since $p_{h_0}(x, \xi) = \sum a_{\alpha \alpha' 00} (\eta^0)^{\alpha'} \xi^\alpha$. Otherwise (3.12) implies (3.14) and we get $v_0 = 0$ from Lemma 3.6 and Condition 2. This is the contrary to (3.18) and (3.23). Then Proposition 3.7 is proved.

Theorem 3.8. *If Condition 1 and 2 hold, we can get the following formulas for $|\mathfrak{Y}| < \delta$, $|\eta| \geq 1$ and $v \in C_0^\infty(\{x; |x| < \delta\})$, where δ is a number which was taken in Lemma 3.3.*

$$(3.24) \quad \sum_{|\alpha: m| \leq 1} \int |(\mu(x, \mathfrak{Y}, \eta) + |\eta| \frac{m}{\rho'})^{1-|\alpha: m|} D_x^\alpha v(x)|^2 dx \leq C \int |L(X, \mathfrak{Y}, D_x, \eta)v(x)|^2 dx.$$

For any k -dimensional multi-index α_1, β_1 we have

$$(3.25) \quad \|\partial_\eta^{\alpha_0} \partial_{\mathfrak{Y}}^{\beta_1} L(X, \mathfrak{Y}, D_x, \eta)v\|_{L^2} \leq C |\eta| |\rho|^{-\rho_0|\alpha_0| + \sigma_0|\beta_1|} \|L(X, \mathfrak{Y}, D_x, \eta)v\|_{L^2}$$

where $\rho_0 = \text{Min}_{1 \leq j, l \leq k} (m'_j \rho'_j / m'_l)$, $\sigma_0 = \text{Max}_{1 \leq j \leq k} (\sigma'_j)$.

Proof. Let $r(x, \mathfrak{Y})$ be a positive root of the equation

$$\sum_{j=1}^n \frac{x_j^2}{r^{2\sigma_j}} + \sum_{j=1}^s \frac{y_j^2}{r^{2\sigma'_j}} = 1.$$

Then $r(x, \mathfrak{Y})$ is a C^∞ -function in $R_x^n \times R_y^s \setminus \{0, 0\}$ and

$$(3.26) \quad r(x, \mathfrak{Y}) \sim |x, \mathfrak{Y}|_{(\sigma, \sigma')}.$$

Let $\chi(x, \mathfrak{Y})$ be a C^∞ -function such that $\chi=1$ if $|x| + |\mathfrak{Y}| \geq 1$ and $\chi=0$ if $|x| + |\mathfrak{Y}| \leq (1/2)$. For any multi-index α ($|\alpha|: m \leq 1$) and $h=(\mathfrak{Y}, \eta)$ ($|\eta|=1$) we define $R_{\alpha h}$ by

$$R_{\alpha h}(x, \xi) = \left(\sum_{j=1}^k \chi(x, \mathfrak{Y}) r(x, \mathfrak{Y})^{(m'_j \rho'_j - m)} |\eta_j|^{m'_j + 1} \right)^{1-|\alpha|: m} \xi^\alpha.$$

Then $\{R_{\alpha h}\}$ is bounded in $\{S_{\lambda_h, 1, 0}^m\}$. From (3.16) we can write for any $v \in C_0^\infty(R_x^n)$

$$R_{\alpha h}(X, D_x) Q_h(X', D_{x'}) p_h(X'', D_{x''}) v = R_{\alpha h}(X, D_x) v - R_{\alpha h}(X, D_x) K_h(X', D_{x'}) v$$

Noting that $\{R_{\alpha h}(X, D_x) Q_h(X', D_{x'})\}$, $\{R_{\alpha h}(X, D_x) K_h(X', D_{x'})\}$ are bounded in $\{S_{\lambda_h, 1, 0}^0\}$, we get from Proposition 3.7

$$\begin{aligned} & \left\| \left(\sum_{j=1}^k \chi(x, \mathfrak{Y}) r(x, \mathfrak{Y})^{m'_j \rho'_j - m} |\eta_j|^{m'_j + 1} \right)^{1-|\alpha|: m} D_x^\alpha v \right\| \\ & \leq \|R_{\alpha h} Q_h P_h v\| + \|R_{\alpha h} K_h v\| \leq C(\|P_h v\| + \|v\|) \leq C\|P_h v\|. \end{aligned}$$

Considering (3.26) we have for $|\eta|=1$

$$\sum_{|\alpha|: m \leq 1} \int |(\mu(x, \mathfrak{Y}, \eta) + |\eta| |\rho|)^{1-|\alpha|: m} D_x^\alpha v|^2 dx \leq C \int |L(X, \mathfrak{Y}, D_x, \eta)v|^2 dx.$$

From the semi-homogeneity we get (3.24). Using Lemma 3.3 and (3.24) we can get (3.25) by the same method as Lemma 3.6 in [4].

Proof of (S) in Theorem 3.1. By the same method as [4] we can prove (S) by using Theorem 3.8.

Proof of (N) of Theorem 3.1 (cf. [3]). Let there exist non-trivial solution $v(x) \in \mathcal{S}$ of $p_h(X, D_x)v(x) = L(X, D_x, \eta)v(x) = 0$ for some $h=\eta$ with $|\eta|=1$. From Proposition 3.5 we can apply Theorem 2.2 and we get that $v(x)$ is analytic, and therefore there exists multi-index α_0 such that

$$(3.27) \quad \partial_{\alpha_0}^{\alpha_0} v(0) \neq 0.$$

We may assume $\eta_1 \neq 0$. We set $m_0 = \text{Max}(m, |\alpha_0|)$ and take even number l_1 and

positive number b such that $\{(\rho, \alpha_0) - (\rho_1 - 1) + b\} / \rho_1'$ is an even number (we denote it by l_2) and $2l_1 \rho_1' \geq m_0 \cdot \text{Max}(\rho_j, \rho_j) + 2 + b$. We define

$$u(x, y) = \int_0^\infty e^{iy \cdot t^{\rho_1'} \eta} \frac{v(t^{\rho_1'} x_1, \dots, t^{\rho_n'} x_n) t^b}{(1 + t^{2\rho_1'})^{l_1}} dt.$$

Then $u \in C^{m_0}$ and $L(X, D_x, D_y)u = 0$. But $u \notin C^\infty$. In fact operating $\partial_x^{\alpha_0}$ and substituting $x=0, y_2=\dots=y_k=0$, we get

$$\partial_x^{\alpha_0} u(0, y_1, 0, \dots, 0) = \int_0^\infty e^{iy_1 t^{\rho_1'} \eta_1} \frac{\partial_x^{\alpha_0} v(0) t^{(\rho, \alpha_0) + b}}{(1 + t^{2\rho_1'})^{l_1}} dt.$$

By changing the variable t by $\theta = t^{\rho_1'}$, we get

$$\partial_x^{\alpha_0} u(0, y_1, 0, \dots, 0) = \frac{\partial_x^{\alpha_0} v(0)}{\rho_1'} \int_0^\infty e^{iy_1 \theta \eta_1} \frac{\theta^{l_2}}{(1 + \theta^2)^{l_1}} d\theta.$$

Noting l_2 is an even number we can write

$$\text{Re} \int_0^\infty e^{iy_1 \theta \eta_1} \frac{\theta^{l_2}}{(1 + \theta^2)^{l_1}} d\theta = P(|y_1|) e^{-|y_1| |\eta_1|}$$

for some polynomial P of order $l_1 - 1$. Therefore we get from (3.27) $\partial_x^{\alpha_0} u(0, y_1, 0, \dots, 0) \notin C^\infty$. Consequently (N) holds.

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