ON THE HYPOTHESIS OF NO "INTERACTION" IN A MULTI-WAY CONTINGENCY TABLE

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1. Summary. In a situation in which the observations are frequencies in a multi-way contingency table such that the observations are supposed to be independent and it is only the total number that is supposed to be fixed from sample to sample, a hypothesis on the structure of the probabilities in the different cells or categories is put forward. This hypothesis, by a certain analogy with the customary terminology of analysis of variance, is defined to be the hypothesis of "no interaction" and a large sample test of this hypothesis in terms of χ^2 is offered. Bartlett's results [1] for the case of a $2 \times 2 \times 2$ table and Norton's results [5] for the case of a $2 \times 2 \times t$ table formally turn out to be special cases of the results given here with these differences: (i) Bartlett's and Norton's results refer to "analysis of variance" situations, with marginal frequencies along at least two ways of the table being fixed, while in this paper, for reasons explained elsewhere [7], it is only the total n that is held fixed. (ii) Bartlett's and Norton's papers do not give any indication of the mechanism behind the formulae for the hypothesis of "no interaction," while this paper attempts to give a definite mathematical (and perhaps also physical) mechanism behind the formulae.

2. Preliminaries and the actual construction of the hypothesis of "no interaction". To fix our ideas, consider a sample of fixed size n of independent observations distributed in a three-way table. Let n_{ijk} denote the observed frequency, and p_{ijk}, the probability in the (ijk)th cell, where $i = 1, 2, \cdots, r$; $j = 1, 2, \cdots, s$; $k = 1, 2, \cdots, t$. Also let the marginals be denoted by $\sum_i n_{ijk} = n_{0jk}$, $\sum_j n_{ijk} = n_{i0k}$, $\sum_k n_{ijk} = n_{ij0}$, $\sum_{i,j} n_{ijk} = n_{00k}$, $\sum_{i,k} n_{ijk} = n_{0j0}$, $\sum_{j,k} n_{ijk} = n_{i00}$, $\sum_{i,j,k} n_{ijk} = n$ (say). Let the corresponding summations over p_{ijk} be denoted by p_{0jk}, p_{i0k}, p_{ij0}, p_{00k}, p_{0j0}, p_{i00}, p_{000}. Since the categories are mutually exclusive and exhaustive, it is easy to see that these are, in fact, the marginal probabilities, so that $p_{000} = 1$. The generalization to more than three variates would be obvious. The likelihood function, which in this case is also the probability of the n_{ijk}'s, is given by

$$\phi(n_{ijk}\text{'s}) = \phi \text{ (say)} = \frac{n!}{\prod_{i,j,k} n_{ijk}!} \prod_{i,j,k} p_{ijk}^{n_{ijk}} \sim \prod_{i,j,k} p_{ijk}^{n_{ijk}}. \tag{2.1}$$

The last expression on the right side of (2.1) is the one we shall need [2, 4] when we are interested in finding the maximum likelihood estimates of the p's.

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Hypothesis of independence between (i, j) and k, that is, the hypothesis of multiple independence. Consider

$$H_0: p_{ijk} = p_{ij0} p_{00k} \quad (\text{for } i = 1, 2, \cdots, r; j = 1, 2, \cdots, s; k = 1, 2, \cdots, t), \tag{2.2}$$

the alternative being, of course, $H \neq H_0$. It is easy to check, by summing over i and j respectively, that (2.2) implies

$$p_{i0k} = p_{i00} p_{00k} \quad \text{and} \quad p_{0jk} = p_{0j0} p_{00k}. \tag{2.3}$$

Summing over k we have merely the consistency condition

$$p_{ij0} = p_{ij0}. \tag{2.4}$$

Notice that although (2.2) implies (2.3), the condition (2.3) will not, in general, imply (2.2). However, for a normal population, (2.3) implies (2.2). Let us ask ourselves what set of conditions is there which, when superimposed on (2.3), will, together, be exactly equivalent to (2.2). One possible set might appear to be

$$H_0: p_{ijk} = \frac{p_{ij0}\, p_{i0k}\, p_{0jk}}{p_{i00}\, p_{0j0}\, p_{00k}} \tag{2.5}$$

$$(\text{for } i = 1, 2, \cdots, r; \quad j = 1, 2, \cdots, s; \quad k = 1, 2, \cdots, t).$$

Check that (2.5) does not imply (2.2), but if on (2.5) we superimpose (2.3), we have (2.2) all right. But (2.5) would be mathematically most difficult to handle, in that the parameters on the right side of this equation are subject to sets of side conditions, typical among them being

$$\sum_k p_{ijk} = p_{ij0} = \sum_k \frac{p_{ij0}\, p_{i0k}\, p_{0jk}}{p_{i00}\, p_{0j0}\, p_{00k}} \tag{2.6}$$

or

$$\sum_k \frac{p_{i0k}\, p_{0jk}}{p_{00k}} = p_{i00}\, p_{0j0},$$

and other such sets obtained by permuting the subscripts. In fact, (2.5) was tried and was found to be intractable.

Physically a less natural and more abstract, but mathematically a much easier, set of conditions seems to be

$$H_0: p_{ijk} = \frac{q_{ij0}\, q_{i0k}\, q_{0jk}}{q_{i00}\, q_{0j0}\, q_{00k}} \tag{2.7}$$

$$(i = 1, 2, \cdots, r; \quad j = 1, 2, \cdots, s; \quad k = 1, 2, \cdots, t),$$

where we do not assume that $q_{ij0} = p_{ij0}$, etc., nor even that $q_{i00} = \sum_j q_{ij0}$, etc. Equation (2.7), after elimination of the q's, leads to a number of constraints on the p's themselves, and it is easier to try to estimate the p's subject to these constraints and to $\sum_{i,j,k} p_{ijk} = 1$, rather than to try to estimate the q's. The only role of the q's and of the hypothesis (2.7) is one of yielding certain constraints on the p's themselves. It will be shown in Sections 3 and 4 that (2.7) is equivalent to just $(r-1)(s-1)(t-1)$ constraints on the p_{ijk}'s, which, together with $\sum_{i,j,k} p_{ijk} = 1$, make just $(r-1)(s-1)(t-1)+1$ constraints. Notice that in this case we do not have constraints like (2.6) which, in practice, turn out to be quite awkward.

It is clear that so far as the functional form is concerned we can, without any loss of generality, replace (2.7) by just

$$(2.8)\quad H_0: p_{ijk} = q_{ij0}q_{i0k}q_{0jk} \quad (i = 1, 2, \cdots, r; j = 1, 2, \cdots, s; k = 1, 2, \cdots, t).$$

THEOREM. (2.3) $\cap$ (2.7) *or* (2.8) $\Rightarrow$ (2.2).

PROOF. A straightforward proof, in which everything is spelled out, is given for the case of a $2 \times 2 \times 2$ table on pages 71 and 72 of reference [3]. A similar proof has been obtained for the general $r \times s \times t$ table and will be shortly published. The following, however, is another proof based on a general type of reasoning.

Starting from (2.8) and summing over j and i and using (2.3) we have, respectively,

$$(2.9)\quad p_{i0k} = p_{i00}p_{00k} = q_{i0k}\sum_j q_{ij0}q_{0jk}$$

and

$$(2.10)\quad p_{0jk} = p_{0j0}p_{00k} = q_{0jk}\sum_{i'} q_{i'j0}q_{i'0k}\,.$$

Substituting in (2.9) for q_{0jk} from (2.10), we have

$$(2.11)\quad \frac{p_{i00}\,p_{00k}}{q_{i0k}} = \sum_j \left[q_{ij0}\,p_{0j0}\,p_{000k}/\sum_{i'} q_{i'j0}\,q_{i'0k}\right].$$

So far as the functional form is concerned, this equation, without any loss of generality, can be replaced by

$$(2.12)\quad \frac{1}{q_{i0k}} = \sum_j \left[q_{ij0} \Big/ \sum_{i'} q_{i'j0}\,q_{i'0k}\right].$$

Now suppose that we regard (2.12) as a set of equations in q_{i0k} (with $i = 1, 2, \cdots, r$ and $k = 1, 2, \cdots, t$) in which q_{ij0}'s act as a set of given parameters, then (i) it is clear from (2.12) that any q_{i0k} will depend on the whole set of q_{ij0}'s, the form of this dependence varying possibly with i but obviously not with k; (ii) if q_{i0k} (with $i = 1, 2, \cdots, r$ and $k = 1, 2, \cdots, t$) is a solution set, $f_3(k)q_{i0k}$ is also a solution set, where $f_3(k)$ is any function of 'k' alone. Together, (i) and (ii) show that so far as the functional form is concerned

$$(2.13)\quad q_{i0k} = f_1(i)f_3(k)$$

and likewise

$$(2.14)\quad q_{0jk} = f_2(j)f_4(k).$$

Combining the two we can, without any loss of generality, write

$$(2.15)\quad p_{ijk} = f(k)q_{ij0} = q_{00k}q_{ij0} \text{ (say)}.$$

Summing over k and over (i, j) we have, respectively,

(2.16) $p_{ij0} = q_{ij0}\sum_k q_{00k} = q_{ij0}q^{(1)}$ (say) and $p_{00k} = q_{00k}q^{(2)}$ (say).

Summing up over any one of these two sets of relations, we have

$$1 = q^{(2)}q^{(1)}. \tag{2.17}$$

Substituting back from (2.16) and (2.17) in (2.15), we have

$$p_{ijk} = p_{ij0}p_{00k}\,. \tag{2.18}$$

Notice that if in (2.7) we were to replace (i, j, k) by (x, y, z), then (2.7) would be found to imply

$$f(x, y, z) = \frac{f_1(x, y)f_2(x, z)f_3(y, z)}{F_1(x)F_2(y)F_3(z)}, \tag{2.19}$$

with nothing else connecting f_1, f_2, f_3, F_1, F_2, F_3 among themselves or with f.

The hypothesis (2.2) is the natural analogue of the hypothesis of "no multiple correlation" between (i, j) and k, while (2.3) is the natural analogue of the hypotheses of no correlation between i and k and between j and k. Thus the hypothesis (2.7) is, as it were, a kind of bridge over the gap between (2.3) and (2.2). By a certain analogy with "normal variate" analysis of variance we can call it the hypothesis of "no interaction" between i and j. "Normal variate" multivariate analysis doesn't have any concept like this, because there this situation does not arise.

3. A large sample test of (2.7) **in terms of** χ^2 [2, 6]. We start from (2.1) and maximize ϕ with respect to p_{ijk}'s subject to $\sum_{i,j,k} p_{ijk} = 1$ and also subject to the constraints that we would get by eliminating the q's between the equations (2.7). We [2] end up with a number of solutions of the maximum likelihood equations subject to constraints, but among these solutions there is one and only one solution set, say $\hat{p}_{ijk}$'s having under (2.7) the property (i) that in large samples ($\hat{p}_{ijk}$'s) $\rightarrow$ (true p^0_{ijk}'s) in probability and (ii) that $\sum_{i,j,k} (n_{ijk} - n\hat{p}_{ijk})^2/n\hat{p}_{ijk}$ has approximately the χ^2-distribution with degrees of freedom equal to the number of constraints on the p's that arise by eliminating q's among (2.7). To fix our ideas we shall first consider the case of a $2 \times 2 \times 2$ table and then the general $r \times s \times t$ table.

"No interaction" in a $2 \times 2 \times 2$ *table.* Consider in this case the hypothesis (2.7), and write it out in full as follows:

$$(3.1)\qquad \begin{aligned} H_0:\ p_{111} &= \frac{q_{110}\,q_{101}\,q_{011}}{q_{100}\,q_{010}\,q_{001}}, & p_{211} &= \frac{q_{210}\,q_{201}\,q_{011}}{q_{200}\,q_{010}\,q_{001}},\\ p_{112} &= \frac{q_{110}\,q_{102}\,q_{012}}{q_{100}\,q_{010}\,q_{002}}, & p_{212} &= \frac{q_{210}\,q_{202}\,q_{012}}{q_{200}\,q_{010}\,q_{002}},\\ p_{121} &= \frac{q_{120}\,q_{101}\,q_{021}}{q_{100}\,q_{020}\,q_{001}}, & p_{221} &= \frac{q_{220}\,q_{201}\,q_{021}}{q_{200}\,q_{020}\,q_{001}},\\ p_{122} &= \frac{q_{120}\,q_{102}\,q_{022}}{q_{100}\,q_{020}\,q_{002}}, & p_{222} &= \frac{q_{220}\,q_{202}\,q_{022}}{q_{200}\,q_{020}\,q_{002}}. \end{aligned}$$

It is easy to check that by eliminating the q's, we have what we will call the "*no interaction*" *constraints*, which in this case represent just one relation among the p's, namely,

$$\frac{p_{111}p_{221}}{p_{211}p_{121}} = \frac{p_{112}p_{222}}{p_{212}p_{122}}. \tag{3.2}$$

This is Bartlett's hypothesis of "no interaction" discussed in [1].

There is, of course, the other side condition on the p's:

$$\textstyle\sum_{i,j,k} p_{ijk} = 1. \tag{3.3}$$

Recalling again Section 2, the likelihood function can be written as

$$\phi \sim \prod_{i,j,k} p_{ijk}^{n_{ijk}}. \tag{3.4}$$

The problem is to estimate the p_{ijk}'s by maximizing ϕ subject to the constraints (3.2) and (3.3). Introducing the usual Lagrangian multipliers on (3.2) and (3.3), we have the maximum likelihood equations

$$\begin{aligned} &\frac{n_{ijk}}{p_{ijk}} + \frac{\lambda}{p_{ijk}} + \mu = 0 \qquad (ijk = 111, 221, 212, 122),\\ &\frac{n_{ijk}}{p_{ijk}} - \frac{\lambda}{p_{ijk}} + \mu = 0 \qquad (ijk = 112, 222, 211, 121). \end{aligned} \tag{3.5}$$

Now multiplying by p_{ijk}, and summing over i, j, k and using (3.3), we have $\mu = -n$, and

$$\begin{aligned} p_{ijk} &= +(n_{ijk} + \lambda)/n \quad (ijk = 111, 221, 212, 122),\\ p_{ijk} &= +(n_{ijk} - \lambda)/n \quad (ijk = 112, 222, 211, 121). \end{aligned} \tag{3.6}$$

Substituting in (8.2), we have for λ the cubic equation

$$\frac{(n_{111} + \lambda)(n_{221} + \lambda)}{(n_{211} - \lambda)(n_{121} - \lambda)} = \frac{(n_{112} - \lambda)(n_{222} - \lambda)}{(n_{212} + \lambda)(n_{122} + \lambda)}. \tag{3.7}$$

There is one and only one root [4] of this equation which will yield an estimate that tends in probability to the true population parameter point and lead to a χ^2-distribution. It will be shown in a later paper that the numerically smallest (real) root of (3.7) is the one which will satisfy this condition.

Solving for λ and substituting in (3.6), we have the estimated $\hat{p}_{ijk}$'s occurring in the usual χ^2. Since

$$\begin{aligned} n_{ijk} - np_{ijk} &= -\lambda \qquad (ijk = 111, 221, 212, 122),\\ n_{ijk} - np_{ijk} &= +\lambda \qquad (ijk = 112, 222, 211, 121), \end{aligned} \tag{3.8}$$

the final χ^2 is given by

$$\chi^2 = \frac{\lambda^2}{n} \sum_{i,j,k=1}^{2} \hat{p}_{ijk}^{-1}. \tag{3.9}$$

This will be a χ^2 with d.f. = the total number of cells (8 here) − [the apparent number of parameters (8 here) − the number of "no interaction" constraints (1 here) − the number of linear relations on the p's coming from the linear constraints on the n's (1 here)] − [the number of linear relations on the n's (1 here)] = the number of "no interaction" constraints = 1, in this case. It was shown in [4] that, in all cases, no matter if i, j, and k are all "variates," or if some are "variates" and some are "ways of classification," or if all are "ways of classification," we are going to end up with a χ^2 with d.f. exactly equal to the number of "no interaction" constraints like those of (3.2).

Notice that in (3.5), the Lagrangian μ goes with the constraint $\sum_{i,j,k} p_{ijk} = 1$ which stems from $\sum_{i,j,k} n_{ijk} = n$, and the Lagrangian λ goes with the "no interaction" constraints (3.2).

4. "No interactions" in an $r \times s \times t$ table. Let us consider here the hypothesis of "no interaction," and try to eliminate the q's. To fix our ideas, consider first the case of a $2 \times 2 \times t$ table. Looking into the mechanics by which (3.2) is obtained from (3.1), it is easy to see that, corresponding to (3.2), we are going to have

$$(4.1) \qquad \frac{p_{11t}\,p_{22t}}{p_{21t}\,p_{12t}} = \frac{p_{11,t-1}\,p_{22,t-1}}{p_{21,t-1}\,p_{12,t-1}} = \frac{p_{11,t-2}\,p_{22,t-2}}{p_{21,t-2}\,p_{12,t-2}} = \cdots = \frac{p_{111}\,p_{221}}{p_{211}\,p_{121}}.$$

For a general $r \times s \times t$ table we can figure out that we are going to have the following "*no interaction constraints*":

$$(4.2) \qquad \frac{p_{rst}\,p_{ijt}}{p_{ist}\,p_{rjt}} = \frac{p_{rsk}\,p_{ijk}}{p_{rsk}\,p_{rjk}}, \qquad \text{for} \begin{cases} k = 1, 2, \cdots, (t-1), \\ j = 1, 2, \cdots, (s-1), \\ i = 1, 2, \cdots, (r-1). \end{cases}$$

This gives us $(t-1)(s-1)(r-1)$ constraints on the p_{ijk}'s. Checking the mechanics of the derivation of (4.2) from (4.1), it will be seen that (4.2) yields a set of independent and exhaustive relations among the p's by eliminating the q's from (2.7). Here p_{rst} is, as it were, a pivotal element, and r, s, and t the pivotal subscripts. We can make any other three subscripts the pivotal ones, and thus obtain another set of independent and exhaustive relations like in (4.2), which would be exactly equivalent to (4.2), and so on.

Our likelihood function is

$$(4.3) \qquad \phi \sim \prod_{i,j,k} p_{ijk}^{n_{ijk}}.$$

Here we have to maximize (4.3) subject to the "no interaction" constraints (4.2), and the further constraint

$$(4.4) \qquad \sum_{i,j,k} p_{ijk} = 1.$$

Introducing for (4.1) the Langrangian multipliers $\lambda_{ijk}[i = 1, 2, \cdots, (r-1);$ $j = 1, 2, \cdots, (s-1); k = 1, 2, \cdots, (t-1)]$, and for (4.4) the Lagrangian multiplier μ, and maximizing (4.3), we have for p_{ijk} the typical equations

$$
(4.5)\quad
\begin{array}{ll}
\dfrac{n_{rst}}{p_{rst}} + \dfrac{\sum_{i=1}^{(r-1)} \sum_{j=1}^{(s-1)} \sum_{k=1}^{(t-1)} \lambda_{ijk}}{p_{rst}} + \mu = 0, & \dfrac{n_{ist}}{p_{ist}} - \dfrac{\sum_{j=1}^{(s-1)} \sum_{k=1}^{(t-1)} \lambda_{ijk}}{p_{ist}} + \mu = 0, \\[2ex]
\dfrac{n_{rjt}}{p_{rjt}} - \dfrac{\sum_{i=1}^{(r-1)} \sum_{k=1}^{(t-1)} \lambda_{ijk}}{p_{rjt}} + \mu = 0, & \dfrac{n_{rsk}}{p_{rsk}} - \dfrac{\sum_{i=1}^{(r-1)} \sum_{j=1}^{(s-1)} \lambda_{ijk}}{p_{rsk}} + \mu = 0, \\[2ex]
\dfrac{n_{ijt}}{p_{ijt}} + \dfrac{\sum_{k=1}^{(t-1)} \lambda_{ijk}}{p_{ijt}} + \mu = 0, & \dfrac{n_{isk}}{p_{isk}} + \dfrac{\sum_{j=i}^{(s-1)} \lambda_{ijk}}{p_{isk}} + \mu = 0 \\[2ex]
\dfrac{n_{rjk}}{p_{rjk}} + \dfrac{\sum_{i=1}^{(r-1)} \lambda_{ijk}}{p_{rjk}} + \mu = 0, & \dfrac{n_{ijk}}{p_{ijk}} - \dfrac{\lambda_{ijk}}{p_{ijk}} + \mu = 0
\end{array}
$$

with, of course, $i = 1, 2, \cdots, (r-1)$; $j = 1, 2, \cdots, (s-1)$; $k = 1, 2, \cdots, (t-1)$. Notice that with the pivotal subscripts (rst) goes a triple summation over the λ's and a positive sign before that expression; with just one subscript changed goes a double summation over the λ's and a negative sign before that expression; with two of the subscripts changed goes a single summation over the λ's and a positive sign before that expression; and finally with all the subscripts changed, we have a single λ_{ijk} with a negative sign before it.

As in the case of the $2 \times 2 \times 2$, it is easy to see by multiplying both sides of (4.5) by p_{ijk} and summing over i, j, k, that $\mu = -n$. Thus solving for the p_{ijk}'s in terms of the n_{ijk}'s and λ_{ijk}'s, and substituting in the "no interaction" constraints (4.2), we have for λ_{ijk} the following equations [for $i = 1, 2, \cdots, (r-1)$; $j = 1, 2, \cdots, (s-1)$; $k = 1, 2, \cdots, (t-1)$]:

$$
(4.10)\qquad \frac{(n_{rst} + \mu_{rst})(n_{ijt} + \mu_{ijt})}{(n_{ist} - \mu_{ist})(n_{rjt} - \mu_{rjt})} = \frac{(n_{rsk} - \mu_{rsk})(n_{ijk} - \mu_{ijk})}{(n_{isk} + \mu_{isk})(n_{rjk} + \mu_{rjk})},
$$

where μ_{rst} stands for the triple summation expression in (4.5), μ_{ist}, μ_{rjt}, μ_{rsk} for the double summation expressions in (4.5), μ_{ijt}, μ_{isk}, μ_{rjk} for single summation expressions in (4.5), and μ_{ijk} is simply λ_{ijk}. As observed in connection with (3.7), here also there is one and only one solution [4] of this equation which will yield an estimate that tends in probability to the true population parameter point and that will lead to a χ^2-distribution; it will be shown in a later paper that the (real) solution for which the distance from the origin in the space of μ_{ijk}'s is the least is the solution leading to a χ^2 distribution. Solving equations (4.10) for the μ_{ijk}'s, and ultimately for the λ_{ijk}'s, in terms of the n_{ijk}'s, we can find the p_{ijk}'s. Substituting these values in the usual expression for χ^2 we have

$$
(4.11)\qquad \sum\nolimits_{i,j,k} \mu_{ijk}^2/(n_{ijk} + \eta_{ijk}\mu_{ijk}),
$$

where $i = 1, 2, \cdots, r; j = 1, 2, \cdots, s; k = 1, 2, \cdots, t$; and where

$\eta_{ijk} = +1$ if $ijk = rst$ (the pivotal subscripts);

$\eta_{ijk} = -1$ if any one subscript differs from the corresponding pivotal subscript;

$\eta_{ijk} = +1$ if any two subscripts differ from the corresponding pivotal subscripts;

$\eta_{ijk} = -1$ if all subscripts differ from the corresponding pivotal subscripts.

Using [2.5] it will be seen that the statistic (4.11) will be distributed as a χ^2 with d.f. = number of "no interaction" constraints on the p's $= (r - 1)(s - 1)(t - 1)$.

For several types of data on effects of exposure to radiation, the equations (4.10) and similar equations for some four-way tables have been solved on an electronic computor at Ann Arbor, Michigan, using the method of steepest descent supplemented by some numerical graphical procedures. The details and the final results for the actual data handled will be reported in a later paper.

5. Concluding remarks. The lines along which the concept and structure of the hypothesis of "no interaction" is to be generalized to multi-way contingency tables of higher dimensions can now be indicated. For example, in a four-way table the hypothesis analogous to (2.7), that is, the hypothesis of no "second-order interaction" seems to be

$$(5.1) \qquad H_0: p_{ijkl} = \frac{q_{ijk0}\, q_{ij0l}\, q_{i0kl}\, q_{0jkl}\, q_{i000}\, q_{0j00}\, q_{00k0}\, q_{000l}}{q_{ij00}\, q_{i0k0}\, q_{i00l}\, q_{0jk0}\, q_{0j0l}\, q_{00kl}}.$$

In this case the hypotheses of four separate "no first-order interactions" follow exactly the same pattern as in Section 4, and need not be separately considered. The extension of (5.1) to higher-order "no interactions," in the case of tables of higher dimensions, forms a certain pattern which has been worked out and which will be discussed in a later paper. The technique of testing (5.1) and "no interaction" hypotheses of higher order is essentially similar, in principle, to what has been discussed in Section 4. The details alone are more complicated. For higher-order "no interactions" there are, however, various intermediate cases of considerable interest which will be discussed later.

Going back to the three-way $r \times s \times t$ table again, it may be remarked [4] that we have an asymptotically equivalent test if we plug into the χ^2-statistic any B. A. N. estimate of the p_{ijk}'s (consistent with (4.2) and (4.4)), and not just the maximum likelihood estimate of the p_{ijk}'s subject to (4.2) and (4.4). In particular, we can estimate [4] the p_{ijk}'s by minimizing the modified χ^2 (sometimes called the χ_1^2) subject to (4.2) and (4.4). However, if we use the χ_1^2-statistic for estimation, a much better procedure would be (i) to define the "no interaction" condition as a "linearized" counterpart of (4.2); (ii) to estimate the p_{ijk}'s by minimizing χ_1^2 subject to (4.4) and the 'linearized' counterpart of (4.2); and (iii) to plug these estimates into χ_1^2 itself and use the χ_1^2-test, which is the same as the χ^2-test. This has been done in [3] and the material will be offered shortly for publication. Notice that (4.2) itself is a logarithmic linear hypothesis of the nature of a set of contrasts.

This paper, unlike most previous work [1, 5, 6], discusses the hypothesis of "no interaction" in relation to the multivariate analysis situation only—that is, where it is only the total n that is fixed and no marginal frequencies. For analysis of variance situations—that is, when marginals along one or more directions are fixed—the authors do not find the "no interaction" concept too meaningful [7]. Nevertheless, starting more or less from the conditional probability set up (which can be justified for the specific cases considered) the authors have discussed in a previous paper [6] the "no interaction" hypothesis and its tests for the analysis of variance situations, too—that is, where the marginal frequencies are fixed along one or more directions of the multiway table. The formal structure of the hypothesis and the formal analysis remain the same as for the multivariate analysis situation discussed in this paper.

We give below a few references which have a direct bearing on this paper.

REFERENCES

[1] M. S. Bartlett, "Contingency table interactions," *J. Roy. Stat. Soc., Suppl.*, Vol. 2 (1935), pp. 248–252.

[2] H. Cramer, *Mathematical Methods of Statistics*, Princeton University Press, 1946, Chap. 30.

[3] S. K. Mitra, "Contributions to the statistical analysis of categorical data," Institute of Statistics, University of North Carolina, Mimeograph Series No. 142, 1955.

[4] J. Neyman, "Contribution to the theory of the χ^2-test," *Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability*, University of California Press, 1949.

[5] W. H. Norton, "Calculation of chi-square from complex contingency tables," *J. Am. Stat. Assn.*, Vol. 40 (1945), pp. 251–258.

[6] S. N. Roy and M. Kastenbaum, "A generalization of analysis of variance and multivariate analysis to data based on frequencies in qualitative categories or class intervals," Institute of Statistics, University of North Carolina, Mimeographed series No. 131, 1955.

[7] S. N. Roy and S. K. Mitra, "An introduction to some nonparametric generalizations of analysis of variance and multivariate analysis," Institute of Statistics, University of North Carolina, Mimeograph series No. 139, 1955.