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Research Article

On the Ideal Convergence of Double Sequences in Locally Solid Riesz Spaces

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The aim of this paper is to define the notions of ideal convergence, I-bounded for double sequences in setting of locally solid Riesz spaces and study some results related to these notions. We also define the notion of I^*-convergence for double sequences in locally solid Riesz spaces and establish its relationship with ideal convergence.

1. Introduction and Preliminaries

In 1951, Fast [1] and Steinhaus [2] introduced the concept of statistical convergence for single sequences, independently. Some basic and important properties of this concept were studied by Buck [3], Šalát [4], Schoenberg [5], and Fridy [6]. Later, the notion of statistical convergence for single sequences was further defined in various spaces; see Çakalli and Khan [7–9], Di Maio et al. [10, 11], Hazarika [12–14], Maddox [15], Mohiuddine et al. [16–19], and so forth. Some application of statistical summability methods is presented in [20, 21]. In 2003, the notion of statistical convergence for single sequences has been extended to double sequences by Mursaleen and Edely [22]. Recently, the statistical convergence and statistical Cauchy for double sequences have been defined in the framework fuzzy and intuitionistic normed spaces by Mohiuddine et al. [23] and Mursaleen and Mohiuddine [24], respectively, and established some interesting results related to the concept of statistical convergence and statistical Cauchy double sequences. Recently, it was defined and studied by Mohiuddine et al. [25] in the setting of locally solid Riesz spaces while for single sequences this concept was first studied by Albayrak and Pehlivan [26] (also see [27–29]). An application of locally solid Riesz spaces in economics can be found in [30].

The notion of ideal convergence for single sequences, which is a generalization of the concept of statistical convergence, was first defined and studied by Kostyrko et al. [31]. Let us recall the notion of ideal convergence and related concepts by Kostyrko et al. [31] as follows. Let $\mathbb{N}$ be a nonempty set. Then a family of sets $I \subseteq P(\mathbb{N})$ (power set of $\mathbb{N}$) is said to be an ideal if I is additive; that is, $A, B \in I \Rightarrow A \cup B \in I$ and $A \in I, B \subseteq A \Rightarrow B \in I$. A family of sets $I \subset P(\mathbb{N})$ (power sets of $\mathbb{N}$) is called an *ideal* if and only if, for each $A, B \in I$, we have $A \cup B \in I$ and, for each $A \in I$ and each $B \subset A$, we have $B \in I$. A nonempty family of sets $\mathcal{F} \subset P(\mathbb{N})$ is a *filter* on $\mathbb{N}$ if and only if $\Phi \notin \mathcal{F}$; for each $A, B \in \mathcal{F}$, we have $A \cap B \in \mathcal{F}$ and for each $A \in \mathcal{F}$ and each $A \subset B$, we have $B \in \mathcal{F}$. An ideal I is called nontrivial ideal if $I \neq \Phi$ and $\mathbb{N} \notin I$. Clearly $I \subset P(\mathbb{N})$ is a nontrivial ideal if and only if $\mathcal{F} = \mathcal{F}(I) = \{\mathbb{N} - A : A \in I\}$ is a filter on $\mathbb{N}$. A nontrivial ideal $I \subset P(\mathbb{N})$ is called *admissible* if and only if $\{\{x\} : x \in \mathbb{N}\} \subset I$. A nontrivial ideal I is *maximal* if there cannot exist any nontrivial ideal $J \neq I$ containing I as a subset.

We remark that if $I = I_f = \{A \subseteq \mathbb{N} : A \text{ is a finite subset}\}$, then the corresponding convergence coincides with the usual convergence. Also, if $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$, then the corresponding convergence coincides with the statistical convergence (where $\delta(A)$ denotes the natural density of the set A). In the above cases, both I_f and I_δ are nontrivial admissible ideals of $\mathbb{N}$.

Kumar [32] defined the notions of I and I^*-convergence of double sequence and studied some properties of these notions. Recently, Das et al. [33] introduced the concepts of I and I^*-convergence of double sequences in the setting of metric space and established some relationship between these types of convergence. Quite recently, Mursaleen and Mohiuddine defined and studied the notion of I-convergence, I^*-convergence, I-limit points, and I-cluster points for single and double sequences, in [34, 35], respectively, in probabilistic normed spaces. Şahiner et al. [36] and Gürdal and Açik [37] introduced the notion of ideal convergence and I-Cauchy sequence in 2-normed spaces, respectively. Mursaleen and Alotaibi [38] introduced the notion of ideal convergence in random 2-normed spaces and later on it was extended by Mohiuddine et al. [39] from single to double sequences. For more details on these concepts, one can be referred to [40–52].

Now we recall the definition of locally solid Riesz spaces and some related concepts as follows. Let X be a real vector space and let $\leq$ be a partial order on this space. X is said to be an *ordered vector space* if it satisfies the following properties:

(1) if $x, y \in X$ and $y \leq x$, then $y + z \leq x + z$ for each $z \in X$;

(2) if $x, y \in X$ and $y \leq x$, then $ay \leq ax$ for each $a \geq 0$.

If, in addition, X is a lattice with respect to the partial ordering, then X is said to be a *Riesz space* (or a *vector lattice*)(see [53]).

For an element x of a Riesz space X, the positive part of x is defined by $x^+ = x \vee \overline{\theta} = \sup\{x, \overline{\theta}\}$, the negative part of x by $x^- = (-x) \vee \overline{\theta}$, and the absolute value of x by $|x| = x \vee (-x)$, where $\overline{\theta}$ is the zero element of X.

A subset S of X is said to be *solid* if $y \in S$ and $|x| \leq |y|$ implies $x \in S$.

A topology τ on a real vector space X that makes the addition and scalar multiplication continuous is said to be a linear topology, that is, when the mappings

$$\begin{gathered}(x, y) \longrightarrow (x + y) \quad (\text{from } (X \times X, \tau \times \tau) \longrightarrow (X, \tau)), \\ (\lambda, x) \longrightarrow (\lambda x) \quad \left(\text{from } \left(\mathbb{R} \times X, \tau' \times \tau\right) \longrightarrow (X, \tau)\right)\end{gathered} \tag{1}$$

are continuous, where τ' is the usual topology on $\mathbb{R}$. In this case the pair (X, τ) is called a *topological vector space*.

Every linear topology τ on a vector space X has a base N for the neighborhoods of $\overline{\theta}$ satisfying the following properties.

(1) Each $Y \in N$ is a *balanced set*; that is, $ax \in Y$ holds for all $x \in Y$ and every $a \in \mathbb{R}$ with $|a| \leq 1$.

(2) Each $Y \in N$ is an *absorbing set*; that is, for every $x \in X$, there exists $a > 0$ such that $ax \in Y$.

(3) For each $Y \in N$ there exists some $E \in N$ with $E + E \subseteq Y$.

A linear topology τ on a Riesz space X is said to be *locally solid* (see [54]) if τ has a base at zero consisting of solid sets. A *locally solid Riesz space* (X, τ) is a Riesz space X equipped with a locally solid topology τ. For more details on these concepts, one can be referred to [55–57].

Throughout the paper, the symbol N_{sol} will stand for a base at zero consisting of solid sets and satisfying conditions (1), (2), and (3) in a locally solid topology. Also we assume that I_2 is a nontrivial admissible ideal of $\mathbb{N} \times \mathbb{N}$.

2. Ideal Convergence of Double Sequences in LSR-Spaces

Throughout the paper X will denote the Hausdorff locally solid Riesz space, which satisfies the first axiom of countability. For our convenience, here and in what follows, we will write an LSR-space instead of a locally solid Riesz space.

The notion of convergence for double sequence was first introduced by Pringsheim [58] as follows. We say that a double sequence $x = (x_{j,k})_{j,k \in \mathbb{N}}$ of reals is convergent to L in Pringsheim's sense (briefly, P-convergent) provided that given $\epsilon > 0$ there exists a positive integer N such that $|x_{j,k} - L| < \epsilon$ whenever $j, k \geq N$.

Let $K \subset \mathbb{N} \times \mathbb{N}$ and $K(m, n)$ denotes the number of (i, j) in K such that $i \leq m$ and $j \leq n$ (see [22]). Then the lower natural density of K is defined by $\underline{\delta}_2(K) = \liminf_{m,n \to \infty}(|K(m, n)|/mn)$. In this case, the sequence $(K(m, n)/mn)$ has a limit in Pringsheim's sense; then we say that K has a *double natural density* and is defined by $P - \lim_{m,n \to \infty}(|K(m, n)|/mn) = \delta_2(K)$.

In the recent past, Mohiuddine et al. [25] introduced the notion of statistical convergence of double sequences in LSR-space as follows. Let (X, τ) be a LSR-space. A double sequence $(x_{k,l})$ of points in X is said to be $S_2(\tau)$-*convergent* to an element x_0 if for each τ-neighborhood V of zero

$$\delta_2\left(\left\{(k, l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin V\right\}\right) = 0. \tag{2}$$

Now we introduce the notions of $I_2(\tau)$-convergence and $I_2(\tau)$-bounded double sequences in LSR-spaces.

Definition 1. Let (X, τ) be a LSR-space. A double sequence $(x_{k,l})$ of points in X is said to be $I_2(\tau)$-*convergent* to an element x_0 of X if for each τ-neighborhood V of zero

$$\left\{(k, l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin V\right\} \in I_2. \tag{3}$$

That is,

$$\left\{(k, l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \in V\right\} \in \mathscr{F}. \tag{4}$$

In this case, one writes $I_2(\tau)\text{-}\lim_{k,l \to \infty} x_{k,l} = x_0$ or $(x_{k,l}) \xrightarrow{I_2(\tau)} x_0$.

Definition 2. Let (X, τ) be a LSR-space. Then, a double sequence $(x_{k,l})$ of points in X is said to be $I_2(\tau)$-*bounded* in X if, for each τ-neighborhood V of zero, there is some $a > 0$,

$$\left\{(k, l) \in \mathbb{N} \times \mathbb{N} : a x_{k,l} \notin V\right\} \in I_2. \tag{5}$$

Definition 3. Let (X, τ) be a LSR-space. One says that a double sequence $x = (x_{k,l})$ is $I_2(\tau)$-*Cauchy* in X if, for each τ-neighborhood V of zero, there exist $p, q \in \mathbb{N}$ such that, for all $k, m \geq p$ and $l, n \geq q$,

$$\left\{(k, l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_{m,n} \notin V\right\} \in I_2. \tag{6}$$

Definition 4. Let (X, τ) be a LSR-space. Then, a double sequence $x = (x_{k,l})$ in X is said to be $I_2^*(\tau)$-convergent to x_0 if there is a set $K = \{(k, l)\} \subseteq \mathbb{N} \times \mathbb{N}$, $k, l = 1, 2, \ldots$, with $K \in F$ such that $\lim_{k,l} x_{k,l} = x_0$. In this case, one writes $I_2^*(\tau)$-$\lim_{k,l} x_{k,l} = x_0$.

Theorem 5. *Let (X,τ) be a LSR-space. Every $I_2(\tau)$-convergent sequence in X has only one limit.*

Proof. Suppose that $x=(x_{k,l})$ is a double sequence in X such that $I_2(\tau)\text{-}\lim_{k,l}x_{k,l}=x_0$ and $I_2(\tau)\text{-}\lim_{k,l}x_{k,l}=y_0$. Let V be any τ-neighborhood of zero. Also for each τ-neighborhood V of zero there is a set $Y\in N_{\text{sol}}$ such that $Y\subseteq V$. Let W in N_{sol} be such that $W+W\subseteq Y$. We define the sets A_1 and A_2 as follows:

$$A_1=\{(k,l)\in\mathbb{N}\times\mathbb{N}:x_{k,l}-x_0\in W\},$$
$$A_2=\{(k,l)\in\mathbb{N}\times\mathbb{N}:x_{k,l}-y_0\in W\}. \tag{7}$$

Since $I_2(\tau)\text{-}\lim x_{k,l}=x_0$ and $I_2(\tau)\text{-}\lim x_{k,l}=y_0$, we get $A_1,A_2\in\mathcal{F}$. Now, let $A=A_1\cap A_2$. Then we have

$$x_0-y_0=x_0-x_{k,l}+x_{k,l}-y_0\in W+W\subseteq Y\subseteq V. \tag{8}$$

As we know, intersection of all τ-neighborhoods V of zero is the singleton set $\{\overline{\theta}\}$ because (X,τ) is Hausdorff. Hence $x_0-y_0=0$; that is, $x_0=y_0$. □

Theorem 6. *Let (X,τ) be a LSR-space and let $(x_{k,l})$ and $(y_{k,l})$ be two double sequences of points in X. Then,*

(i) *if $I_2(\tau)\text{-}\lim_{k,l}x_{k,l}=x_0$ and $I_2(\tau)\text{-}\lim_{k,l}y_{k,l}=y_0$, then $I_2(\tau)\text{-}\lim_{k,l}(x_{k,l}+y_{k,l})=x_0+y_0$;*

(ii) *if $I_2(\tau)\text{-}\lim_{k,l}x_{k,l}=x_0$, then $I_2(\tau)\text{-}\lim_{k,l}ax_{k,l}=ax_0$ for $a\in\mathbb{R}$.*

Proof. Assume that $I_2(\tau)\text{-}\lim_{k,l}x_{k,l}=x_0$ and $I_2(\tau)\text{-}\lim_{k,l}y_{k,l}=y_0$. Suppose that V is an arbitrary τ-neighborhood of zero. Then there exists $Y\in N_{\text{sol}}$ such that $Y\subseteq V$. Let $W\in N_{\text{sol}}$ such that $W+W\subseteq Y$. Thus, we can write

$$B_1=\{(k,l)\in\mathbb{N}\times\mathbb{N}:x_{k,l}-x_0\in W\},$$
$$B_2=\{(k,l)\in\mathbb{N}\times\mathbb{N}:y_{k,l}-y_0\in W\}. \tag{9}$$

Then we have $B_1,B_2\in\mathcal{F}$.

Let $B=B_1\cap B_2$. Hence we have $B\in\mathcal{F}$ and

$$(x_{k,l}+y_{k,l})-(x_0+y_0)=(x_{k,l}-x_0)+(y_{k,l}-y_0)\in W+W\subseteq Y\subseteq V. \tag{10}$$

Therefore

$$\{(k,l)\in\mathbb{N}\times\mathbb{N}:(x_{k,l}+y_{k,l})-(x_0+y_0)\in V\}\in\mathcal{F}. \tag{11}$$

Since V is arbitrary, we have $I_2(\tau)\text{-}\lim(x_{k,l}+y_{k,l})=x_0+y_0$.

(ii) Suppose that $I_2(\tau)\text{-}\lim_{k,l}x_{k,l}=x_0$ and also suppose that V is an arbitrary τ-neighborhood of zero. Then there exists $Y\in N_{\text{sol}}$ such that $Y\subseteq V$, so we have

$$\{(k,l)\in\mathbb{N}\times\mathbb{N}:x_{k,l}-x_0\in Y\}\in\mathcal{F}. \tag{12}$$

Since Y is balanced, $a(x_{k,l}-x_0)\in Y$ holds for all $x_{k,l}-x_0\in Y$ and for every $a\in\mathbb{R}$ with $|a|\le 1$. Therefore

$$\{(k,l)\in\mathbb{N}\times\mathbb{N}:x_{k,l}-x_0\in Y\}\subseteq\{(k,l)\in\mathbb{N}\times\mathbb{N}:ax_{k,l}-ax_0\in Y\}\subseteq\{(k,l)\in\mathbb{N}\times\mathbb{N}:x_{k,l}-x_0\in V\}. \tag{13}$$

Thus, we have

$$\{(k,l)\in\mathbb{N}\times\mathbb{N}:x_{k,l}-x_0\in V\}\in\mathcal{F} \tag{14}$$

for each τ-neighborhood V of zero. Now let $|a|>1$ and $[|a|]$ be the smallest integer greater than or equal to $|a|$. Then there exists $W\in N_{\text{sol}}$ such that $[|a|]W\subseteq Y$. From our assumption that $I_2(\tau)\text{-}\lim_{k,l}x_{k,l}=x_0$, we obtain that

$$K=\{(k,l)\in\mathbb{N}\times\mathbb{N}:x_{k,l}-x_0\in W\}\in\mathcal{F}. \tag{15}$$

Therefore

$$|ax_{k,l}-ax_0|=|a|\,|x_{k,l}-x_0|\le[|a|]\,|x_{k,l}-x_0|\in[|a|]\,W\subseteq Y\subseteq V. \tag{16}$$

Since Y is solid, $ax_k-ax_0\in Y$. It follows that $ax_{k,l}-ax_0\in V$. Thus,

$$\{(k,l)\in\mathbb{N}\times\mathbb{N}:ax_{k,l}-ax_0\in V\}\in\mathcal{F}, \tag{17}$$

for each τ-neighborhood V of zero. We conclude that $I_2(\tau)\text{-}\lim_{k,l}ax_{k,l}=ax_0$. □

Theorem 7. *Let (X,τ) be a LSR-space. If a double sequence $(x_{k,l})$ in X is $I_2(\tau)$-convergent, then it is $I_2(\tau)$-bounded.*

Proof. Assume that $I_2(\tau)\text{-}\lim_{k,l\to\infty}x_{k,l}=x_0$. Suppose V is an arbitrary τ-neighborhood of zero. Then, there exists $Y\in N_{\text{sol}}$ such that $Y\subseteq V$. Let $W\in N_{\text{sol}}$ such that $W+W\subseteq Y$. Using our assumption, we obtain that

$$A=\{(k,l)\in\mathbb{N}\times\mathbb{N}:x_{k,l}-x_0\notin W\}\in I_2. \tag{18}$$

Since W is absorbing, there exists $a>0$ such that $ax_0\in W$. Let b be such that $|b|\le 1$ and $b\le a$. Since W is solid and $|bx_0|\le|ax_0|$, we have $bx_0\in W$. Also, since W is balanced, $x_{k,l}-x_0\in W$ implies $b(x_{k,l}-x_0)\in W$. Then we have

$$bx_{k,l}=b(x_{k,l}-x_0)+bx_0\in W+W\subseteq V,\quad\text{for each } k,l\in\mathbb{N}-A. \tag{19}$$

Thus

$$\{(k,l)\in\mathbb{N}\times\mathbb{N}:bx_{k,l}\notin W\}\in I_2. \tag{20}$$

Hence $(x_{k,l})$ is $I_2(\tau)$-bounded. □

Theorem 8. *Let (X,τ) be a LSR-space and let $(x_{k,l})$, $(y_{k,l})$, and $(z_{k,l})$ be three double sequences of points in X such that*

(i) *$x_{k,l}\le y_{k,l}\le z_{k,l}$, for all $k,l\in\mathbb{N}$,*

(ii) *$I_2(\tau)\text{-}\lim_{k,l}x_{k,l}=x_0=I_2(\tau)\text{-}\lim_{k,l}z_{k,l}$.*

Then $I_2(\tau)\text{-}\lim_{k,l}y_{k,l}=x_0$.

Proof. Suppose that the given conditions (i) and (ii) hold for the double sequences $(x_{k,l})$, $(y_{k,l})$, and $(z_{k,l})$. Suppose V is an arbitrary τ-neighborhood of zero. Then, there exists $Y\in N_{\text{sol}}$

such that $Y \subseteq V$. Let $W \in N_{\text{sol}}$ such that $W + W \subseteq Y$. It follows from (ii) that $P, Q \in F$, where

$$P = \{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \in W\}, \quad Q = \{(k,l) \in \mathbb{N} \times \mathbb{N} : z_{k,l} - x_0 \in W\}. \tag{21}$$

Also from the given condition (i), we have

$$x_{k,l} - x_0 \le y_{k,l} - x_0 \le z_{k,l} - x_0 \Longrightarrow |y_{k,l} - x_0| \le |x_{k,l} - x_0| + |z_{k,l} - x_0| \in W + W \subseteq Y. \tag{22}$$

Since Y is solid, we have $y_{k,l} - x_0 \in Y \subseteq V$. Thus,

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : y_{k,l} - x_0 \in V\} \in \mathcal{F}, \tag{23}$$

for each τ-neighborhood V of zero. Thus $I_2(\tau)\text{-}\lim_{k,l} y_{k,l} = x_0$. □

Theorem 9. *Let (X,τ) be a LSR-space. A double sequence $(x_{k,l})$ is $I_2(\tau)$-convergent to x_0 in X if and only if for each τ-neighborhood V of zero there exists a subsequence $(x_{k'(r),l'(s)})$ of $(x_{k,l})$ such that $\lim_{r,s\to\infty} x_{k'(r),l'(s)} = x_0$ and*

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_{k'(r),l'(s)} \notin V\} \in I_2. \tag{24}$$

Proof. Suppose that $I_2(\tau)\text{-}\lim_{k,l\to\infty} x_{k,l} = x_0$. Also, suppose that V is an arbitrary τ-neighborhood of zero. Let $\{V_i\}$ be a sequence of nested base of τ-neighborhoods of zero. For each $i \in \mathbb{N}$, put

$$E^{(i)} = \{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin V_i\}. \tag{25}$$

Then, $E^{(i+1)} \subset E^{(i)}$ and $E^{(i)} \in F$. Let $m(1)$ and $n(1)$ be such that $r > m(1)$ and $s > n(1)$, respectively. Then $E^{(1)} \neq \phi$. For $r, s \in \mathbb{N}$ such that $m(1) \le r < m(2)$ and $n(1) \le s < n(2)$, choose $k'(r), l'(s) \in E^{(i)}$; that is, $x_{k'(r),l'(s)} - x_0 \in V_1$. In general, choose $m(p+1) > m(p)$ and $n(p+1) > n(p)$ such that $r > m(p+1)$ and $s > n(p+1)$ hold. Then $E^{(p+1)} \neq \phi$. Therefore for all r, s which satisfy $m(p) \le r < m(p+1)$ and $n(p) \le s < n(p+1)$, choose $k'(r), l'(s) \in E^{(p)}$; that is, $x_{k'(r),l'(s)} - x_0 \in V_p$. Hence, it follows that $\lim_{r,s} x_{k'(r),l'(s)} = x_0$.

Since V is an arbitrary τ-neighborhood of zero, there exists $Y \in N_{\text{sol}}$ such that $Y \subseteq V$. Let $W \in N_{\text{sol}}$ such that $W + W \subseteq Y$. Now

$$x_{k,l} - x_{k'(r),l'(s)} = x_{k,l} - x_0 + x_{k'(r),l'(s)} - x_0 \in W + W \subseteq Y \subseteq V. \tag{26}$$

Also $I_2(\tau)\text{-}\lim_{k,l\to\infty} x_{k,l} = x_0$ and $\lim_{r\to\infty} x_{k'(r),l'(s)} = x_0$ imply that

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_{k'(r),l'(s)} \notin V\} \in I_2. \tag{27}$$

Next suppose for an arbitrary τ-neighborhood V of zero that there exists a subsequence $(x_{k'(r),l'(s)})$ of $(x_{k,l})$ such that $\lim_{r,s\to\infty} x_{k'(r),l'(s)} = x_0$ and

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_{k'(r),l'(s)} \notin V\} \in I_2. \tag{28}$$

Since V is any τ-neighborhood of zero, we choose $W \in N_{\text{sol}}$ such that $W + W \subseteq V$. Then we have

$$x_{k,l} - x_0 = x_{k,l} - x_{k'(r),l'(s)} + x_{k'(r),l'(s)} - x_0 \in W + W \subseteq V. \tag{29}$$

That is,

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin V\} \subseteq \{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_{k'(r)} \notin W\} \cup \{(r,s) \in \mathbb{N} \times \mathbb{N} : x_{k'(r),l'(s)} - x_0 \notin W\}. \tag{30}$$

Therefore

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin V\} \in I_2. \tag{31}$$

□

Theorem 10. *If $\lim_{k,l\to\infty} x_{k,l} = x_0$ and $I_2(\tau)\text{-}\lim_{k,l\to\infty} y_{k,l} = 0$, then $I_2(\tau)\text{-}\lim_{k,l\to\infty}(x_{k,l} + y_{k,l}) = \lim_{k,l\to\infty} x_{k,l}$.*

Proof. Let V be any τ-neighborhood of 0. Then there exists $Y \in N_{\text{sol}}$ such that $Y \subseteq V$. Let $W \in N_{\text{sol}}$ such that $W + W \subseteq Y$. Since $\lim_{k,l\to\infty} x_{k,l} = x_0$, then there exist integers n_0, m_0 such that $k \ge n_0, l \ge m_0$ implies that $x_{k,l} - x_0 \in W$. Hence

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin W\} \subseteq \mathbb{N} \times \mathbb{N} - \{(n_0, m_0)\}. \tag{32}$$

By the assumption $I_2(\tau)\text{-}\lim_{k,l\to\infty} y_{k,l} = 0$, $\{(k,l) \in \mathbb{N} \times \mathbb{N} : y_{k,l} \notin W\} \in I_2$. Thus

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : (x_{k,l} - x_0) + y_{k,l} \notin V\} \subseteq \{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin W\} \cup \{(k,l) \in \mathbb{N} \times \mathbb{N} : y_{k,l} \notin W\}. \tag{33}$$

That is,

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : (x_{k,l} - x_0) + y_{k,l} \notin V\} \in I_2. \tag{34}$$

This implies that $I_2(\tau)\text{-}\lim_{k,l\to\infty}(x_{k,l} + y_{k,l}) = \lim_{k,l\to\infty} x_{k,l}$. □

Theorem 11. *Let (X,τ) be a LSR-space and let $x = (x_{k,l})$ be a double sequence in X. If there is a $I_2(\tau)$-convergent sequence $y = (y_{k,l})$ in X such that $\{(k,l) \in \mathbb{N} \times \mathbb{N} : y_{k,l} \neq x_{k,l} \notin V\} \in I_2$ then x is also $I_2(\tau)$-convergent.*

Proof. Suppose that $\{(k,l) \in \mathbb{N} \times \mathbb{N} : y_{k,l} \neq x_{k,l} \notin V\} \in I_2$ and $I_2(\tau)\text{-}\lim_{k,l} y_{k,l} = x_0$. Then for an arbitrary τ-neighborhood V of zero, we have

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : y_{k,l} - x_0 \notin V\} \in I_2. \tag{35}$$

Now,

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin V\} \subseteq \{(k,l) \in \mathbb{N} \times \mathbb{N} : y_{k,l} \neq x_{k,l} \notin V\} \cup \{(k,l) \in \mathbb{N} \times \mathbb{N} : y_{k,l} - x_0 \notin V\}. \tag{36}$$

Therefore, we have

$$\{(k,l) \in \mathbb{N}\times\mathbb{N} : x_{k,l} - x_0 \notin V\} \in I_2. \tag{37}$$

□

Theorem 12. *Let (X,τ) be a LSR-space. If a double sequence $x = (x_{k,l})$ is $I_2^*(\tau)$-convergent to x_0, then it is $I_2(\tau)$-convergent to x_0.*

Proof. Suppose that $I_2^*(\tau)$-$\lim_k x_{k,l} = x_0$. Let V be an arbitrary τ-neighborhood V of zero. Since $I_2^*(\tau)$-$\lim_{k,l} x_{k,l} = x_0$, there is a set $K = \{(k,l)\} \subseteq \mathbb{N}\times\mathbb{N}$, $(k,l \in \mathbb{N})$ with $K \in F$ such that $k \geq n$, $l \geq m$ and $(k,l) \in K$ implies $x_{k,l} - x_0 \in V$. Then

$$\begin{aligned} K_1 &= \{(k,l) \in \mathbb{N}\times\mathbb{N} : x_{k,l} - x_0 \notin V\} \\ &\subseteq \mathbb{N}\times\mathbb{N} - \{(k_{n+1}, l_{m+1}), (k_{n+2}, l_{m+2}), \ldots\}. \end{aligned} \tag{38}$$

Therefore

$$K_1 \in I_2. \tag{39}$$

Hence x is $I_2(\tau)$-convergent to x_0. □

Theorem 13. *The sequential method $I_2(\tau)$ is regular.*

Proof of the theorem is straightforward, so it is omitted.

From Theorem 12, we can easily obtain the following useful result.

Theorem 14. *The sequential method $I_2(\tau)$ is subsequential.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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