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Research Article

On the Ideal Convergence of Double Sequences in Locally Solid Riesz Spaces

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The aim of this paper is to define the notions of ideal convergence, I-bounded for double sequences in setting of locally solid Riesz spaces and study some results related to these notions. We also define the notion of I^* -convergence for double sequences in locally solid Riesz spaces and establish its relationship with ideal convergence.

1. Introduction and Preliminaries

In 1951, Fast [1] and Steinhaus [2] introduced the concept of statistical convergence for single sequences, independently. Some basic and important properties of this concept were studied by Buck [3], Šalát [4], Schoenberg [5], and Fridy [6]. Later, the notion of statistical convergence for single sequences was further defined in various spaces; see Çakalli and Khan [7-9], Di Maio et al. [10, 11], Hazarika [12-14], Maddox [15], Mohiuddine et al. [16-19], and so forth. Some application of statistical summability methods is presented in [20, 21]. In 2003, the notion of statistical convergence for single sequences has been extended to double sequences by Mursaleen and Edely [22]. Recently, the statistical convergence and statistical Cauchy for double sequences have been defined in the framework fuzzy and intuitionistic normed spaces by Mohiuddine et al. [23] and Mursaleen and Mohiuddine [24], respectively, and established some interesting results related to the concept of statistical convergence and statistical Cauchy double sequences. Recently, it was defined and studied by Mohiuddine et al. [25] in the setting of locally solid Riesz spaces while for single sequences this concept was first studied by Albayrak and Pehlivan [26] (also see [27–29]). An application of locally solid Riesz spaces in economics can be found in [30].

The notion of ideal convergence for single sequences, which is a generalization of the concept of statistical convergence, was first defined and studied by Kostyrko et al. [31]. Let

us recall the notion of ideal convergence and related concepts by Kostyrko et al. [31] as follows. Let N be a nonempty set. Then a family of sets $I \subseteq P(\mathbb{N})$ (power set of \mathbb{N}) is said to be an ideal if I is additive; that is, $A, B \in I \Rightarrow A \cup B \in I$ and $A \in I, B \subseteq A \Rightarrow B \in I$. A family of sets $I \subset P(\mathbb{N})$ (power sets of \mathbb{N}) is called an *ideal* if and only if, for each $A, B \in I$, we have $A \cup B \in I$ and, for each $A \in I$ and each $B \subset A$, we have $B \in I$. A nonempty family of sets $\mathcal{F} \subset P(\mathbb{N})$ is a *filter* on \mathbb{N} if and only if $\Phi \notin \mathcal{F}$; for each $A, B \in \mathcal{F}$, we have $A \cap B \in \mathcal{F}$ and for each $A \in \mathcal{F}$ and each $A \subset B$, we have $B \in \mathcal{F}$. An ideal I is called nontrivial ideal if $I \neq \Phi$ and $\mathbb{N} \notin I$. Clearly $I \subset P(\mathbb{N})$ is a nontrivial ideal if and only if $\mathcal{F} = \mathcal{F}(I) = \{ \mathbb{N} - A : A \in I \}$ is a filter on \mathbb{N} . A nontrivial ideal $I \subset P(\mathbb{N})$ is called *admissible* if and only if $\{\{x\}: x \in \mathbb{N}\} \subset I$. A nontrivial ideal I is maximal if there cannot exist any nontrivial ideal $J \neq I$ containing I as a subset.

We remark that if $I=I_f=\{A\subseteq\mathbb{N}:A \text{ is a finite subset}\}$, then the corresponding convergence coincides with the usual convergence. Also, if $I=I_\delta=\{A\subseteq\mathbb{N}:\delta(A)=0\}$, then the corresponding convergence coincides with the statistical convergence (where $\delta(A)$ denotes the natural density of the set A). In the above cases, both I_f and I_δ are nontrivial admissible ideals of \mathbb{N} .

Kumar [32] defined the notions of I and I^* -convergence of double sequence and studied some properties of these notions. Recently, Das et al. [33] introduced the concepts of I and I^* -convergence of double sequences in the setting of metric space and established some relationship between these

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types of convergence. Quite recently, Mursaleen and Mohiuddine defined and studied the notion of I-convergence, I^* -convergence, I-limit points, and I-cluster points for single and double sequences, in [34, 35], respectively, in probabilistic normed spaces. Şahiner et al. [36] and Gürdal and Açik [37] introduced the notion of ideal convergence and I-Cauchy sequence in 2-normed spaces, respectively. Mursaleen and Alotaibi [38] introduced the notion of ideal convergence in random 2-normed spaces and later on it was extended by Mohiuddine et al. [39] from single to double sequences. For more details on these concepts, one can be referred to [40-52].

Now we recall the definition of locally solid Riesz spaces and some related concepts as follows. Let X be a real vector space and let \leq be a partial order on this space. X is said to be an *ordered vector space* if it satisfies the following properties:

- (1) if $x, y \in X$ and $y \le x$, then $y + z \le x + z$ for each $z \in X$:
- (2) if $x, y \in X$ and $y \le x$, then $ay \le ax$ for each $a \ge 0$.

If, in addition, X is a lattice with respect to the partial ordering, then X is said to be a *Riesz space* (or a *vector lattice*)(see [53]).

For an element x of a Riesz space X, the positive part of x is defined by $x^+ = x \vee \overline{\theta} = \sup\{x, \overline{\theta}\}$, the negative part of x by $x^- = (-x) \vee \overline{\theta}$, and the absolute value of x by $|x| = x \vee (-x)$, where $\overline{\theta}$ is the zero element of X.

A subset *S* of *X* is said to be *solid* if $y \in S$ and $|x| \le |y|$ implies $x \in S$.

A topology τ on a real vector space X that makes the addition and scalar multiplication continuous is said to be a linear topology, that is, when the mappings

$$(x, y) \longrightarrow (x + y)$$
 (from $(X \times X, \tau \times \tau) \longrightarrow (X, \tau)$),
 $(\lambda, x) \longrightarrow (\lambda x)$ (from $(\mathbb{R} \times X, \tau' \times \tau) \longrightarrow (X, \tau)$) (1)

are continuous, where τ' is the usual topology on \mathbb{R} . In this case the pair (X, τ) is called a *topological vector space*.

Every linear topology τ on a vector space X has a base N for the neighborhoods of $\overline{\theta}$ satisfying the following properties.

- (1) Each $Y \in N$ is a *balanced set*; that is, $ax \in Y$ holds for all $x \in Y$ and every $a \in \mathbb{R}$ with $|a| \le 1$.
- (2) Each $Y \in N$ is an *absorbing set*; that is, for every $x \in X$, there exists a > 0 such that $ax \in Y$.
- (3) For each $Y \in N$ there exists some $E \in N$ with $E + E \subseteq Y$

A linear topology τ on a Riesz space X is said to be *locally solid* (see [54]) if τ has a base at zero consisting of solid sets. A *locally solid Riesz space* (X, τ) is a Riesz space X equipped with a locally solid topology τ . For more details on these concepts, one can be referred to [55–57].

Throughout the paper, the symbol $N_{\rm sol}$ will stand for a base at zero consisting of solid sets and satisfying conditions (1), (2), and (3) in a locally solid topology. Also we assume that I_2 is a nontrivial admissible ideal of $\mathbb{N} \times \mathbb{N}$.

2. Ideal Convergence of Double Sequences in LSR-Spaces

Throughout the paper *X* will denote the Hausdorff locally solid Riesz space, which satisfies the first axiom of countability. For our convenience, here and in what follows, we will write an LSR-space instead of a locally solid Riesz space.

The notion of convergence for double sequence was first introduced by Pringsheim [58] as follows. We say that a double sequence $x=(x_{j,k})_{j,k\in\mathbb{N}}$ of reals is convergent to L in Pringsheim's sense (briefly, P-convergent) provided that given $\epsilon>0$ there exists a positive integer N such that $|x_{j,k}-L|<\epsilon$ whenever $j,k\geq N$.

Let $K \subset \mathbb{N} \times \mathbb{N}$ and K(m,n) denotes the number of (i,j) in K such that $i \leq m$ and $j \leq n$ (see [22]). Then the lower natural density of K is defined by $\underline{\delta}_2(K) = \liminf_{m,n \to \infty} (|K(m,n)|/mn)$. In this case, the sequence (K(m,n)/mn) has a limit in Pringsheim's sense; then we say that K has a *double natural density* and is defined by $P - \lim_{m,n \to \infty} (|K(m,n)|/mn) = \delta_2(K)$.

In the recent past, Mohiuddine et al. [25] introduced the notion of statistical convergence of double sequences in LSR-space as follows. Let (X, τ) be a LSR-space. A double sequence $(x_{k,l})$ of points in X is said to be $S_2(\tau)$ -convergent to an element x_0 if for each τ -neighborhood V of zero

$$\delta_2\left(\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:x_{k,l}-x_0\notin V\right\}\right)=0. \tag{2}$$

Now we introduce the notions of $I_2(\tau)$ -convergence and $I_2(\tau)$ -bounded double sequences in LSR-spaces.

Definition 1. Let (X, τ) be a LSR-space. A double sequence $(x_{k,l})$ of points in X is said to be $I_2(\tau)$ -convergent to an element x_0 of X if for each τ -neighborhood V of zero

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin V\} \in I_2. \tag{3}$$

That is,

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \in V\} \in \mathcal{F}. \tag{4}$$

In this case, one writes $I_2(\tau)$ - $\lim_{k,l\to\infty} x_{k,l} = x_0$ or $(x_{k,l}) \xrightarrow{I_2(\tau)} x_0$.

Definition 2. Let (X, τ) be a LSR-space. Then, a double sequence $(x_{k,l})$ of points in X is said to be $I_2(\tau)$ -bounded in X if, for each τ -neighborhood V of zero, there is some a > 0,

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : ax_{k,l} \notin V\} \in I_2. \tag{5}$$

Definition 3. Let (X, τ) be a LSR-space. One says that a double sequence $x = (x_{k,l})$ is $I_2(\tau)$ -Cauchy in X if, for each τ -neighborhood V of zero, there exist $p, q \in \mathbb{N}$ such that, for all $k, m \ge p$ and $l, n \ge q$,

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_{m,n} \notin V\} \in I_2. \tag{6}$$

Definition 4. Let (X, τ) be a LSR-space. Then, a double sequence $x = (x_{k,l})$ in X is said to be $I_2^*(\tau)$ -convergent to x_0 if there is a set $K = \{(k,l)\} \subseteq \mathbb{N} \times \mathbb{N}, k, l = 1, 2, \ldots$, with $K \in F$ such that $\lim_{k,l} x_{k,l} = x_0$. In this case, one writes $I_2^*(\tau)$ - $\lim_{k,l} x_{k,l} = x_0$.

Theorem 5. Let (X, τ) be a LSR-space. Every $I_2(\tau)$ -convergent sequence in X has only one limit.

Proof. Suppose that $x=(x_{k,l})$ is a double sequence in X such that $I_2(\tau)$ - $\lim_{k,l} x_{k,l} = x_0$ and $I_2(\tau)$ - $\lim_{k,l} x_{k,l} = y_0$. Let V be any τ -neighborhood of zero. Also for each τ -neighborhood V of zero there is a set $Y \in N_{\text{sol}}$ such that $Y \subseteq V$. Let W in N_{sol} be such that $W+W \subseteq Y$. We define the sets A_1 and A_2 as follows:

$$A_{1} = \{(k, l) \in \mathbb{N} \times \mathbb{N} : x_{k, l} - x_{0} \in W\},$$

$$A_{2} = \{(k, l) \in \mathbb{N} \times \mathbb{N} : x_{k, l} - y_{0} \in W\}.$$
(7)

Since $I_2(\tau)$ -lim $x_{k,l}=x_0$ and $I_2(\tau)$ -lim $x_{k,l}=y_0$, we get $A_1,A_2\in \mathcal{F}$. Now, let $A=A_1\cap A_2$. Then we have

$$x_0 - y_0 = x_0 - x_{k,l} + x_{k,l} - y_0 \in W + W \subseteq Y \subseteq V.$$
 (8)

As we know, intersection of all τ -neighborhoods V of zero is the singleton set $\{\overline{\theta}\}$ because (X, τ) is Hausdorff. Hence $x_0 - y_0 = 0$; that is, $x_0 = y_0$.

Theorem 6. Let (X, τ) be a LSR-space and let $(x_{k,l})$ and $(y_{k,l})$ be two double sequences of points in X. Then,

- (i) if $I_2(\tau)$ - $\lim_{k,l} x_{k,l} = x_0$ and $I_2(\tau)$ - $\lim_{k,l} y_{k,l} = y_0$, then $I_2(\tau)$ - $\lim_{k,l} (x_{k,l} + y_{k,l}) = x_0 + y_0$;
- (ii) if $I_2(\tau)$ - $\lim_{k,l} x_{k,l} = x_0$, then $I_2(\tau)$ - $\lim_{k,l} ax_{k,l} = ax_0$ for $a \in \mathbb{R}$.

Proof. Assume that $I_2(\tau)$ - $\lim_{k,l} x_{k,l} = x_0$ and $I_2(\tau)$ - $\lim_{k,l} y_{k,l} = y_0$. Suppose that V is an arbitrary τ -neighborhood of zero. Then there exists $Y \in N_{\text{sol}}$ such that $Y \subseteq V$. Let $W \in N_{\text{sol}}$ such that $W + W \subseteq Y$. Thus, we can write

$$B_{1} = \{(k, l) \in \mathbb{N} \times \mathbb{N} : x_{k, l} - x_{0} \in W\},$$

$$B_{2} = \{(k, l) \in \mathbb{N} \times \mathbb{N} : y_{k, l} - y_{0} \in W\}.$$
(9)

Then we have $B_1, B_2 \in \mathcal{F}$.

Let $B = B_1 \cap B_2$. Hence we have $B \in \mathcal{F}$ and

$$(x_{k,l} + y_{k,l}) - (x_0 + y_0) = (x_{k,l} - x_0) + (y_{k,l} - y_0) \in W + W \subseteq Y \subseteq V.$$
(10)

Therefore

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : (x_{k,l} + y_{k,l}) - (x_0 + y_0) \in V\} \in \mathcal{F}. \tag{11}$$

Since *V* is arbitrary, we have $I_2(\tau)$ -lim $(x_{k,l} + y_{k,l}) = x_0 + y_0$.

(ii) Suppose that $I_2(\tau)$ - $\lim_{k,l} x_{k,l} = x_0$ and also suppose that V is an arbitrary τ -neighborhood of zero. Then there exists $Y \in N_{\text{sol}}$ such that $Y \subseteq V$, so we have

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \in Y\} \in \mathcal{F}. \tag{12}$$

Since *Y* is balanced, $a(x_{k,l} - x_0) \in Y$ holds for all $x_{k,l} - x_0 \in Y$ and for every $a \in \mathbb{R}$ with $|a| \le 1$. Therefore

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \in Y\}$$

$$\subseteq \{(k,l) \in \mathbb{N} \times \mathbb{N} : ax_{k,l} - ax_0 \in Y\}$$

$$\subseteq \{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \in V\}.$$

$$(13)$$

Thus, we have

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \in V\} \in \mathcal{F} \tag{14}$$

for each τ -neighborhood V of zero. Now let |a| > 1 and [|a|] be the smallest integer greater than or equal to |a|. Then there exists $W \in N_{\text{sol}}$ such that $[|a|]W \subseteq Y$. From our assumption that $I_2(\tau)$ -lim $_{k,l}x_{k,l} = x_0$, we obtain that

$$K = \{(k, l) \in \mathbb{N} \times \mathbb{N} : x_{k, l} - x_0 \in W\} \in \mathcal{F}.$$
 (15)

Therefore

$$|ax_{k,l} - ax_0| = |a| |x_{k,l} - x_0|$$

$$\leq [|a|] |x_{k,l} - x_0| \in [|a|] W \subseteq Y \subseteq V.$$
(16)

Since *Y* is solid, $ax_k - ax_0 \in Y$. It follows that $ax_{k,l} - ax_0 \in V$. Thus,

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : ax_{k,l} - ax_0 \in V\} \in \mathcal{F},\tag{17}$$

for each τ -neighborhood V of zero. We conclude that $I_2(\tau)$ - $\lim_{k,l} ax_{k,l} = ax_0$.

Theorem 7. Let (X, τ) be a LSR-space. If a double sequence $(x_{k,l})$ in X is $I_2(\tau)$ -convergent, then it is $I_2(\tau)$ -bounded.

Proof. Assume that $I_2(\tau)$ - $\lim_{k,l\to\infty} x_{k,l} = x_0$. Suppose V is an arbitrary τ -neighborhood of zero. Then, there exists $Y\in N_{\mathrm{sol}}$ such that $Y\subseteq V$. Let $W\in N_{\mathrm{sol}}$ such that $W+W\subseteq Y$. Using our assumption, we obtain that

$$A = \{ (k, l) \in \mathbb{N} \times \mathbb{N} : x_{k, l} - x_0 \notin W \} \in I_2.$$
 (18)

Since W is absorbing, there exists a>0 such that $ax_0\in W$. Let b be such that $|b|\leq 1$ and $b\leq a$. Since W is solid and $|bx_0|\leq |ax_0|$, we have $bx_0\in W$. Also, since W is balanced, $x_{k,l}-x_0\in W$ implies $b(x_{k,l}-x_0)\in W$. Then we have

$$bx_{k,l} = b(x_{k,l} - x_0) + bx_0 \in W$$

+ $W \subseteq V$, for each $k, l \in \mathbb{N} - A$. (19)

Thus

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : bx_{k,l} \notin W\} \in I_2. \tag{20}$$

Hence
$$(x_{k,l})$$
 is $I_2(\tau)$ -bounded.

Theorem 8. Let (X, τ) be a LSR-space and let $(x_{k,l})$, $(y_{k,l})$, and $(z_{k,l})$ be three double sequences of points in X such that

(i)
$$x_{k,l} \leq y_{k,l} \leq z_{k,l}$$
, for all $k, l \in \mathbb{N}$,

(ii)
$$I_2(\tau)$$
- $\lim_{k,l} x_{k,l} = x_0 = I_2(\tau)$ - $\lim_{k,l} z_{k,l}$.

Then $I_2(\tau)$ - $\lim_{k,l} y_{k,l} = x_0$.

Proof. Suppose that the given conditions (i) and (ii) hold for the double sequences $(x_{k,l})$, $(y_{k,l})$, and $(z_{k,l})$. Suppose V is an arbitrary τ -neighborhood of zero. Then, there exists $Y \in N_{\text{sol}}$

such that $Y \subseteq V$. Let $W \in N_{sol}$ such that $W + W \subseteq Y$. It follows from (ii) that $P, Q \in F$, where

$$\begin{split} P &= \left\{ (k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \in W \right\}, \\ Q &= \left\{ (k,l) \in \mathbb{N} \times \mathbb{N} : z_{k,l} - x_0 \in W \right\}. \end{split} \tag{21}$$

Also from the given condition (i), we have

$$x_{k,l} - x_0 \le y_{k,l} - x_0 \le z_{k,l} - x_0$$

$$\implies |y_{k,l} - x_0| \le |x_{k,l} - x_0|$$

$$+ |z_{k,l} - x_0| \in W + W \subseteq Y.$$
(22)

Since *Y* is solid, we have $y_{k,l} - x_0 \in Y \subseteq V$. Thus,

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : y_{k,l} - x_0 \in V\} \in \mathcal{F},\tag{23}$$

for each τ -neighborhood V of zero. Thus $I_2(\tau)$ - $\lim_{k,l} y_{k,l} = x_0$.

Theorem 9. Let (X, τ) be a LSR-space. A double sequence $(x_{k,l})$ is $I_2(\tau)$ -convergent to x_0 in X if and only if for each τ -neighborhood V of zero there exists a subsequence $(x_{k'(\tau),l'(s)})$ of $(x_{k,l})$ such that $\lim_{r,s\to\infty} x_{k'(r),l'(s)} = x_0$ and

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_{k'(r),l'(s)} \notin V\} \in I_2.$$
 (24)

Proof. Suppose that $I_2(\tau)$ - $\lim_{k,l\to\infty} x_{k,l} = x_0$. Also, suppose that V is an arbitrary τ -neighborhood of zero. Let $\{V_i\}$ be a sequence of nested base of τ -neighborhoods of zero. For each $i\in\mathbb{N}$, put

$$E^{(i)} = \{ (k, l) \in \mathbb{N} \times \mathbb{N} : x_{k, l} - x_0 \notin V_i \}.$$
 (25)

Then, $E^{(i+1)} \subset E^{(i)}$ and $E^{(i)} \in F$. Let m(1) and n(1) be such that r > m(1) and s > n(1), respectively. Then $E^{(1)} \neq \phi$. For $r, s \in \mathbb{N}$ such that $m(1) \leq r < m(2)$ and $n(1) \leq s < n(2)$, choose $k'(r), l'(s) \in E^{(i)}$; that is, $x_{k'(r), l'(s)} - x_0 \in V_1$. In general, choose m(p+1) > m(p) and n(p+1) > n(p) such that r > m(p+1) and s > n(p+1) hold. Then $E^{(p+1)} \neq \phi$. Therefore for all r, s which satisfy $m(p) \leq r < m(p+1)$ and $n(p) \leq s < n(p+1)$, choose $k'(r), l'(s) \in E^{(p)}$; that is, $x_{k'(r), l'(s)} - x_0 \in V_p$. Hence, it follows that $\lim_{r,s} x_{k'(r), l'(s)} = x_0$.

Since V is an arbitrary τ -neighborhood of zero, there exists $Y \in N_{\text{sol}}$ such that $Y \subseteq V$. Let $W \in N_{\text{sol}}$ such that $W + W \subseteq Y$. Now

$$x_{k,l} - x_{k'(r),l'(s)} = x_{k,l} - x_0 + x_{k'(r),l'(s)} - x_0 \in W + W \subseteq Y \subseteq V.$$
(26)

Also $I_2(\tau)$ - $\lim_{k,l\to\infty} x_{k,l} = x_0$ and $\lim_{r\to\infty} x_{k'(r),l'(s)} = x_0$ imply that

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_{k'(r),l'(s)} \notin V\} \in I_2.$$
 (27)

Next suppose for an arbitrary τ -neighborhood V of zero that there exists a subsequence $(x_{k'(r),l'(s)})$ of $(x_{k,l})$ such that $\lim_{r,s\to\infty}x_{k'(r),l'(s)}=x_0$ and

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_{k'(r),l'(s)} \notin V\} \in I_2.$$
 (28)

Since V is any τ -neighborhood of zero, we choose $W \in N_{\text{sol}}$ such that $W + W \subseteq V$. Then we have

$$\begin{aligned} x_{k,l} - x_0 &= x_{k,l} - x_{k'(r),l'(s)} \\ &+ x_{k'(r),l'(s)} - x_0 \in W + W \subseteq V. \end{aligned} \tag{29}$$

That is,

$$\begin{aligned} & \left\{ (k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin V \right\} \\ & \subseteq \left\{ (k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_{k'(r)} \notin W \right\} \\ & \cup \left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : x_{k'(r),l'(s)} - x_0 \notin W \right\}. \end{aligned} \tag{30}$$

Therefore

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin V\} \in I_2.$$
 (31)

Theorem 10. If $\lim_{k,l\to\infty} x_{k,l} = x_0$ and $I_2(\tau)$ - $\lim_{k,l\to\infty} y_{k,l} = 0$, then $I_2(\tau)$ - $\lim_{k,l\to\infty} (x_{k,l} + y_{k,l}) = \lim_{k,l\to\infty} x_{k,l}$.

Proof. Let V be any τ -neighborhood of 0. Then there exists $Y \in N_{\text{sol}}$ such that $Y \subseteq V$. Let $W \in N_{\text{sol}}$ such that $W + W \subseteq Y$. Since $\lim_{k,l \to \infty} x_{k,l} = x_0$, then there exist integers n_0, m_0 such that $k \ge n_0, l \ge m_0$ implies that $x_{k,l} - x_0 \in W$. Hence

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin W\} \subseteq \mathbb{N} \times \mathbb{N} - \{(n_0, m_0)\}. \tag{32}$$

By the assumption $I_2(\tau)$ - $\lim_{k,l\to\infty}y_{k,l}=0,$ $\{(k,l)\in\mathbb{N}\times\mathbb{N}:y_{k,l}\notin W\}\in I_2$. Thus

$$\begin{aligned} \left\{ (k,l) \in \mathbb{N} \times \mathbb{N} : \left(x_{k,l} - x_0 \right) + y_{k,l} \notin V \right\} \\ &\subseteq \left\{ (k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin W \right\} \\ &\cup \left\{ (k,l) \in \mathbb{N} \times \mathbb{N} : y_{k,l} \notin W \right\}. \end{aligned} \tag{33}$$

That is,

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : (x_{k,l} - x_0) + y_{k,l} \notin V\} \in I_2.$$
 (34)

This implies that $I_2(\tau)$ - $\lim_{k,l\to\infty}(x_{k,l}+y_{k,l})=\lim_{k,l\to\infty}x_{k,l}$.

Theorem 11. Let (X, τ) be a LSR-space and let $x = (x_{k,l})$ be a double sequence in X. If there is a $I_2(\tau)$ -convergent sequence $y = (y_{k,l})$ in X such that $\{(k,l) \in \mathbb{N} \times \mathbb{N} : y_{k,l} \neq x_{k,l} \notin V\} \in I_2$ then x is also $I_2(\tau)$ -convergent.

Proof. Suppose that $\{(k,l) \in \mathbb{N} \times \mathbb{N} : y_{k,l} \neq x_{k,l} \notin V\} \in I_2$ and $I_2(\tau)$ - $\lim_{k,l} y_{k,l} = x_0$. Then for an arbitrary τ -neighborhood V of zero, we have

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : y_{k,l} - x_0 \notin V\} \in I_2. \tag{35}$$

Now,

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin V\}$$

$$\subseteq \{(k,l) \in \mathbb{N} \times \mathbb{N} : y_{k,l} \neq x_{k,l} \notin V\}$$

$$\cup \{(k,l) \in \mathbb{N} \times \mathbb{N} : y_{k,l} - x_0 \notin V\}.$$

$$(36)$$

Therefore, we have

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin V\} \in I_2. \tag{37}$$

Theorem 12. Let (X, τ) be a LSR-space. If a double sequence $x = (x_{k,l})$ is $I_2^*(\tau)$ -convergent to x_0 , then it is $I_2(\tau)$ -convergent to x_0 .

Proof. Suppose that $I_2^*(\tau)$ - $\lim_k x_{k,l} = x_0$. Let V be an arbitrary τ -neighborhood V of zero. Since $I_2^*(\tau)$ - $\lim_{k,l} x_{k,l} = x_0$, there is a set $K = \{(k,l)\} \subseteq \mathbb{N} \times \mathbb{N}$, $(k,l \in \mathbb{N})$ with $K \in F$ such that $k \geq n, l \geq m$ and $(k,l) \in K$ implies $x_{k,l} - x_0 \in V$. Then

$$K_{1} = \{(k, l) \in \mathbb{N} \times \mathbb{N} : x_{k, l} - x_{0} \notin V\}$$

$$\subseteq \mathbb{N} \times \mathbb{N} - \{(k_{n+1}, l_{m+1}), (k_{n+2}, l_{m+2}), \ldots\}.$$
(38)

Therefore

$$K_1 \in I_2. \tag{39}$$

Hence *x* is $I_2(\tau)$ -convergent to x_0 .

Theorem 13. *The sequential method* $I_2(\tau)$ *is regular.*

Proof of the theorem is straightforward, so it is omitted. From Theorem 12, we can easily obtain the following useful result.

Theorem 14. *The sequential method* $I_2(\tau)$ *is subsequential.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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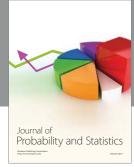
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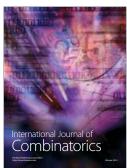






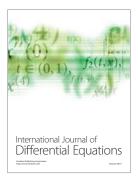


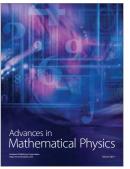


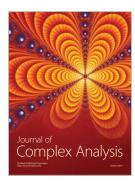


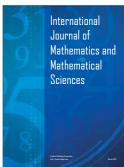


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