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ON THE IDEAL STRUCTURE OF THE SEMIGROUP OF CLOSED SUBSETS OF A TOPOLOGICAL SEMIGROUP

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Among the many semigroups which can be derived from a given compact (jointly continuous) semigroup S is the semigroup 2^S consisting of its non-empty compact subsets; the product is the usual one defined by the rule $EF=\{xy:x\in E, y\in F\}$. The Vietoris or finite topology on 2^S (in which a base for the open sets is obtained by taking all sets of the form $\langle V_1, V_2,\ldots,V_n\rangle=\{E:E\subseteq V_1\cup V_2\cup\cdots\cup V_n$ and $E\cap V_1\neq\emptyset$ for $1\leqq i\leqq n\}$ as $V_1, V_2,\ldots,V_n$ run over all finite collections of open subsets of S) makes 2^S a compact, jointly continuous semigroup. The topology has a long history, having been introduced by Vietoris in 1923 and studied by Michael [**4**]. The utility of the topological semigroup was established by Hofmann and Mostert [**3**; see especially Section 3.7]; in fact they prefer to produce directly the uniform structure on 2^S rather than the topology.

The semigroup 2^S can be structurally complex even when S is quite simple, although the relationship between the two semigroups can provide an effective tool for attacking problems about 2^S. We shall illustrate these points by a discussion of the prime ideals of 2^S for a particular class of semigroups S. The importance of prime ideals is that they are the kernels of semicharacters—see Hofmann and Keimel [**1**] or Hofmann, Mislove and Stralka [**2**; Section II.2].

To be precise, we shall take S to be a direct product $I_1\times I_2$ of two totally ordered spaces I_1, I_2 compact in their order topologies. The semigroup operation will be minimum, explicitly

$$(x_1,x_2)(y_1,y_2)=(\min\{x_1,y_1\},\min\{x_2,y_2\}).$$

Then S is commutative and idempotent; 2^S inherits commutativity, but it is not idempotent. However, for each E in 2^S we have $E^3=E^2$, as is easily seen, and $\{E^2:E\in 2^S\}$ therefore forms an idempotent subsemigroup. Another immediate consequence of the relationship $E^3=E^2$ is that each semicharacter on 2^S takes only the values 0 and 1. Topological properties of semicharacters are therefore immediately reflected in properties of the prime ideals which are their kernels—for example, continuous semicharacters have clopen (closed and open) kernels.

We shall obtain in Section 2 explicit descriptions of the open and of the clopen prime ideals in 2^S. Although it follows from Lemma 1.1 that every prime ideal is an intersection of a decreasing family of open prime ideals, a description of the general prime ideal, or even of the general closed prime ideal, becomes much too involved. However, in Section 3 we shall characterize the general prime ideal in the special case in which S is totally ordered (obtained by taking I_1 or I_2 to be a singleton). The analysis carried out here could have been made for any finite product of totally ordered spaces, but the case of just two factors illustrates all the problems involved.

1. Preliminaries

Here we shall make a few remarks about more general commutative semigroups. Recall that an ideal of a semigroup S is *prime* if its complement in S is a subsemigroup; we call a subset T of S a *prime subsemigroup* if either $S\backslash T$ is a prime ideal or $T=S$.

We define a quasi-order $\prec$ on a commutative semigroup S with identity by writing $x \prec y$ if and only if $x \in yS$; if y is idempotent this is equivalent to $yx=x$. The *upper* (resp. *lower*) set of $x \in S$ is then

$$\uparrow x=\{y \in S: x \prec y\} \text{ (resp. } \downarrow x=\{y: y \prec x\}).$$

The following lemma is established by elementary algebraic means, and we leave the proof to the reader.

Lemma 1.1. *Let S be a commutative semigroup with the property that there exists an integer n such that $x^{n+1}=x^n$ for all $x \in S$. Then x^n is idempotent for each $x \in S$. Also, for any idempotent e in S, $\uparrow e$ is a prime subsemigroup. Moreover, if (e_i) is a family of idempotents directed downwards by $\succ$ (i.e. given i, j there is k such that $e_k e_j = e_i e_k = e_k$) then $\bigcup_i \uparrow e_i$ is again a prime subsemigroup. Conversely, if T is a prime subsemigroup then the set $\{e_i: e_i$ is an idempotent in $T\}$ is directed downwards by $\succ$, and $T=\bigcup_i \uparrow e_i$.*

We shall also need the following elementary lemma concerning prime ideals.

Lemma 1.2. (i) *The union of any non-empty family of prime ideals is a prime ideal.*

(ii) *Any non-empty intersection of a family of prime ideals which is directed downwards by $\supseteq$ is a prime ideal.*

2. The case $S=I_1 \times I_2$

In this section we take S to be $I_1 \times I_2$ as in the introduction; the minimum (resp. maximum) element of each of I_1 and I_2 will be denoted by 0 (resp. 1).

Lemma 2.1. *Let $E \in 2^S$. Then E is idempotent if and only if whenever $(x_1, x_2) \notin E$ either $(x_1, y) \notin E$ for all $y \in I_2$ with $y > x_2$ or $(y, x_2) \in E$ for all $y \in I_1$ with $y > x_1$.*

Proof. For any $E \in 2^S$, $E^2 \supseteq E$. If $x=(x_1, x_2) \in E^2 \backslash E$ then $x=uv$ with $u, v \in E$, $u \neq x$ and $v \neq x$. Clearly one coordinate of u must be the same as that of x and the other coordinate of v must then be the same as that of x. So the result is now clear.

We can now state the main theorem of this section. Recall that a subset M of S is an *antilattice* if there do not exist distinct elements m, m' in M with $m < m'$. Also if $x \in S$ and

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$U \subseteq S$ we write

$$J(x) = \{F \in 2^S : F \cap (\uparrow x) = \emptyset\},$$

$$K(U) = \{F \in 2^S : F \cap U \neq \emptyset\}.$$

Theorem 2.2 *Let P be a subset of 2^S. Then P is an open prime ideal if and only if it is of the form*

$$P = \bigcup_{m \in M} J(m) \cup K(U)$$

where M is a closed non-empty antilattice in S with $M \neq \{(0,0)\}$ and U is an open subset of S disjoint from M such that each $x = (x_1, x_2) \in U$ satisfies this condition:

$$\left.\begin{array}{l}\text{either there are } y=(x_1,y_2)\in U \text{ and } m\in M \text{ such that } y\leqq m,\ y\leqq x \text{ and}\\ (x_1,z)\in U \text{ for all } z>y_2 \text{ in } I_2, \text{ or there are } y=(y_1,x_2)\in U \text{ and } m\in M\\ \text{such that } y\leqq m, y\leqq x \text{ and } (z,x_2)\in U \text{ for all } z>y_1 \text{ in } I_1.\end{array}\right\}(*)$$

Remark 1. The condition $(*)$ is given in the form in which we shall most often apply it. To understand what it means, imagine $S = I_1 \times I_2$ represented as a closed rectangle in the plane. For $(x_1, y_2) \in S$, the set $\{(x_1, z) : z > y_2\}$ is the vertical line segment from (x_1, y_2) to $(x_1, 1)$ without the lower end point. Similarly, $\{(z, x_2) : z > y_1\}$ is the horizontal line segment from (y_1, x_2) to $(1, x_2)$ without the left-hand endpoint. Observe that an open set U is a union of such line segments if and only if it has the property that $uv \in U$ implies that either $u \in U$ or $v \in U$ (for U is not the union of such segments if and only if we can find $(x_1, x_2) \in U$, $u = (x_1, z_2) \notin U$ with $z_2 > x_2$, and $v = (z_1, x_2) \notin U$ with $z_2 > x_1$).

Now the condition $(*)$ asserts that U is a union of line segments of the above forms. It asserts further that if $x = (x_1, x_2) \in U$, then either x is on a segment in U from (x_1, y_2) to $(x_1, 1)$ with (x_1, y_2) less than or equal to some element of M, or x is on a segment in U from (y_1, x_2) to $(1, x_2)$ with (y_1, x_2) less than or equal to some element of M.

Remark 2. By taking U to be empty and M to be a singleton, we see that each $J(m)$ is itself a prime ideal. By contrast, $K(U)$ is an ideal only in exceptional circumstances.

Proof. Let P be an open prime ideal of 2^S. Since 2^S is a compact semigroup, it is clear from Lemma 1.1 and the compactness of $2^S \backslash P$ that there exists an idempotent $E \in 2^S$ with

$$P = \{F \in 2^S : E \not\leq F\} = \{F \in 2^S : EF \neq E\}.$$

We take M to be the set of maximal elements of E ($m \in M$ iff $m \in E$ and $e \in E, e \geqq m \Rightarrow e = m$), and take U to be the set of all elements x of $S \backslash E$ such that statement $(*)$ is true with U replaced by $S \backslash E$.

Observe that M is a closed antilattice, and $M \cap U = \phi$ since $U \subseteq S \backslash E$. Also, since $P \neq 2^S$, $E \neq \{(0,0)\}$ so $M \neq \{(0,0)\}$, and since $E \neq \phi$, $M \neq \phi$. It is clear that U satisfies $(*)$

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for all $x \in U$ and that U is open (since $S \backslash E$ is). We must show that

$$\{F \in 2^S : EF \neq E\} = \bigcup_{m \in M} J(m) \cup K(U).$$

Suppose that $EF \nsubseteq E$. Choose $f \in F$, $e \in E$ with $ef \notin E$. So $ef \neq e$. If $ef = f$ then $f \leqq e$. Choose $m \in M$, $m \geqq e$ so $f \leqq m$. Since $f \in S \backslash E$ and E is idempotent it is clear that $f \in U$ (taking $x = y = f$). On the other hand, suppose that $ef \neq f$. Since E is idempotent, either the line from ef to e or the line from ef to f must lie in $S \backslash E$. Since $e \in E$, it must be the latter. Again, choose $m \in M$, $m \geqq e$; then $m \geqq ef$. In this case it is also clear that $f \in U$ (taking $x = f, y = ef$). So in both cases $U \cap F \neq \phi$. So if $EF \nsubseteq E$ then $F \in K(U)$.

Suppose that $E \nsubseteq EF$. Choose $e \in E \backslash EF$. Clearly $F \cap (\uparrow e) = \phi$. Choose $m \in M, m \geqq e$. Then $F \cap (\uparrow m) = \phi$. So $F \in J(m)$.

We have now shown that $P \subseteq \bigcup_{m \in M} J(m) \cup K(U)$, and proceed to the reverse inequality.

If $F \in K(U)$, say $x \in F \cap U$, then choose y, m as in the definition of U. Then mx will be on the line from y to x, since $y \leqq mx \leqq x$. So $mx \in U$. So $EF \cap U \supseteq MF \cap U \neq \phi$. Since $U \cap E = \phi$, $EF \neq E$. Alternatively, if $m \in M$ and $F \in J(m)$ then $m \notin F$. Therefore $m \in E \backslash EF$, so $EF \neq E$. It follows that $P = \bigcup_{m \in M} J(m) \cup K(U)$.

To prove the converse, suppose we are given sets U_0, M_0 which satisfy the conditions (on U and M) of the theorem. Let

$$E = \{e \in S : e \notin U_0 \text{ and } e \leqq m_0 \text{ for some } m_0 \in M_0\}.$$

Obviously E is closed. We prove E is idempotent. For e, f in E it is clear that $ef \in E$ unless $ef \in U_0$. But by the remark after the statement of the theorem, $ef \in U_0$ implies that either $e \in U_0$ of $f \in U_0$, and neither of these holds. Thus $E^2 \subseteq E$ and so $E^2 = E$.

Because M_0 is an antilattice, the set of maximal elements of E is precisely M_0. We now show that the set U constructed from E by the method of the first part of the proof is just U_0. Indeed, suppose $x = (x_1, x_2) \in S \backslash E$ and that there is $y = (x_1, y_2) \in S \backslash E$ such that $y_2 < x_2$ and $y \leqq m_0$ for some $m_0 \in M_0$. Because $y \notin E$, we see that $y \in U_0$, and hence since U_0 satisfies $*$, $x \in U_0$. This, and a parallel proof with $y = (y_1, x_2)$, achieves our end.

The first part of the proof now assures us that

$$\bigcup_{m \in M_0} J(m) \cup K(U_0) = \{F \in 2^S : EF \neq E\},$$

and the right-hand side is an open prime ideal.

Corollary 2.3. *Let P be an open prime ideal as in Theorem 2.2. Then P is closed if and only if U is closed, M is finite and $\uparrow m$ is open for every $m \in M$.*

Remark. This result characterises the continuous semicharacters of 2^S since they take the form $1 - \chi_P$ for such P.

Proof. Suppose that P is closed. Choose a net (u_α) in U converging to x. Then each doubleton $\{u_\alpha,1\}$ is in $K(U)\subseteq P$ and $(\{u_\alpha,1\})$ converges to $\{x,1\}$ in 2^S. Therefore $\{x,1\}\in P$. Now $1\in\uparrow e$ for all e in S so $\{x,1\}\notin J(e)$. This means that $\{x,1\}\in K(U)$. Now condition $(*)$ ensures that $1\notin U$ since $U\cap M=\phi$. So $x\in U$. Therefore U must be closed.

Suppose that M is infinite, so that it contains a non-isolated point m_0. Put $X=S\backslash(\uparrow m_0)$. Since M is an antilattice, $M\backslash\{m_0\}\subseteq X$. Now every closed set which is a subset of X is in $J(m_0)\subseteq P$, and in particular every closed subset of $M\backslash\{m_0\}$ is in P. Therefore $M=\overline{M\backslash\{m_0\}}\in\bar{P}=P$ (because, for example, the limit of the increasing net of finite subsets of a set Y is the closure of Y). But $M\notin K(U)$ since M is disjoint from U, and $M\notin J(m)$ for any m in M because $m\notin S\backslash(\uparrow m)$, so that $m\notin P$. This contradiction establishes that M is finite.

Finally, if $m\in M$ and $S\backslash(\uparrow m)$ is not closed there exists a net (e_α) in $S\backslash(\uparrow m)$ converging to m. Put $M_1=M\backslash\{m\}$. Then $M_1\cup\{e_\alpha\}\in P$ so $M=\lim_\alpha M_1\cup\{e_\alpha\}\in\bar{P}=P$. As in the last paragraph, this is impossible.

Conversely, if $\uparrow m$ is open then $J(m)$ is closed, and if U is closed then so is $K(U)$. So, in that case, if M is finite then P is a finite union of closed sets and so is closed.

We conclude with a lemma to be used in the next section.

Lemma 2.4. *With the above notation,*

$$\bigcup_{m\in M} J(m)\cup K(U)\subseteq \bigcup_{m\in M'} J(m)\cup K(U')$$

if and only if $U\subseteq U'$ and for each $m\in M$ there exists $m'\in M'$ with $m'\geqq m$.

Proof. The sufficiency of the conditions is obvious. Suppose we have the inclusion between the prime ideals. Let $u\in U$. So the doubleton $\{u,1\}$ is in the ideals. Now $\{u,1\}\notin J(e)$ for all $e\in S$. So $\{u,1\}\cap U'\neq\phi$. But $1\notin U'$, so $u\in U'$. Therefore $U\subseteq U'$.

Suppose that $m\in M$ and that $(\uparrow m)\cap M'=\phi$. Then $M'\in J(m)$ so $M'\in\bigcup_{m\in M'} J(m)\cup K(U')$. Clearly $M'\notin J(m')$ for all $m'\in M'$. So $M'\cap U'\neq\phi$. This is impossible, so $(\uparrow m)\cap M'\neq\phi$, as required.

Remark. Since M has to be an antilattice, this shows that the representation in Theorem 2.2 is unique.

3. The case $S=I$

We now turn to the case where S is totally ordered, say $S=I$. We recover this from the two-dimensional case by putting $I_1=I$ and $I_2=\{1\}$. By Theorem 2.2 we see that an open prime ideal has the form

$$J(m)\cup K(U)$$

where

$$J(m)=\{F\subseteq I: F\subseteq[0,m)\},\quad K(U)=\{F\in 2^S: F\cap U\neq\phi\},$$

U is open, $m>0$ and $m\geqq\sup U$, since the antilattice M clearly must be a single point.

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Now, by Lemma 1.1, any prime ideal is an intersection of a directed family of such ideals, say $P=\bigcap_i P_i=\bigcap_i[J(m_i)\cup K(U_i)]$. Then (m_i) must be decreasing (from Lemma 2.4) and so must converge, to m say. We write

$$\bar{J}(m)=\{F:F\subseteq[0,m]\}.$$

Now

$$P=\bigcap_i (J(m_i)\cup\{F:F\cap U_i\neq\phi\}).$$

The first possibility is that (m_i) is eventually constant. In this case we can assume that $m_i=m>0$. Then we have

$$P=J(m)\cup\{F:F\cap U_i\neq\phi \text{ for all } i\}$$

where U_i is open and $m\geqq\sup U_i$ for all i. The alternative is that (m_i) is not eventually constant, in which case $m<m_i$ for all i, so that $\bar{J}(m)\subseteq P$. If $F\in P$ then $F\subseteq[0,m_i]$ for all i or $F\cap U_i\neq\phi$ for all i since (U_i) is decreasing. So

$$F\in\bar{J}(m)\cup\{F:F\cap U_i\neq\phi \text{ for all } i\}.$$

Conversely $\bar{J}(m)\subseteq P$ and so

$$P=\bar{J}(m)\cup\{F:F\cap U_i\neq\phi \text{ for all } i\}.$$

So we have proved the following.

Theorem 3.1. *Every prime ideal of 2^S is either of the form*

$$J(m)\cup\{F:F\cap U_i\neq\phi \text{ for all } i\}$$

where (U_i) is a decreasing net of open sets, $m>0$ and $m\geqq\sup U_i$ for all i, or of the form

$$\bar{J}(m)\cup\{F:F\cap U_i\neq\phi \text{ for all } i\}$$

where (U_i) is a decreasing net of open sets, $m\in S$ and $m\geqq\inf_i(\sup U_i)$.

We now turn to a a discussion of closed prime ideals. There are two possibilities for the above element m. One is that $m\in\overline{[0,m)}$, when $\bar{J}(m)=\overline{J(m)}$. Otherwise $[0,m)=[0,m']$ for some $m'\in S$ and then $J(m)=\bar{J}(m')$. So we can assume that a closed ideal P is of the second kind in Theorem 3.1.

The idea in the following analysis is that each U_i can be split into two parts, one consisting of elements larger than m, the other of elements smaller than m. Each part gives rise to an ideal; we write

$$P_1=\{F\in 2^S:F\cap U_i\cap(m,1]\neq\phi \text{ for all } i\}\cup\bar{J}(m),$$

$$P_2=\{F\in 2^S:F\cap U_i\cap[0,m]\neq\phi \text{ for all } i\}\cup\bar{J}(m).$$

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Because (U_i) is a decreasing net, $P=P_1\cup P_2$. We shall show that each of these ideals is determined by a set of filters of closed sets, and that the second case is particularly simple because the filters are fixed.

We begin with P_1 and put

$$\mathscr{I}_1=\{F\cap[m,1]:F\in P_1\}.$$

Observe that $\mathscr{I}_1$ consists of just the empty set unless $m<\sup U_i$ for each i, in which case $m=\inf_i(\sup U_i)$. Then $\mathscr{I}_1$ has the following properties:

(i) $F\in\mathscr{I}_1$ implies $m\in F$;
(ii) $F\in\mathscr{I}_1, F\subseteq G$ implies $G\in\mathscr{I}_1$;
(iii) $F\in\mathscr{I}_1, e>m$ implies $F\cap[m,e]\in\mathscr{I}_1$;
(iv) $F\cup G\in\mathscr{I}_i$ implies that either $F\in\mathscr{I}_1$ or $G\in\mathscr{I}_1$ (to see this observe that if $F\cap U_i=\phi$ then $F\cap U_j=\phi$ for $j>i$ so $G\cap U_i\neq\phi$ for $j>i$, and it follows that $G\cap U_i\neq\phi$ for all i).

Finally, since P is closed, if (F_k) is a decreasing net of sets in $\mathscr{I}_1$ then $F_k\in J_1$ for each k so $\lim_k F_k=\bigcap_k F_k\in P$. Since $F_k\subseteq[m,1]$ for all k, we see that either $\bigcap_k F_k=\{m\}$ or $\bigcap_k F_k\in\mathscr{I}_1$. So we have a final condition,

(v) if (F_k) is a decreasing net in $\mathscr{I}_1$, then either $\bigcap_k F_k=\{m\}$ or $\bigcap_k F_k\in\mathscr{I}_1$.

A converse to these statements also holds.

Lemma 3.2. *Let $\mathscr{I}_1$ be a family of compact subsets of* $[m,1]$ *satisfying* (i) *to* (v). *Then.*

$$P_1=\bar{J}(m)\cup\{F:F\cap[m,1]\in\mathscr{I}_1\}$$

is a closed prime ideal of 2^S.

The verification that P_1 is a closed prime ideal is left to the reader.

We can discuss P_2 in a similar manner. Begin by observing that if $F\subseteq[0,m)$ and $G\subseteq[m,1]$ then $F\cup G\in P_2$ if and only if $F\cup\{1\}\in P_2$. Write $\mathscr{I}_2=\{F\subseteq[0,m]:F\cup\{1\}\in P_2\}=\{F\subseteq[0,m]:F\cap U_i\neq\phi \text{ for all } i\}$.

Then $\mathscr{I}_2$ has the properties:

(vi) $F\in\mathscr{I}_2$ and $F\subseteq G\subseteq[0,m]$ implies $G\in\mathscr{I}_2$;
(vii) $F\cup G\in\mathscr{I}_2$ implies $F\in\mathscr{I}_2$ or $G\in\mathscr{I}_2$.

Now let (F_k) be a decreasing net of closed sets in $\mathscr{I}_2$. Then $(F_k\cup\{1\})$ is a decreasing net in $\mathscr{I}_2\subseteq P$. Since P is closed, $\bigcap_k F_k\cup\{1\}$ is in P. This implies that $\bigcap_k F_k\in\mathscr{I}_2$. But here we can go further. Suppose that (F_k) is a maximal chain of closed sets in $\mathscr{I}_2$, ordered by $\supseteq$, and that $\bigcap_k F_k$ has at least two points; then using (vii) we can choose a set in $\mathscr{I}_2$ strictly contained in $\bigcap_k F_k$, which is a contradiction. Thus every set in $\mathscr{I}_2$ contains a singleton set in $\mathscr{I}_2$. Write $E_2=\{x:\{x\}\in\mathscr{I}_2\}$. Note that $E_2\subseteq[0,m]$ and that $F\in\mathscr{I}_2$ if and only if $F\cap E_2\neq\phi$. Since P is closed, E_2 is closed. This leads to the following Lemma.

Lemma 3.3 *Let $E_2 \subseteq [0, m]$ be compact. Write*

$$P_2 = \{F \subseteq S : F \cap E_2 \neq \phi\} \cup \bar{J}(m).$$

Then P_2 is a closed prime ideal of 2^S.

Again, the proof that P_2 is a closed prime ideal is left to the reader.

Putting the above facts together we have the following theorem.

Theorem 3.4. *Let S be totally ordered. The closed prime ideals of 2^S are those of the form $P = P_1 \cup P_2$ where P_1 and P_2 are closed ideals of the types described in Lemmas* 3.2 *and* 3.3 *respectively.*

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