

## On the Identification of Material Symmetry for Anisotropic Elastic Materials

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Stephen C. Cowin, M. M. Mehrabadi

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ON THE IDENTIFICATION OF MATERIAL  
SYMMETRY FOR ANISOTROPIC ELASTIC MATERIALS

BY  
STEPHEN C. COWIN  
AND  
MORTEZA M. MEHRABADI

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**UNIVERSITY OF MINNESOTA**  
**514 Vincent Hall**  
**206 Church Street S.E.**  
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On the Identification of Material  
Symmetry for Anisotropic Elastic Materials

by

Stephen C. Cowin  
Department of Biomedical Engineering  
Tulane University  
New Orleans, LA 70118

and

Morteza M. Mehrabadi  
Department of Mechanical Engineering  
Tulane University  
New Orleans, LA 70118

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## SUMMARY

The problem considered here is that of identifying the type of elastic material symmetry of a material, given the values of the components of the fourth rank elasticity tensor of the material relative to a known, but arbitrary, coordinate system. Four simple eigenvalue problems are posed for the determination of the normals to the planes of reflective material symmetry of the elastic material. The solution of the eigenvalue problems will determine the number and orientation of the normals to the planes of reflective material symmetry. This information is then used to determine if the material has triclinic, monoclinic, orthotropic, tetragonal, transversely isotropic or isotropic material symmetry. It is suggested that this restricted method of classification of elastic material symmetries should suffice for most materials where the elastic symmetry is not determined by the crystalline structure alone. This should include all materials except for single crystals. It would include structural materials, geomaterials and biological materials.

## 1. Introduction

### (a) The General Problem

The problem considered here is that of determining the material symmetry of an anisotropic elastic material. The constitutive relation for linear anisotropic elasticity is the generalized Hooke's law,

$$T_{ij} = C_{ijkl} E_{kl}, \quad (1.1)$$

which is the most general linear relation between the stress tensor whose components are  $T_{ij}$  and the strain tensor whose components are  $E_{ij}$ , where the strain has been assumed to be measured from an unstressed reference state. The coefficients of linearity,  $C_{ijkl}$ , are the components of the fourth rank elasticity tensor. There are three important symmetry restrictions on the elasticity tensor  $C_{ijkl}$ , restrictions that are independent of those imposed by material symmetry. These are the symmetries

$$C_{ijkl} = C_{jikl}, \quad C_{ijkl} = C_{ijlk}, \quad C_{ijkl} = C_{klij}, \quad (1.2)$$

which follow from the symmetry of the stress tensor, the symmetry of the strain tensor, and the thermodynamic requirement that no work be produced by the elastic material in a closed loading cycle, respectively. The number of independent components of a fourth rank tensor in three dimensions is 81, but the restrictions (1.2) reduce the number of independent components of  $C_{ijkl}$  to 21. As is customary in the discussion of linear anisotropic elasticity, we introduce the single index notation for stress ( $\alpha_1 = T_{11}$ ,  $\alpha_2 = T_{22}$ ,  $\alpha_3 = T_{33}$ ,  $\alpha_4 = T_{23}$ ,  $\alpha_5 = T_{13} = T_{31}$ ,  $\alpha_6 = T_{12} = T_{21}$ ) and strain ( $\epsilon_1 = E_{11}$ ,  $\epsilon_2 = E_{22}$ ,  $\epsilon_3 = E_{33}$ ,  $\epsilon_4 = 2E_{23} = 2E_{32}$ ,  $\epsilon_5 = 2E_{13} = 2E_{31}$ ,  $\epsilon_6 = 2E_{12} = 2E_{21}$ ); thus (1.1) is replaced by

$$\sigma_{\alpha} = c_{\alpha\beta} \epsilon_{\beta}, \quad (1.3)$$

where the 6x6 matrix with components  $c_{\alpha\beta}$  represents the components of  $C_{ijkl}$ . The shift from an index system with a range of three ( $i, j = 1, 2, 3$ ) to one with a range of 6 ( $\alpha, \beta = 1, 2, 3, 4, 5, 6$ ) is accomplished by the following rules for replacing a pair of indices  $ij$  by a single index: 11 to 1, 22 to 2, 33 to 3, 23 to 4, 13 to 5 and 12 to 6. This change of notation incorporates the first two symmetries of (1.2) and the third is reflected in the symmetry of the 6x6 matrix  $c$ ,

$$c = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix}. \quad (1.4)$$

For specific material symmetries the matrix (1.4) will have fewer than 21 non-zero and distinct components in special cartesian coordinate systems associated with the specific material symmetry involved. These special cartesian coordinate systems are referred to here as symmetry cartesian coordinate systems. As an example, there is a symmetry cartesian coordinate system for orthotropic material symmetry relative to which the components of  $C_{ijkl}$  reduce to nine distinct components. These components are represented by a  $c$  of the form

$$c = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & c & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} . \quad (1.5)$$

However, in general, in a coordinate system other than its symmetry coordinate system there will be 21 non-zero components of  $C_{ijklm}$  for orthotropic symmetry. The relationship between the representation of  $C_{ijklm}$  relative to an arbitrary coordinate system and relative to its symmetry coordinate system is analogous to the relationship between the representation of a symmetric second rank tensor  $T_{ij}$  relative to an arbitrary coordinate system,

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{bmatrix} , \quad (1.6)$$

and relative to its principal coordinate system,

$$\begin{bmatrix} T_{11} & 0 & 0 \\ 0 & T_{22} & 0 \\ 0 & 0 & T_{33} \end{bmatrix} . \quad (1.7)$$

Thus, for example, the components of  $C_{ijklm}$  for orthotropic symmetry relative to an arbitrary coordinate system will, in general, have a representation of the form (1.4), but relative to the symmetry



coordinate system it will have the representation (1.5). For each type of material symmetry the representation (1.5) and its associated symmetry coordinate system or systems will be different.

The general problem posed here assumes that one is given the numerical values of  $C_{ijkl}$  relative to an arbitrarily selected but known coordinate system  $x_k$  for a material of unknown symmetry. Methods for the measurement of the values of  $C_{ijkl}$  are described by Hayes (1) and by Van Buskirk, Cowin and Carter (2). Hayes (1) outlines a program of mechanical tests for the determination of the values. Van Buskirk, Cowin and Carter (2) suggest a method employing ultrasonic wave propagation which employs only one small material specimen. The general problem considered in this paper is to use this information to determine the symmetry coordinate systems for the material. Once the symmetry coordinate system has been determined the fourth rank tensor transformation law can be used to obtain the numerical values of  $C_{ijkl}$  in the symmetry coordinate system.

#### (b) The Restricted Problem

The general problem just described has alternative methods of solution in the case of a single crystal. The techniques of crystal physics can be used to identify crystallographic directions which, in turn, can be used to identify the symmetry coordinate systems. Measurements of the elastic constants can then be accomplished with a prior knowledge of the crystallographic directions and symmetry coordinate systems. In this way the general problem described above is avoided in the case of single crystals.

Two restrictions are placed on the general problem described above. Both of these restrictions tend to exclude the consideration of single crystals. The first restriction is a restriction on the class of real materials to which the following analysis is addressed. The second restriction is that the set of material symmetries considered is a subset of those permitted in linear anisotropic elasticity. These two restrictions are discussed in the following two paragraphs.

The real materials of interest here are described as mesomorphic because their elastic symmetry is not determined by their chemical constituents, but rather by an intermediate structural level of the material. Examples of mesomorphic materials include almost all geomaterials, all plant and animal tissue and almost all manufactured materials. A single crystal would not be mesomorphic because its elastic symmetry is generally determined by its chemical constituents. Structural metals, being polycrystals, are mesomorphic because the source of their material symmetry is their method of manufacture (extrusion, rolling, etc.) and not the crystalline microstructure required by their chemical composition.

The elastic material symmetries of interest are a subset described as the pure reflective symmetries. The pure reflective symmetries of elasticity are six in number and include isotropy, transverse isotropy, tetragonal symmetry, orthotropy, monoclinic and triclinic symmetry. Isotropy is the maximum material symmetry possible and triclinic symmetry is the minimum possible. Table 1 is a table of the pure reflective symmetries. The classification of the pure reflective symmetries is by the number of distinct planes of

symmetry possible in a given material symmetry. The number of the planes of symmetry associated with the six pure reflective symmetries are listed in the second column of Table 1. A plane of symmetry at a point in an elastic material is a plane with respect to which the material has reflective symmetry. The concept of the plane of symmetry is discussed by Love (3) and Spencer (4). It is one of the geometric elements, along with the concept of axis of symmetry, used in describing the groups that characterize the elastic material symmetries. The terminology pure reflective symmetries was introduced to describe the six elastic symmetries listed in Table 1 because it will be shown in section 4 that these symmetries are completely classified by the number of planes of reflective symmetry they possess.

The total number of material symmetries possible in linear anisotropic elasticity is 34. These include isotropy, transverse isotropy and the 32 crystal classes. The pure reflective symmetries include isotropy, transverse isotropy and 4 of the 32 crystal classes. Thus the sets of pure reflective symmetries and symmetries of the crystal classes contain all of the material symmetries possible in linear anisotropic elasticity, but they are intersecting subsets.

## 2. The Conditions for the Existence of a Plane of Symmetry

The main theorem of this paper is that the conditions

$$C_{irpq} a_r a_p a_q = (C_{rspq} a_r a_s a_p a_q) a_i, \quad (2.1)$$

$$C_{ikkj} a_j = (C_{pkkq} a_p a_q) a_i, \quad (2.2)$$

$$C_{ijkk} a_j = (C_{pqkk} a_p a_q) a_i, \quad (2.3)$$

and

$$C_{ijkl} b_j b_k a_m = (C_{rspq} b_s b_p a_r a_q) a_i, \quad (2.4)$$

constitute a set of necessary and sufficient conditions for the vector  $\underline{a}$  to be the normal to a plane of symmetry of a material of given elasticities  $C_{ijkl}$ . The vector  $\underline{b}$  is any vector perpendicular to  $\underline{a}$ .

We will prove necessity first. Let  $\underline{R}$  be a symmetric improper orthogonal tensor representing the reflection in a plane with unit normal  $\underline{a}$ , then (see, e.g., Spencer (4))

$$R_{ij} = \delta_{ij} - 2a_i a_j. \quad (2.5)$$

The orthogonality of  $\underline{R}$  is expressed by the conditions

$$R_{ki} R_{kj} = R_{ik} R_{kj} = \delta_{ij}. \quad (2.6)$$

The effect of  $\underline{R}$  operating on the vector  $\underline{a}$  and on  $\underline{b}$ , which is any vector perpendicular to  $\underline{a}$ , are easily calculated from (2.5), thus

$$R_{ij} a_j = R_{ji} a_j = -a_i, \quad (2.8)$$

$$R_{ij} b_j = R_{ji} b_j = b_i, \quad \underline{a} \cdot \underline{b} = 0. \quad (2.9)$$

The condition that  $\underline{R}$  belongs to the symmetry group of the material whose elasticity tensor is  $C_{ijkl}$  is expressed by

$$C_{ijkl} = R_{ir} R_{js} R_{kp} R_{mq} C_{rspq}. \quad (2.10)$$

This is a special case of equation (e) on p. 70 of Gurtin (5) and it is a condition discussed in some detail by Spencer (4).

To prove the result (2.1) one contracts both sides of (2.10) with the triplet  $a_j a_k a_m$  and then employs (2.8) three times and (2.5) once. To prove (2.2) one first contracts (2.10) with respect to the  $j$  and  $k$  indices and with  $a_m$ , thus from (2.5)

$$C_{ikkm} a_m = R_{ir} R_{mq} a_m C_{rppq} \quad (2.11)$$

Employing (2.5) and (2.8), the result (2.2) follows. The proof of (2.3) is very similar to the proof of (2.2) and we omit its description. To prove (2.4) one contracts (2.10) with  $b_j b_k a_m$  and then employs (2.8) and (2.5) once and (2.9) twice.

To prove sufficiency, it is shown that if  $\underline{a}$  and  $\underline{b}$  are solutions of (2.1) through (2.4), then  $\underline{a}$  is the normal to a plane of symmetry. Let  $\underline{a}$  and  $\underline{b}$  be the solutions of these equations. Then, without a loss in generality, one can take the coordinate axes  $x_1$  and  $x_2$  along the directions of  $\underline{a}$  and  $\underline{b}$ , respectively. With respect to such a coordinate system, (2.1) through (2.4) yield

$$\begin{aligned} C_{i111} &= C_{1111} \delta_{i1} & , & & C_{ikkl} &= C_{1kk1} \delta_{i1} & , \\ C_{i1kk} &= C_{11kk} \delta_{i1} & , & & C_{i221} &= C_{1221} \delta_{i1} & . \end{aligned} \quad (2.12)$$

The explicit form of these results is

$$C_{1112} = C_{1113} = C_{2212} = C_{2213} = C_{2321} = C_{2331} = C_{3312} = C_{3313} = 0,$$

or equivalently,

$$c_{16} = c_{15} = c_{25} = c_{26} = c_{46} = c_{45} = c_{36} = c_{35} = 0. \quad (2.13)$$

From Table 1 it can be seen that the conditions (2.13) are the requirements for monoclinic material symmetry where the  $x_1$  co-ordinate direction is the normal to the plane of symmetry. It follows then that any solution of (2.1) through (2.4) is the normal to a plane of symmetry. This completes the proof of the theorem.

The conditions (2.2) and (2.3) of this theorem show that  $\underline{a}$  must be an eigenvector of the symmetric second rank tensors  $C_{ikkk}$  and  $C_{ijkk}$ , respectively. These two conditions allow one to calculate a set of possible  $\underline{a}$ 's. If one of the members of the set of  $\underline{a}$ 's also

satisfies (2.1) and (2.4), that is to say that it is also an eigenvector of  $C_{irpq}^a p^a q^a$  (or  $C_{irpq}^a r^a q^a$  or  $C_{irpq}^a r^a p^a$ ) and  $C_{ijklm}^b j^b k^b$ , then it represents a normal to a plane of symmetry of a material characterized by  $C_{ijklm}$ .

As a numerical example of the application of this theorem, consider the following symmetric 6x6 matrix to be a given  $\underline{c}$  matrix of the type (1.4):

$$\underline{c} = \frac{1}{81} \begin{bmatrix} 1769.48 & 873.50 & 838.22 & -17.68 & -110.32 & 144.92 \\ 873.50 & 1846.64 & 836.66 & -37.50 & -32.32 & 153.80 \\ 838.22 & 836.66 & 1603.28 & -29.68 & -93.52 & 22.40 \\ -17.68 & -37.60 & -29.68 & 438.77 & 57.68 & -50.50 \\ -110.32 & -32.32 & -93.52 & 57.68 & 439.79 & -34.78 \\ 144.92 & 153.80 & 22.70 & -50.50 & -37.78 & 501.80 \end{bmatrix} \quad (2.14)$$

We first seek the eigenvectors of  $C_{ijkk}$  and  $C_{ikkj}$  to determine if a plane of symmetry exists. That is to say, we first seek to satisfy equations (2.2) and (2.3). These calculations are summarized as follows:

$$\begin{aligned} C_{ijkk} &= \begin{bmatrix} c_{11}+c_{12}+c_{13} & c_{16}+c_{26}+c_{36} & c_{15}+c_{25}+c_{35} \\ c_{16}+c_{26}+c_{36} & c_{12}+c_{22}+c_{23} & c_{14}+c_{24}+c_{34} \\ c_{15}+c_{25}+c_{35} & c_{14}+c_{24}+c_{34} & c_{13}+c_{23}+c_{33} \end{bmatrix} = \\ &= \frac{1}{9} \begin{bmatrix} 386.80 & 35.68 & -26.24 \\ 35.68 & 395.20 & -9.44 \\ -26.24 & -9.44 & 364.24 \end{bmatrix} = \underline{Q} \begin{bmatrix} 38.08 & 0 & 0 \\ 0 & 40.88 & 0 \\ 0 & 0 & 48.40 \end{bmatrix} \underline{Q}^T, \end{aligned} \quad (2.15)$$

$$\begin{aligned}
C_{ikkk} &= \begin{bmatrix} c_{11}+c_{55}+c_{66} & c_{16}+c_{26}+c_{45} & c_{15}+c_{46}+c_{35} \\ c_{16}+c_{26}+c_{45} & c_{22}+c_{44}+c_{66} & c_{24}+c_{34}+c_{56} \\ c_{15}+c_{46}+c_{35} & c_{24}+c_{34}+c_{56} & c_{33}+c_{44}+c_{55} \end{bmatrix} = \\
&= \begin{bmatrix} 33.47 & 4.4 & -3.14 \\ -4.4 & 34.41 & -1.26 \\ -3.14 & -1.26 & 30.64 \end{bmatrix} = \underset{\sim}{Q} \begin{bmatrix} 28.13 & 0 & 0 \\ 0 & 30.95 & 0 \\ 0 & 0 & 39.44 \end{bmatrix} \underset{\sim}{Q}^T,
\end{aligned} \tag{2.16}$$

where  $\underset{\sim}{Q}$  is an orthogonal matrix given by

$$\underset{\sim}{Q} = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -1/3 & -2/3 & 2/3 \\ 2/3 & -2/3 & -1/3 \end{bmatrix}. \tag{2.17}$$

These results show that, since the eigenvalues are distinct, not more than three planes of symmetry can exist. Further, since the eigenvectors of  $C_{ijkk}$  and  $C_{ikkk}$  are coincident, all three could be normals to planes of symmetry. These three eigenvectors are

$$\begin{aligned}
\underset{\sim}{e}'_1 &= (2/3)\underset{\sim}{e}_1 - (1/3)\underset{\sim}{e}_2 + (2/3)\underset{\sim}{e}_3, \\
\underset{\sim}{e}'_2 &= (1/3)\underset{\sim}{e}_1 - (2/3)\underset{\sim}{e}_2 - (2/3)\underset{\sim}{e}_3, \\
\underset{\sim}{e}'_3 &= (2/3)\underset{\sim}{e}_1 + (2/3)\underset{\sim}{e}_2 - (1/3)\underset{\sim}{e}_3,
\end{aligned} \tag{2.18}$$

where  $\underset{\sim}{e}_k$  are unit vectors along the coordinate axes of  $C_{ijkl}$  corresponding to  $\underset{\sim}{c}$  given by (2.14). The next step is to determine if the vectors  $\underset{\sim}{e}'_1$ ,  $\underset{\sim}{e}'_2$  and  $\underset{\sim}{e}'_3$  are also solutions of (2.1) and (2.4). They are. Since there are three mutually orthogonal planes of symmetry for the material whose  $\underset{\sim}{c}$  is given by (2.14), it follows that the material has orthotropic symmetry.

The three eigenvectors (2.18) form a symmetry cartesian coordinate system for the material whose  $\underset{\sim}{c}$  is given by (2.14). We

denote the components of  $\underline{c}$  referred to its symmetry coordinate system by  $\underline{c}'$ . To calculate  $\underline{c}'$  we recall the cartesian tensor transformation law for the fourth rank elasticity tensor

$$C_{i'j'k'm'} = Q_{i'i} Q_{j'j} Q_{k'k} Q_{m'm} C_{ijklm} , \quad (2.19)$$

where  $Q_{i'k}$  is an orthogonal transformation whose numerical components are given by (2.17) and which transforms from the coordinate system whose base vectors are  $\underline{e}_1, \underline{e}_2, \underline{e}_3$  to the symmetry coordinate system of the material. From (2.14), (2.19) and (2.17) it follows that

$$\underline{c}' = \begin{bmatrix} 18 & 9.98 & 10.1 & 0 & 0 & 0 \\ 9.98 & 20.2 & 10.7 & 0 & 0 & 0 \\ 10.1 & 10.7 & 27.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6.23 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5.61 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4.52 \end{bmatrix} . \quad (2.20)$$

This completes the numerical example. The values of  $\underline{c}'$  given by (2.20) are the elasticities of human femoral bone reported by Ashman, Cowin, Van Buskirk and Rice (5). The numerical example was generated by applying the inverse transformation to (2.19) to (2.20) using (2.17), producing the matrix (2.14) which was the start of the numerical example.

### 3. Generation of the Set of Pure Reflective Material Symmetries

It has been noted that a material with triclinic material symmetry has no plane of symmetry and that a material with monoclinic symmetry has one. In order to continue this classification scheme



for material symmetries by planes of symmetry, a material with more than one plane of symmetry is considered next.

First assume that one plane of symmetry exists and choose  $x_1$ -axis along its unit normal  $\underline{a}$ . With respect to such a coordinate system  $C_{ijkk}$  and  $C_{ikkj}$  will have the form

$$C_{ijkk} = \begin{bmatrix} C_{11kk} & 0 & 0 \\ 0 & C_{22kk} & C_{23kk} \\ 0 & C_{23kk} & C_{33kk} \end{bmatrix} = \begin{bmatrix} c_{11}+c_{12}+c_{13} & 0 & 0 \\ 0 & c_{12}+c_{22}+c_{23} & c_{14}+c_{24}+c_{34} \\ 0 & c_{14}+c_{24}+c_{34} & c_{13}+c_{23}+c_{33} \end{bmatrix}, \quad (3.1)$$

and

$$C_{ikkj} = \begin{bmatrix} C_{1kk1} & 0 & 0 \\ 0 & C_{2kk2} & C_{2kk3} \\ 0 & C_{2kk3} & C_{3kk3} \end{bmatrix} = \begin{bmatrix} c_{11}+c_{55}+c_{66} & 0 & 0 \\ 0 & c_{22}+c_{44}+c_{55} & c_{24}+c_{34}+c_{55} \\ 0 & c_{24}+c_{34}+c_{55} & c_{33}+c_{44}+c_{55} \end{bmatrix}. \quad (3.1)$$

where the conditions (2.13) have been employed. Now if a second plane of symmetry exists, since it has to be a principal plane of  $C_{ijkk}$  and  $C_{ikkj}$ , it must be orthogonal to the first plane of symmetry. Selecting the  $x_2$ -axis along the normal to the second plane of symmetry, it follows from (3.1) that

$$C_{23kk} = c_{14} + c_{24} + c_{34} = 0, \quad (3.2)$$

$$C_{2kk3} = c_{24} + c_{34} + c_{55} = 0, \quad (3.3)$$

while (2.1) and (2.4) require that

$$C_{1222} = c_{24} = 0, \quad (3.4)$$

$$C_{1332} = c_{55} = 0, \text{ and } C_{3332} = c_{34} = 0. \quad (3.5)$$

Equations (3.2) through (3.5) are equivalent to the following

$$c_{14} = c_{24} = c_{34} = c_{55} = 0. \quad (3.6)$$

These conditions, in conjunction with the conditions for monoclinic symmetry (2.13), will admit a third plane of symmetry mutually orthogonal to the first two. To verify this one need only note that since the  $x_1$  and  $x_2$  axes have been chosen to coincide with the principal directions of  $C_{ijkk}$  and  $C_{ikkj}$ ,  $\underline{e}_3$ , the unit vector along  $x_3$ -axis, is the third eigenvector of  $C_{ijkk}$  and  $C_{ikkj}$ . Thus  $\underline{a} = \underline{e}_3$  is a solution of (2.2) and (2.3). It is easy to show that  $\underline{a} = \underline{e}_3$  also satisfies (2.1) and (2.4). Thus, a third plane of symmetry with the unit normal  $\underline{e}_3$  exists. Therefore, it can be concluded that if two orthogonal planes of symmetry exist, then there exists a third mutually orthogonal plane of symmetry and the material has orthotropic symmetry.

Suppose a fourth plane of symmetry exists. Since this plane should also be a principal plane of  $C_{ijkk}$ , two of the eigenvalues of  $C_{ijkk}$  have to be equal, for example,

$$C_{11kk} = C_{22kk} . \quad (3.7)$$

This makes any direction in  $x_1$ - $x_2$  plane a principal direction of  $C_{ijkk}$ . A similar argument holds for eigenvalues of  $C_{ikkj}$ , thus

$$C_{1kk1} = C_{2kk2} . \quad (3.8)$$

The equations (3.7) and (3.8) yield the following results:

$$C_{11} - C_{22} = C_{23} - C_{13} = C_{44} - C_{55} . \quad (3.9)$$

Now since  $\underline{a}$ , the unit normal to the fourth plane of symmetry, is in  $x_1$ - $x_2$  plane, it could be represented as

$$\underline{a} = \cos\theta \underline{e}_1 + \sin\theta \underline{e}_2 , \quad (3.10)$$

and  $\underline{b}$ , the vector normal to  $\underline{a}$ , could be represented by

$$\underline{b} = -\sin\theta \underline{e}_1 + \cos\theta \underline{e}_2 . \quad (3.11)$$

With these definitions for  $\underline{a}$  and  $\underline{b}$ , the matrices  $C_{ijpq} a^p a^q$  and

$C_{ipqj}b_p b_q$ , appearing in (2.1) and (2.4), would have the form

$$C_{ijpq}a_p a_q = \begin{bmatrix} c_{11}\cos^2\theta + c_{12}\sin^2\theta & c_{66}\sin 2\theta & 0 \\ c_{66}\sin 2\theta & c_{12}\cos^2\theta + c_{22}\sin^2\theta & 0 \\ 0 & 0 & c_{13}\cos^2\theta + c_{23}\sin^2\theta \end{bmatrix},$$

and

$$C_{ipqj}b_p b_q = \begin{bmatrix} c_{11}\sin^2\theta + c_{66}\cos^2\theta & -(c_{12} + c_{66})\sin 2\theta & 0 \\ -(c_{12} + c_{66})\sin 2\theta & c_{66}\sin^2\theta + c_{22}\cos^2\theta & 0 \\ 0 & 0 & c_{55}\sin^2\theta + c_{44}\cos^2\theta \end{bmatrix}.$$

Now recalling (2.1) and (2.7),  $\underline{a}$  has to be an eigenvector of these two matrices, thus

$$\begin{aligned} \tan 2\theta &= \frac{2c_{66}\sin 2\theta}{c_{11}\cos^2\theta + c_{12}(\sin^2\theta - \cos^2\theta) - c_{22}\sin^2\theta} = \\ &= \frac{-(c_{12} + c_{66})\sin 2\theta}{c_{11}\sin^2\theta + c_{66}(\cos^2\theta - \sin^2\theta) - c_{22}\cos^2\theta}. \end{aligned}$$

A little algebraic manipulation shows that these two conditions are equivalent to the following two equations:

$$c_{11} - c_{22} = 0, \quad (3.12)$$

and

$$[c_{66} - (c_{11} - c_{12})/2]\cos 2\theta = 0. \quad (3.13)$$

Combining (3.9), (3.12) and (3.13), we find that for four planes of symmetry to exist, we should have either

$$\begin{aligned} c_{11} &= c_{22}, \quad c_{44} = c_{55}, \quad c_{13} = c_{23}, \\ \theta &= 45^\circ \quad \text{or} \quad 135^\circ, \end{aligned} \quad (3.14)$$

or

$$\begin{aligned} c_{11} &= c_{22}, \quad c_{44} = c_{55}, \quad c_{13} = c_{23}, \\ c_{66} &= (c_{11} - c_{12})/2, \\ \theta &\neq 45^\circ \quad \text{or} \quad 135^\circ. \end{aligned} \quad (3.15)$$

A material whose elasticity matrix satisfies the conditions (2.13), (3.6) and (3.14) has tetragonal symmetry which is characterized by five planes of symmetry. The five planes consist of four whose normals lie in one plane, each making an angle of  $45^\circ$  with its nearest neighbors, and a fifth whose normal lies in the plane perpendicular to the plane of the first four normals. These are the three mutually orthogonal planes of orthotropic symmetry plus two more mutually orthogonal planes making an angle of  $45^\circ$  with the first set of three.

When the elasticity matrix satisfies (2.13), (3.6), and (3.15), then any vector in the plane of  $\underline{e}_1$  and  $\underline{e}_2$  is the normal to a plane of symmetry and it follows that the  $\underline{e}_1, \underline{e}_2$  plane is a plane of isotropy. Note that the discussion following (3.6) shows that any plane of isotropy is also a plane of symmetry. An elastic material with one plane of isotropy has transversely isotropic material symmetry.

Suppose now that an elastic material with transversely isotropic material symmetry has an additional, distinct plane of symmetry. It will be shown that such a material necessarily has isotropic material symmetry. To this end consider a transversely isotropic material whose plane of isotropy is the  $\underline{e}_1, \underline{e}_2$  plane and whose elasticity matrix satisfies (2.13), (3.6) and (3.15). Since any additional, distinct plane of symmetry should be a principal plane of  $C_{ijkl}$  and  $C_{ikkk}$ , we must have, in addition to (3.7) and (3.8),

$$C_{22kk} = C_{33kk}, \quad C_{2kk2} = C_{33kk},$$

or equivalently

$$C_{22} - C_{33} = C_{13} - C_{12} = C_{55} - C_{66}. \quad (3.16)$$

Again since  $\underline{a}$ , the unit normal to the additional plane of symmetry,

is in  $x_2$ - $x_3$  plane, it could be represented as

$$\underline{a} = \cos\phi \underline{e}_2 + \sin\phi \underline{e}_3 .$$

Requiring that  $\underline{a}$  be further an eigenvector of  $C_{ijpq} a_p a_q$  and  $C_{ipqj} b_p b_q$  and following an argument similar to that which led to (3.12) and (3.13), we find

$$c_{22} - c_{33} = 0, \quad (3.17)$$

and

$$[c_{44} - (c_{22} - c_{23})/2] \cos 2\phi = 0 . \quad (3.18)$$

These equations can be obtained from (3.12) and (3.13) by changing  $1 \rightarrow 2$ ,  $2 \rightarrow 3$ , and  $6 \rightarrow 4$  in (3.12) and (3.13). Combining (3.16) and (3.17), we have

$$c_{22} = c_{33} , \quad c_{12} = c_{13} , \quad c_{55} = c_{66} . \quad (3.19)$$

It follows from (3.15)<sub>2</sub>, (3.19)<sub>3</sub>, (3.15)<sub>4</sub>, (3.19)<sub>2</sub> and (3.15)<sub>3</sub> that

$$c_{44} = (c_{22} - c_{23})/2 , \quad (3.20)$$

and thus (3.18) is identically satisfied for any  $\phi$ . The conditions (3.19) imposed on the elasticity coefficients for a transversely isotropic material require that it be isotropic.

Substitution of elasticity tensor  $C_{ijkl}$  for isotropic material symmetry

$$C_{ijkl} = \lambda \delta_{ij} \delta_{km} + \mu (\delta_{ik} \delta_{jm} + \delta_{im} \delta_{jk}) , \quad (3.21)$$

where  $\lambda$  and  $\mu$  are the Lamé elastic constants, into the condition (2.10) for the existence of a normal for a plane of symmetry satisfies (2.10) identically for any  $\underline{a}$ . Thus, for an isotropic material it follows that any plane passing through a point is both a plane of symmetry and a plane of isotropy. The results developed in this section are summarized in the following theorem:

Theorem (The Classification Theorem for Anisotropic Elastic Material Symmetries by Planes of Symmetry): The following classification of elastic material symmetries by planes of symmetry is exhaustive. Anisotropic elastic material symmetry is characterized by either no planes of symmetry (triclinic), one plane of symmetry (monoclinic), three mutually orthogonal planes of symmetry (orthotropic), five planes of symmetry (one set of three which are mutually orthogonal and a second set of two which are mutually orthogonal but make an angle of  $45^{\circ}$  with respect to two planes of the first set of three) (tetragonal), one plane of isotropy (transverse isotropy) or by every plane passing through a point being a plane of symmetry and a plane of isotropy (isotropy).

These elastic material symmetries have been called the pure reflective material symmetries in this paper.

#### 4. Discussion

The pure reflective material symmetries of elasticity represent the elastic symmetries of most materials that are not single crystals such as structural materials, biological materials and geological materials. The elastic symmetries of the 32 crystal classes, on the other hand, represent the elastic symmetries of single crystals. This division is not rigorous because, for example, there are four material symmetries that are contained in both the set of pure reflective material symmetries and the set of elastic symmetries of the 32 crystal classes. However, it does appear that one can broadly associate most single crystals with the material symmetries of the crystal classes and most mesomorphic materials with the pure

reflective material symmetries. Since this is the case, perhaps these two classes of material symmetry should receive equal treatment in general elasticity volumes, that is to say, elasticity volumes not devoted to crystal physics. It appears that tradition in elasticity has followed the work of Voigt (7) in relying heavily on the crystal classes in the discussion of anisotropy, while the applications of elasticity have been to non pure crystalline materials as well as to single crystals. The set of pure reflective material symmetries are a much simpler to understand set of material symmetries than those of the crystal classes and they apply to a broader class of materials.

Our initial calculations indicate that an exhaustive and complete solution to the problem of the identification of material symmetry is possible. That is to say, the restrictions in the present paper to mesomorphic materials and pure reflective symmetries are removable. We note that although there are 32 crystal classes as well as transverse isotropy and isotropy, these 34 symmetries correspond to only 10 distinct elasticity tensors of the form (1.5). These ten tensors are those of isotropy plus the nine associated with crystals originally worked out by Voigt and repeated in the works of Voigt (7), Love (3) and Gurtin (5). The present work develops a method for the identification of 6 of these 10 distinct elasticity tensors. It can be extended to identify the other four. For example, the results of this paper can be extended to identify the form of the elasticity tensor associated with the cubic crystal classes. If a tensor  $C_{ijklm}$  is found to represent orthotropic symmetry by the method of this paper and, in addition, is invariant under  $90^\circ$  rotations about the normals to each plane of symmetry, then

it is an elasticity tensor associated with the cubic crystal classes. A similar additional rotational requirement, if satisfied by a system determined to be monoclinic by the methods of this paper, will produce the elasticity tensor that represents the seven constant tetragonal crystal classes. Finally, it appears that the six and seven constant elasticity tensors characterizing some of the hexagonal crystal classes can be identified by seeking axes of symmetry corresponding to rotations of  $120^\circ$ . These results, combined with those presented in the present paper, will provide the structure of an algebraic sieve that will permit the identification of any one of the ten elasticity tensors that characterize the elastic symmetries of the thirty two crystal classes and the mesomorphic symmetries of isotropy, and transverse isotropy given the numerical values of  $C_{ijklm}$  relative to an arbitrary coordinate system. The present writers have undertaken the construction of the algebraic sieve described and will report on it in the future.

It is our ultimate goal to apply the method described to experimentally determined values of  $C_{ijklm}$  to identify the appropriate material symmetry. Experimental errors will be reflected as errors in eigenvectors which in turn, will cause the system of equations (2-1) to (2.4) not be satisfied, even when the elastic symmetry is one that should satisfy these equations. This statistical-numerical aspect of the algebraic sieve has not yet been studied.



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Type of Material Symmetry	Number of Planes of Symmetry	Number of Planes of Isotropy	Number of Independent Elastic Coefficients	Restrictions on the Elastic Coefficients $c_{\alpha\beta}$
Triclinic (1)	0	0	21	NONE
Monoclinic (2)	1	0	13	$c_{16} = c_{15} = c_{25} = c_{26} = 0,$ $c_{46} = c_{45} = c_{35} = c_{36} = 0,$
Orthotropic (3)	3	0	9	All of the above plus $c_{14} = c_{24} = c_{34} = c_{56} = 0,$
Tetragonal (4)	5	0	6	All of the above plus $c_{11} = c_{22}, c_{44} = c_{55}, c_{23} = c_{13},$
Transverse Isotropy (5)	$\infty^1 + 1$	1	5	All of the above plus $c_{66} = \frac{1}{2}(c_{22} - c_{12}),$
Isotropy (6)	$\infty^2$	$\infty^2$	2	All of the above plus $c_{11} = c_{33}, c_{13} = c_{12}, c_{44} = \frac{1}{2}(c_{11} - c_{13})$

TABLE 1 THE PURE REFLECTIVE MATERIAL SYMMETRIES OF LINEAR ANISOTROPIC ELASTICITY

The following is a description of the symmetry coordinate systems for the pure reflective symmetries.

- (1) Any coordinate system is a symmetry coordinate system for the triclinic symmetry because it is, in effect, the absence of symmetry. (2) Any coordinate system that contains the normal to the plane of symmetry as a coordinate direction is a symmetry coordinate system for monoclinic symmetry. (3,4) Any coordinate system formed from the normals to three perpendicular planes of symmetry as coordinate directions is a symmetry coordinate system for orthotropic symmetry or tetragonal symmetry. (5) Any coordinate system that contains the normal to the plane of isotropy as a coordinate direction is a symmetry coordinate system for transverse isotropy. (6) Any coordinate system is a symmetry coordinate system for isotropic symmetry.

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